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VOLTERRA DIFFERENTIAL CONSTITUTIVE OPERATORS AND LOCALITY CONSIDERATIONS IN ELECTROMAGNETIC THEORY

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Abstract—Macroscopic Maxwell's theory for electrodynamics is an indeterminate set of coupled, vector, partial differential equations. This infrastructure requires the supplement of constitutive equations.

Recently a general framework has been suggested, taking into account dispersion, inhomogeneity and nonlinearity, in which the constitutive equations are posited as differential equations involving the differential operators based on the Volterra functional series.

The validity of such representations needs to be examined. Here it is shown that for such representations to be effective, the spatiotemporal functions associated with the Volterra differential operators must be highly localized, or equivalently, widely extended in the transform space.

This is achieved by exploiting Delta-function expansions, leading in a natural way to polynomial differential operators.

The Four-vector Minkowski space is used throughout, facilitating general results and compact notation.

1. INTRODUCTION

Although our discussion applies to other models of physics based on an Eulerian [1] spatiotemporal description, we specifically focus on the macroscopic Maxwell model of electromagnetism, given by

$$\begin{aligned} \partial_{\mathbf{x}} \times \mathbf{E} &= -\partial_t \mathbf{B} - \mathbf{j}_m \\ \partial_{\mathbf{x}} \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\ \partial_{\mathbf{x}} \cdot \mathbf{D} &= \rho_e \\ \partial_{\mathbf{x}} \cdot \mathbf{B} &= \rho_m \end{aligned} \tag{1}$$

where $\partial_{\mathbf{x}}$ and ∂_t denote the space derivative (usually referred to as "Del" or "Nabla"), and the partial time derivative, operators, respectively. All the fields are space and time dependent, e.g., $\mathbf{E} = \mathbf{E}(\mathbf{X})$. Here

$$\mathbf{X} = (\mathbf{x}; ict) \tag{2}$$

symbolizes the space-time dependence, where the Minkowski-space location vector notation is used, with c denoting the universal constant of the speed of light, and i is the unit imaginary complex number $i^2 = -1$, but in the present study there is no attempt to include any relativistic considerations, and the same notation could also apply to other models, e.g., continuum mechanics. For symmetry and completeness, in the present representation, the Maxwell equations include the conventional electric (index e), as well as the fictitious magnetic (index m), current and charge density sources.

The Spatiotemporal functions (1) dependent on $\mathbf{X}$ can be represented in an associated spectral space $\mathbf{K}$ in terms of the four-dimensional Fourier transform pair, e.g., for some function $f(\mathbf{X})$ we have

$$f(\mathbf{X}) = \alpha \int (d^4\mathbf{K}) \overline{f}(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} \tag{3}$$

where $\alpha = (2\pi)^{-4}$, and the bar denotes the transformed function, and we effect a four-fold integration over the infinite spectral space,

$$\mathbf{K} = (\mathbf{k}, i\omega/c) \tag{4}$$

where $\mathbf{K}$ is a new quadruplet, once again constituting a Minkowski four-vector where $\mathbf{k}$ is the propagation vector and ω is the (angular) frequency. Associated with (3) is its inverse transformation

$$\overline{f}(\mathbf{K}) = \int (d^4\mathbf{X}) f(\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{5}$$

In (3), (5), $\mathbf{K}\cdot\mathbf{X} = \mathbf{k}\cdot\mathbf{x} - \omega t$ is a plane wave appropriate phase, but care must be exercised, because thus far (3), (5) refer to arbitrary functions, without reference to any specific wave equation. In an obvious manner, derivatives affect the exponential, yielding

$$\partial_{\mathbf{X}} f(\mathbf{X}) = \alpha \int (d^4\mathbf{K}) i\mathbf{K}\overline{f}(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} \tag{6}$$

where (e.g., see [2]), the four-gradient vector

$$\partial_{\mathbf{X}} = \left(\partial_{\mathbf{x}}, -\frac{i}{c}\partial_t\right) \tag{7}$$

once again constitutes a Minkowski space four-vector, due to the fact that in (6) we have a factor $\mathbf{K}$ on the right-hand side, which is a four-vector. Subsequently we use the same symbol for the function and its transform, and the distinction between $f(\mathbf{X})$, $\overline{f}(\mathbf{K})$ will be based on the argument, or on the context where the function is used. Thus by comparing (4), (7), we derive from (1) the Fourier transformed Maxwell's equations

$$\begin{aligned} i\mathbf{k}\times\mathbf{E} &= i\omega\mathbf{B} - \mathbf{j}_m \\ i\mathbf{k}\times\mathbf{H} &= -i\omega\mathbf{D} + \mathbf{j}_e \\ i\mathbf{k}\cdot\mathbf{D} &= \rho_e \\ i\mathbf{k}\cdot\mathbf{B} &= \rho_m \end{aligned} \tag{8}$$

where now the transformed fields, e.g., $\mathbf{E} = \mathbf{E}(\mathbf{K})$ etc. are dependent on $\mathbf{K}$ space coordinates, as can be deduced from the $\mathbf{K}$ space coordinates involved in (8).

2. DISPERSIVE HOMOGENEOUS LINEAR MEDIA

Consider first the case of linear dispersive homogeneous media. Starting from (8), constitutive relations must be propounded. As a prototypical case, an anisotropic dielectric constitutive relation is stipulated, given in $\mathbf{K}$ space by

$$\mathbf{D}(\mathbf{K}) = \tilde{\varepsilon}(i\mathbf{K})\cdot\mathbf{E}(\mathbf{K}) \tag{9}$$

Where "˜" denotes a dyadic (in the present case a three-dyadic) or generically higher tensor entities as displayed subsequently. For convenience, the argument of the constitutive parameter is written as $i\mathbf{K}$. Inasmuch as (9) constitutes a product, in the time domain we have a four-fold convolution integral

$$\mathbf{D}(\mathbf{X}) = \int (d^4\mathbf{X}_1)\tilde{\varepsilon}^*(\mathbf{X}_1) \cdot \mathbf{E}(\mathbf{X} - \mathbf{X}_1) \tag{10}$$

Note that $\tilde{\varepsilon}(i\mathbf{K})$, $\tilde{\varepsilon}^*(\mathbf{X})$ constitute a Fourier transform pair.

The convolution (10) constitutes a functional, or an integral operator acting on the field $\mathbf{E}(\mathbf{X})$, which yields $\mathbf{D}(\mathbf{X})$. We are now going to replace (10) with a *symbolic* differential operator. Although this formal transition has been shown before [3], the range of validity of the so-called Volterra differential operators has not been discussed thus far.

From the definition (3), we have

$$\begin{aligned}
\mathbf{D}(\mathbf{X}) &= \alpha \int (d^4\mathbf{K})\tilde{\varepsilon}(i\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}} \cdot \mathbf{E}(\mathbf{K}) \\
&= \alpha \int (d^4\mathbf{K})\tilde{\varepsilon}(i\mathbf{K} \Rightarrow \partial_{\mathbf{X}})e^{i\mathbf{K}\cdot\mathbf{X}} \cdot \mathbf{E}(\mathbf{K}) \\
&= \alpha\tilde{\varepsilon}(i\mathbf{K} \Rightarrow \partial_{\mathbf{X}}) \int (d^4\mathbf{K}) \cdot \mathbf{E}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}} \qquad (11)
\end{aligned}$$

in which we have replaced $i\mathbf{K}$ by the corresponding four-gradient operator acting on the exponential, and pulled it outside the integral sign. Note carefully that the new operator $\tilde{\varepsilon}(i\mathbf{K} \Rightarrow \partial_{\mathbf{X}})$ possesses the functional structure of $\tilde{\varepsilon}(i\mathbf{K})$ in (9) and not its transform $\tilde{\varepsilon}^*(\mathbf{X}_1)$ given in (10). Henceforth we will use the notation $\tilde{\varepsilon}(i\mathbf{K} \Rightarrow \partial_{\mathbf{X}}) \equiv \tilde{\varepsilon}(\partial_{\mathbf{X}})$ in the following form

$$\mathbf{D}(\mathbf{X}) = \tilde{\varepsilon}(\partial_{\mathbf{X}}) \cdot \mathbf{E}(\mathbf{X}) \tag{12}$$

understanding that the new operator is obtained by formally substituting the components of $i\mathbf{K}$ with the corresponding components of $\partial_{\mathbf{X}}$.

A direct transition from (12) to (9) is not feasible. Thus we can inverse-transform (12) and in accordance with (5) obtain

$$\mathbf{D}(\mathbf{K}) = \int (d^4\mathbf{X}) \left\{\tilde{\varepsilon}(\partial_{\mathbf{X}}) \cdot \mathbf{E}(\mathbf{X})\right\} e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{13}$$

but if we wish to avoid substituting (3) for $\mathbf{E}(\mathbf{X})$ inside the integral (13), then we are stuck. On the other hand, if we knew the detailed structure of $\tilde{\varepsilon}(\partial_{\mathbf{X}})$, then we could effect a repeated integration by parts.

For a simple example, let us assume that $\tilde{\boldsymbol{\varepsilon}}(\partial_{\mathbf{X}}) = \partial_{\mathbf{X}}\mathbf{F}$, where $\mathbf{F}$ is an arbitrary four-vector, then (13) can be recast as

$$\mathbf{D}(\mathbf{K}) = \int (d^4\mathbf{X}) \left\{ \partial_{\mathbf{X}}\mathbf{F} \cdot \mathbf{E}(\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}} \right\} - (d^4\mathbf{X}) \left\{ \partial_{\mathbf{X}} e^{-i\mathbf{K}\cdot\mathbf{X}} \right\} \mathbf{F} \cdot \mathbf{E}(\mathbf{X}) \tag{14}$$

and if the first integral (14) can be assumed to vanish, then we finally obtain

$$\mathbf{D}(\mathbf{K}) = i\mathbf{K}\mathbf{F} \cdot \mathbf{E}(\mathbf{K}) \tag{15}$$

which is consistent with (9).

Consequently one expects that for the representation (12) to be valid, the functions appearing in (10) must be appropriately qualified.

3. DELTA EXPANSIONS AND THE CONVOLUTION INTEGRAL

We start by defining the four-dimensional delta function (distribution) in Minkowski space

$$\delta(\mathbf{X}) = \alpha \int (d^4\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}_1} = \frac{i}{c} \delta(x)\delta(y)\delta(z)\delta(t) \tag{16}$$

Following Lindell [4] (see also Van Bladel [5]), an arbitrary three-dyadic function $\tilde{\varepsilon}(\mathbf{X})$ is recast in terms of differential operations on the delta-function

$$\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{X}) = \tilde{\mathbf{F}}_0 \cdot \delta(\mathbf{X}) + \tilde{\mathbf{F}}_1 : \partial_{\mathbf{X}} \delta(\mathbf{X}) + \cdots + \Big(\tilde{\mathbf{F}}_n \cdot\!\cdot\!\cdot \underbrace{\partial_{\mathbf{X}} \ldots \partial_{\mathbf{X}}}_{n} \Big) \delta(\mathbf{X}) + \cdots \tag{17}$$

where the constant coefficients $\tilde{\mathbf{F}}_0, \tilde{\mathbf{F}}_1, \ldots, \tilde{\mathbf{F}}_n$, are a four-dyadic, a double-four-dyadic, and higher tensors, as required by the four-gradient factors for the multiplication (denoted by the cluster $\cdot\!\cdot\!\cdot$) to render a three-dyadic on the left-hand side of (17). For brevity of notation (17) will be compacted in the form

$$\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{X}) = \left(\tilde{\mathbf{F}}_n \cdot\!\cdot\!\cdot \partial_{\mathbf{X}}^{(n)} \right) \delta(\mathbf{X}) \tag{18}$$

with the understanding that $\tilde{\mathbf{F}}_n$ is associated with the four-gradient operator $\partial_{\mathbf{X}}$ appearing n times and that (18) describes a series, wherein a summation on n (in the sense of the Einstein convention for repeated indices in summations) is effected.

Now substitute (18) into (5)

$$\tilde{\boldsymbol{\varepsilon}}(i\mathbf{K}) = \int (d^4\mathbf{X}) \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}} = \int (d^4\mathbf{X}) \left[\left(\tilde{\mathbf{F}}_n \cdot\!\cdot\!\cdot \partial_{\mathbf{X}}^{(n)} \right) \delta(\mathbf{X}) \right] e^{-i\mathbf{K}\cdot\mathbf{X}} \tag{19}$$

The first term (17) yields the constant dyadic $\tilde{\mathbf{F}}_0$. Similarly to the one-dimensional formula for an integral involving the derivative of the delta function, here the second term is dealt by effecting integration by parts yielding

$$\begin{aligned}&\int (d^4\mathbf{X}) \left[\left(\tilde{\mathbf{F}}_1 : \partial_{\mathbf{X}}\right) \delta(\mathbf{X})\right] e^{-i\mathbf{K}\cdot\mathbf{X}} \\ &= \int (d^4\mathbf{X}) \left[\left(\tilde{\mathbf{F}}_1 : \partial_{\mathbf{X}}\right) \delta(\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}}\right] - \int (d^4\mathbf{X}) \left[\left(\tilde{\mathbf{F}}_1 : \partial_{\mathbf{X}}\right) e^{-i\mathbf{K}\cdot\mathbf{X}}\right] \delta(\mathbf{X})\end{aligned} \tag{20}$$

Obviously, like in the one-dimensional case, the first integral on the right-hand side (20) vanishes, while the second yields $\tilde{\mathbf{F}}_1 : i\mathbf{K}$. Consistently we find

$$\tilde{\varepsilon}(i\mathbf{K}) = \tilde{\mathbf{F}}_0 \cdot + \tilde{\mathbf{F}}_1 : i\mathbf{K} + \cdots + \tilde{\mathbf{F}}_n \because \underbrace{i\mathbf{K} \ldots i\mathbf{K}}_{n} + \cdots = \tilde{\mathbf{F}}_n \because (i\mathbf{K})^{(n)} \tag{21}$$

as the Fourier transform of (17), (18).

Substituting (18) in (10), we obtain an integral similar to (19), however the exponential $e^{-i\mathbf{K}\cdot\mathbf{X}}$ is now replaced by $\mathbf{E}(\mathbf{X} - \mathbf{X}_1)$, thusly

$$\begin{aligned}\mathbf{D}(\mathbf{X}) &= \int (d^4\mathbf{X}_1) \tilde{\varepsilon}^*(\mathbf{X}_1) \cdot \mathbf{E}(\mathbf{X} - \mathbf{X}_1) \\ &= \int (d^4\mathbf{X}_1) \left[\left(\tilde{\mathbf{F}}_n \because \partial^{(n)}_{\mathbf{X}_1}\right) \delta(\mathbf{X}_1)\right] \mathbf{E}(\mathbf{X} - \mathbf{X}_1)\end{aligned} \tag{22}$$

and applying the sum of operators in brackets (22), and repeating the repetitive integration by parts scheme, we finally derive

$$\begin{aligned}\mathbf{D}(\mathbf{X}) &= \tilde{\varepsilon}(\partial_{\mathbf{X}}) \cdot \mathbf{E}(\mathbf{X}) \\ &= \left(\tilde{\mathbf{F}}_0 \cdot + \tilde{\mathbf{F}}_1 : \partial_{\mathbf{X}} + \cdots + \tilde{\mathbf{F}}_n \because \underbrace{\partial_{\mathbf{X}} \ldots \partial_{\mathbf{X}}}_{n} + \cdots + \right) \mathbf{E}(\mathbf{X}) \\ &= \left(\tilde{\mathbf{F}}_n \because \partial^{(n)}_{\mathbf{X}}\right) \mathbf{E}(\mathbf{X})\end{aligned} \tag{23}$$

clearly demonstrating the validity of the symbolic representation (12).

4. INTERIM DISCUSSION (1)

We have demonstrated the transition from the spectral domain dispersive constitutive relation (9) to the spatiotemporal Volterra differential operator representation (12), in the context of the delta expansion representation (17).

Obviously the (generalized) function (17), by virtue of its composition of the delta-function and its derivatives, is highly localized in the $\mathbf{X}$ domain. This means that to effectively handle an associated physical situation the series must be truncated after a small number of leading terms. The same situation applies to Taylor expansions: While expressed in terms of derivatives, which act locally, the expansion (for "smooth" functions) can (within the mathematical limitations on the functions at hand) yields values at other locations. But to be efficiently exploited, the series must be used as an approximation, truncated after a small number of leading significant terms.

Corresponding to the high localization of (17) in the $\mathbf{X}$ domain, the Fourier transform (21) is "flat", i.e., widely extended in the transform space $\mathbf{K}$. Clearly a polynomial-like expression (21) is not efficient for describing highly localized functions in the spectral domain $\mathbf{K}$, because it requires many terms, which in turn prescribes many terms in the delta expansion (17). Nevertheless, many physically meaningful constitutive relations (e.g., a magnetized cold plasma), display resonances, which are best described by the poles of rational functions. This implies that our prototypical choice (9) is inadequate. Indeed, it has been argued [1], and shown on some simple examples, that in general we should address a differential relation with operators on both sides of the equation, thusly

$$\tilde{\boldsymbol{\varepsilon}}_{\mathbf{D}}(i\mathbf{K}) \cdot \mathbf{D}(\mathbf{K}) = \tilde{\boldsymbol{\varepsilon}}_{\mathbf{E}}(i\mathbf{K}) \cdot \mathbf{E}(\mathbf{K}) \tag{24}$$

Consequently the poles expected on the right-hand side of (24) appear as zeroes of the polynomial on the left-hand side. All the results given above for (9) follow.

As an example for (9), consider the simple case of an unmagnetized cold plasma, characterized in the spectral domain by

$$\tilde{\boldsymbol{\varepsilon}}(i\mathbf{K}) = \varepsilon_0 \left(1 - \omega_p^2/\omega^2\right) = \frac{\varepsilon_{\mathbf{E}}}{\varepsilon_{\mathbf{D}}} = \frac{\varepsilon_0 \left(\omega^2 - \omega_p^2\right)}{\omega^2} \tag{25}$$

Where ω_p is the plasma frequency. In the $\mathbf{X}$ domain, corresponding to (12), the structure of (25) prescribes

$$\varepsilon(\partial_{\mathbf{X}}) = \frac{\varepsilon_{\mathbf{E}}(\partial_{\mathbf{X}})}{\varepsilon_{\mathbf{D}}(\partial_{\mathbf{X}})} = \frac{\varepsilon_0 \left(\partial_t^2 + \omega_p^2\right)}{\partial_t^2} \tag{26}$$

This is a simple case where we deal with scalar constitutive parameters, and the representation (25) in terms of a ratio of polynomials, the transition to (26), and the ensuing differential form

$$\tilde{\boldsymbol{\varepsilon}}_{\mathbf{D}}(\partial_{\mathbf{X}}) \cdot \mathbf{D}(\mathbf{X}) = \tilde{\boldsymbol{\varepsilon}}_{\mathbf{E}}(\partial_{\mathbf{X}}) \cdot \mathbf{E}(\mathbf{X}) \tag{27}$$

is straightforward. In general, we can continue the discussion using our original forms (9), (12), as long as we understand that they are prototypical for the corresponding (24), (27), for dispersive, linear materials. In a more general context, any constitutive expression which supplements (1) or (8), rendering them into a determinate set of equations, can be discussed in this context.

Finally, we introduce a new notation applying to the differential-operator constitutive relations,

$$\mathbf{D}(\mathbf{X}) = \tilde{\varepsilon}(\partial_{\mathbf{X}}) \cdot \mathbf{E}(\mathbf{X}) \equiv \tilde{\varepsilon}(\partial_{\mathbf{X}_1}) \cdot \mathbf{E}(\mathbf{X}_1)\Big|_{\mathbf{X}_1 \Rightarrow \mathbf{X}} \tag{28}$$

where the last expression (28), in an obvious way, prescribes that first the differential operations are carried out in terms of $\mathbf{X}_1$ and afterwards the substitution $\mathbf{X}_1 \Rightarrow \mathbf{X}$ is performed. What in (28) appears to be a superfluous notation, will become significant for the nonlinear cases discussed below.

5. DISPERSIVE INHOMOGENEOUS LINEAR MEDIA

In a quest for a generalized approach to the problem of constitutive relations [3], and as exemplified by some problems [1], it seems advantageous to define inhomogeneous dispersive media in the $\mathbf{X}$ domain, in terms of spatiotemporally dependent coefficients in the following manner:

$$\mathbf{D}(\mathbf{X}) = \tilde{\varepsilon}(\mathbf{X}, \partial_{\mathbf{X}}) \cdot \mathbf{E}(\mathbf{X}) \equiv \tilde{\varepsilon}(\mathbf{X}, \partial_{\mathbf{X}_1}) \cdot \mathbf{E}(\mathbf{X}_1)\Big|_{\mathbf{X}_1 \Rightarrow \mathbf{X}} \tag{29}$$

cf. (28). The analog of (23) is

$$\begin{aligned}
\mathbf{D}(\mathbf{X}) \;&=\; \Big[\tilde{\mathbf{F}}_0(\mathbf{X}) \cdot + \tilde{\mathbf{F}}_1(\mathbf{X}) : \partial_{\mathbf{X}_1} + \cdots + \tilde{\mathbf{F}}_n(\mathbf{X}) \cdot\!\!\cdot\!\!\cdot \underbrace{\partial_{\mathbf{X}_1} \ldots \partial_{\mathbf{X}_1}}_{n} + \cdots\Big] \\
&\qquad \cdot \mathbf{E}(\mathbf{X}_1)\Big|_{\mathbf{X}_1 \Rightarrow \mathbf{X}} \\
&=\; \Big[\tilde{\mathbf{F}}_n(\mathbf{X}) \cdot\!\!\cdot\!\!\cdot\, \partial_{\mathbf{X}_1}^{(n)}\Big] \mathbf{E}(\mathbf{X}_1)\Big|_{\mathbf{X}_1 \Rightarrow \mathbf{X}}
\end{aligned} \tag{30}$$

6. INTERIM DISCUSSION (2)

In a strict sense, in (3), (5), elements of the $\mathbf{K}$ and $\mathbf{X}$ spaces are mutually exclusive, i.e., a function is expressed in either one of these spaces, but not both. The exception is the eikonal approximation (sometimes referred to as the WKB approximation, or method of characteristics) where expressions like

$$\mathbf{D}(\mathbf{X},\mathbf{K}) = \tilde{\varepsilon}(\mathbf{X}, i\mathbf{K}) \cdot \mathbf{E}(\mathbf{X},\mathbf{K}) \tag{31}$$

can be justified, but then $\mathbf{K}$ and $\mathbf{X}$ are not associated Fourier transform spaces. Indeed, one finds analyses of electromagnetic problems involving either dispersive systems or inhomogeneous (including time transients), but not both simultaneously. This is a puzzling situation, because spatial inhomogeneity due to the presence of structures in an otherwise homogeneous medium, are referred to as "geometrical dispersion", while macroscopic dispersion of a medium can often be traced to its microscopic structure, involving certain structural elements (atoms, molecules, etc.). The problem is solved by using the definition (29), (30).

Now consider the associated representation of (29), (30), in the spectral domain. Applying (3) to (29), (30), we obtain

$$\begin{aligned}
\mathbf{D}(\mathbf{X}) &= \alpha \int (d^4\mathbf{K})\mathbf{D}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}} \\
&= \tilde{\varepsilon}(\mathbf{X}, \partial_{\mathbf{X}_1})\alpha \int (d^4\mathbf{K}_1)\cdot \mathbf{E}(\mathbf{K}_1)e^{i\mathbf{K}_1\cdot\mathbf{X}_1}\Big|_{\mathbf{X}_1\Rightarrow\mathbf{X}} \\
&= \alpha \int (d^4\mathbf{K}_1)\tilde{\varepsilon}(\mathbf{X}, \partial_{\mathbf{X}_1})e^{i\mathbf{K}_1\cdot\mathbf{X}_1}\Big|_{\mathbf{X}_1\Rightarrow\mathbf{X}}\cdot\mathbf{E}(\mathbf{K}_1) \\
&= \alpha \int (d^4\mathbf{K}_1)\tilde{\varepsilon}(\mathbf{X}, i\mathbf{K}_1)e^{i\mathbf{K}_1\cdot\mathbf{X}}\cdot\mathbf{E}(\mathbf{K}_1)
\end{aligned} \tag{32}$$

Transforming $\tilde{\varepsilon}(\mathbf{X}, i\mathbf{K}_1)$ with respect to $\mathbf{X}$, indicated by the diamond, renders (32) as a double-fold four-integral

$$\mathbf{D}(\mathbf{X}) = \alpha^2 \iint (d^4\mathbf{K}_1)(d^4\mathbf{K}_2)\tilde{\varepsilon}^{\diamondsuit}(\mathbf{K}_2, i\mathbf{K}_1)\cdot\mathbf{E}(\mathbf{K}_1)e^{i(\mathbf{K}_1+\mathbf{K}_2)\cdot\mathbf{X}} \tag{33}$$

Applying the inverse transform (5) to (33), and identifying the delta-function (16), we obtain

$$\begin{aligned}
\mathbf{D}(\mathbf{K}) &= \alpha \iint (d^4\mathbf{K}_1)(d^4\mathbf{K}_2)\tilde{\varepsilon}^{\diamondsuit}(\mathbf{K}_2, i\mathbf{K}_1)\cdot\mathbf{E}(\mathbf{K}_1) \\
&\quad\cdot\left[\alpha\int (d^4\mathbf{X})e^{i(\mathbf{K}_1+\mathbf{K}_2-\mathbf{K})\cdot\mathbf{X}}\right] \\
&= \alpha \iint (d^4\mathbf{K}_1)(d^4\mathbf{K}_2)\tilde{\varepsilon}^{\diamondsuit}(\mathbf{K}_2, i\mathbf{K}_1)\cdot\mathbf{E}(\mathbf{K}_1)\delta(\mathbf{K}_1+\mathbf{K}_2-\mathbf{K})
\end{aligned} \tag{34}$$

which can be collapsed to a single four-integral,

$$\mathbf{D}(\mathbf{K}) = \alpha \int (d^4\mathbf{K}_1)\tilde{\varepsilon}^{\diamondsuit}(\mathbf{K}-\mathbf{K}_1, i\mathbf{K}_1)\cdot\mathbf{E}(\mathbf{K}_1) \tag{35}$$

or alternatively written as

$$\mathbf{D}(\mathbf{K}) = \alpha \int (d^4\mathbf{K}_1)\tilde{\varepsilon}^{\diamondsuit}(\mathbf{K}_2, i\mathbf{K}_1) \cdot \mathbf{E}(\mathbf{K}_1)$$
$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 \tag{36}$$

Clearly, the indicated constraint on the second line (36) must be substituted into the integral to facilitate a solution, but written in this form it describes the relation between the interaction parameters $\mathbf{K}_1$, $\mathbf{K}_2$ and the resulting $\mathbf{K}$.

The results (35), (36) add a new dimension to our understanding of the role of inhomogeneity in electromagnetic materials. Usually, when material properties are discussed, the bulk homogeneous medium is considered. When such a medium is homogeneous and linear throughout space, no new spectral components can be created. If the medium under consideration is homogeneous and nonlinear (as discussed below) new spectral components, both frequencies and propagation vectors, might emerge. This so called "mixing" phenomenon is already known from electrical circuit theory. In a sense, (35) constitutes a convolution in $\mathbf{K}$ space, which does not appear in homogeneous linear media (9) where a local interaction in the spectral domain is indicated. It suggests that a non-local interaction exists in the spectral domain, i.e., in (35) the value of $\mathbf{D}(\mathbf{K})$ for some $\mathbf{K}$ depends on the value of $\mathbf{E}(\mathbf{K})$ throughout the spectral domain. The same conclusion follows from (36) and the associated constraint. This seems to be a paradox: We are familiar with spectral interactions in nonlinear systems, where new frequencies are created due to the interaction of signals at other frequencies, so why does inhomogeneity as well imply new spectral components? Some reflection reveals that we have actually encountered such phenomena in linear systems [6]. Consider a lens embedded in an otherwise homogeneous medium, or a reflection and reflection mechanism consisting of a dielectric plane interface (governed by Snell's law). Such a system is inhomogeneous in space, new spectral components, possessing new wavenumbers, are created everywhere in space. Similarly, moving objects or moving boundaries can be regarded as inhomogeneities in time, leading to the familiar Doppler effect, by which new frequencies are created due to such inhomogeneities, in spite of the fact that the system is linear. Consequently the paradox is not only resolved, but the results also lend credibility to the model (29), (30), and the spectral representation (34)–(36).

By inspection of (30), the details of the integrations (34)–(36)

become clearer: For the first term in (30) we obtain

$$\mathbf{D}_0(\mathbf{K}) = \alpha \int (d^4\mathbf{K}_1)\tilde{\mathbf{F}}_0^{\diamondsuit}(\mathbf{K}-\mathbf{K}_1)\cdot\mathbf{E}(\mathbf{K}_1) \tag{37}$$

which displays a four-convolution in $\mathbf{K}$ space. Inasmuch as the first term does not involve a four-gradient operator, this corresponds to a dispersion-less case. The next term yields

$$\mathbf{D}_1(\mathbf{K}) = \alpha \int (d^4\mathbf{K}_1)\tilde{\mathbf{F}}_1^{\diamondsuit}(\mathbf{K}-\mathbf{K}_1) : [i\mathbf{K}_1\mathbf{E}(\mathbf{K}_1)] \tag{38}$$

where in the square brackets the two lumped together terms constitute a four-dyadic and the double multiplication finally yields a three-vector. Similarly to (37), the structure of (38) once again corresponds to a four-convolution in the spectral domain, and this will be characteristic of all terms, in general,

$$\mathbf{D}(\mathbf{K}) = \alpha \int (d^4\mathbf{K}_1)\tilde{\mathbf{F}}_n^{\diamondsuit}(\mathbf{K}-\mathbf{K}_1) \because [(i\mathbf{K}_1)^{(n)}\mathbf{E}(\mathbf{K}_1)] \tag{39}$$

where the summation notation stated above is understood.

With that we are now ready to extend the present results to nonlinear systems.

7. DISPERSIVE HOMOGENEOUS NONLINEAR MEDIA

Weak or moderate nonlinear media can be modeled by the Volterra functional series. Rewritten for the Minkowski four-space [3],

$$\begin{aligned}
\mathbf{D}(\mathbf{X}) &= \sum_n \mathbf{D}^{(m)}(\mathbf{X}) \\
\mathbf{D}^{(m)}(\mathbf{X}) &= \int (d^4\mathbf{X}_1)\ldots\int (d^4\mathbf{X}_m)\tilde{\varepsilon}^{*(m)}(\mathbf{X}_1,\ldots,\mathbf{X}_m) \qquad (40)\\
&\quad \because \mathbf{E}(\mathbf{X}-\mathbf{X}_1)\ldots\mathbf{E}(\mathbf{X}-\mathbf{X}_m)
\end{aligned}$$

clearly displaying various products of fields, typical of nonlinear interactions, as well as an extended convolution structure which provides a consistent extension from the linear case (10).

The Fourier transform of (40) is given by [3]

$$\begin{aligned}
\mathbf{D}^{(m)}(\mathbf{K}) &= \alpha^{(m-1)}\int (d^4\mathbf{K}_1)\ldots\int (d^4\mathbf{K}_{m-1})\tilde{\varepsilon}^{(m)}(i\mathbf{K}_1,\ldots,i\mathbf{K}_m) \\
&\quad \because \mathbf{E}(\mathbf{K}_1)\ldots\mathbf{E}(\mathbf{K}_m) \qquad (41)
\end{aligned}$$

where $\tilde{\varepsilon}^{*(m)}(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ and $\tilde{\varepsilon}^{(m)}(i\mathbf{K}_1, \ldots, i\mathbf{K}_m)$ are m-fold four-space Fourier transform pairs, and in (41) the constraint

$$\mathbf{K} = \mathbf{K}_1 + \cdots + \mathbf{K}_m \tag{42}$$

is incorporated. Although here we are dealing with an infinite homogeneous medium, (42) acquires the same structure as in (36).

In terms of the delta-expansions analogous to (18), we now have

$$\tilde{\varepsilon}^{*(m)}(\mathbf{X}_1, \ldots, \mathbf{X}_m) = \left[\left(\tilde{\mathbf{F}}_{n_1} \because \partial_{\mathbf{X}_1}^{(n_1)}\right)\delta(\mathbf{X}_1)\right] \ldots \left[\left(\tilde{\mathbf{F}}_{n_m} \because \partial_{\mathbf{X}_m}^{(n_m)}\right)\delta(\mathbf{X}_m)\right] \tag{43}$$

where the book-keeping of the tensorial rank and the associated indices must be separately considered for specific problems. The notation (43) can be compacted by assuming that expressions in brackets in (43) are multiplied out and the terms regrouped according to the number (zero or any positive integer) of differentiations involved. The general term will serve as symbolizing the whole series, thus (43) can be rewritten as.

$$\tilde{\varepsilon}^{*(m)}(\mathbf{X}_1, \ldots, \mathbf{X}_m) = \left[\tilde{\mathbf{G}}_{n_1 \ldots n_m} \because \partial_{\mathbf{X}_1}^{(n_1)} \ldots \partial_{\mathbf{X}_m}^{(n_m)}\right]\delta(\mathbf{X}_1) \ldots \delta(\mathbf{X}_m) \tag{44}$$

Consistent with (21) and (43), (44), we have

$$\begin{aligned} \tilde{\varepsilon}^{(m)}(i\mathbf{K}_1, \ldots, i\mathbf{K}_m) &= \left[\tilde{\mathbf{F}}_{n_1} \because (i\mathbf{K}_1)^{(n_1)}\right] \ldots \left[\tilde{\mathbf{F}}_{n_m} \because (i\mathbf{K}_m)^{(n_m)}\right] \\ &= \left[\tilde{\mathbf{G}}_{n_1 \ldots n_m} \because (i\mathbf{K}_1)^{(n_1)} \ldots (i\mathbf{K}_m)^{(n_m)}\right] \end{aligned} \tag{45}$$

describing the kernel of (41) in terms of polynomials in $i\mathbf{K}_1, \ldots, i\mathbf{K}_m$, where $\tilde{\mathbf{G}}_{n_1 \ldots n_m}$ are the new coefficients after the multiplication and regrouping..

Analogously to (12) we now stipulate that

$$\mathbf{D}^{(m)}(\mathbf{X}) = \tilde{\varepsilon}^{(m)}(\partial_{\mathbf{X}_1}, \ldots, \partial_{\mathbf{X}_m}) \because \mathbf{E}(\mathbf{X}_1) \ldots \mathbf{E}(\mathbf{X}_m)\Big|_{\mathbf{X}_1, \ldots, \mathbf{X}_m \Rightarrow \mathbf{X}} \tag{46}$$

which is understood as an instruction to apply the various differentiation operations to the fields given in terms of the corresponding arguments, and finally substitute $\mathbf{X}$ coordinates for all arguments.

In terms of the polynomial representations, corresponding to (43)–(45), we now have in (46)

$$\begin{aligned} \tilde{\varepsilon}^{(m)}(\partial_{\mathbf{X}_1}, \ldots, \partial_{\mathbf{X}_m}) &= \left(\tilde{\mathbf{F}}_{n_1} \because \partial_{\mathbf{X}_1}^{(n_1)}\right) \ldots \left(\tilde{\mathbf{F}}_{n_m} \because \partial_{\mathbf{X}_m}^{(n_m)}\right) \\ &= \tilde{\mathbf{G}}_{n_1 \ldots n_m} \because \partial_{\mathbf{X}_1}^{(n_1)} \ldots \partial_{\mathbf{X}_m}^{(n_m)} \end{aligned} \tag{47}$$

8. INTERIM DISCUSSION (3)

The basic equations (40)–(42) and (46) of the last subsection have been given before [3]. It is easy to derive (46) formally: Applying the Fourier transform (3) yields

$$\begin{aligned}
\mathbf{D}^{(m)}(\mathbf{X}) &= \alpha^m \int (d^4\mathbf{K}_1)\ldots\int (d^4\mathbf{K}_m)\tilde{\varepsilon}^{(m)}(\partial_{\mathbf{X}_1},\ldots,\partial_{\mathbf{X}_m}) \\
&\quad \because\ \mathbf{E}(\mathbf{K}_1)\ldots\mathbf{E}(\mathbf{K}_m)e^{i(\mathbf{K}_1\cdot\mathbf{X}_1+\cdots+\mathbf{K}_m\cdot\mathbf{X}_m)}\Big|_{\mathbf{X}_1,\ldots,\mathbf{X}_m\Rightarrow\mathbf{X}} \\
&= \alpha^m \int (d^4\mathbf{K}_1)\ldots\int (d^4\mathbf{K}_m)\tilde{\varepsilon}^{(m)}(i\mathbf{K}_1,\ldots,i\mathbf{K}_m) \\
&\quad \because\ \mathbf{E}(\mathbf{K}_1)\ldots\mathbf{E}(\mathbf{K}_m)e^{i(\mathbf{K}_1+\cdots+\mathbf{K}_m)\cdot\mathbf{X}}
\end{aligned} \tag{48}$$

Transforming (48) according to (5) and using (16) yields

$$\begin{aligned}
\mathbf{D}^{(m)}(\mathbf{K}) &= \alpha^m \int (d^4\mathbf{X})\int (d^4\mathbf{K}_1)\ldots\int (d^4\mathbf{K}_m)\tilde{\varepsilon}^{(m)}(i\mathbf{K}_1,\ldots,i\mathbf{K}_m) \\
&\quad \because\ \mathbf{E}(\mathbf{K}_1)\ldots\mathbf{E}(\mathbf{K}_m)e^{i(\mathbf{K}_1+\cdots+\mathbf{K}_m-\mathbf{K})\cdot\mathbf{X}} \\
&= \alpha^{(m-1)} \int (d^4\mathbf{K}_1)\ldots\int (d^4\mathbf{K}_m)\tilde{\varepsilon}^{(m)}(i\mathbf{K}_1,\ldots,i\mathbf{K}_m) \\
&\quad \because\ \mathbf{E}(\mathbf{K}_1)\ldots\mathbf{E}(\mathbf{K}_m)\delta(\mathbf{K}_1+\cdots+\mathbf{K}_m-\mathbf{K})
\end{aligned} \tag{49}$$

in agreement with (41), (42).

Evidently the more explicit polynomial structure (47), corresponding to the delta-function expansion (43), (44) could have been applied to (48). Once again this demonstrates that the formalism is valid for highly localized functions in the spatiotemporal domain $\mathbf{X}$, or equivalently, widely extended in the transform space $\mathbf{K}$.

Next, we recast (49) in the form

$$\begin{aligned}
\mathbf{D}^{(m)}(\mathbf{K}) &= \alpha^{(m-1)} \int (d^4\mathbf{K}_1)\ldots \\
&\quad \int (d^4\mathbf{K}_{m-1})\tilde{\varepsilon}^{(m)}(i\mathbf{K}_1,\ldots,i(\mathbf{K}-(\mathbf{K}_1+\cdots+\mathbf{K}_{m-1}))) \\
&\quad \because\ \mathbf{E}(\mathbf{K}_1)\ldots\mathbf{E}(\mathbf{K}-(\mathbf{K}_1+\cdots+\mathbf{K}_{m-1}))
\end{aligned} \tag{50}$$

Upon freezing all integrations, except with regard to $\mathbf{K}_1$, (50) acquires a structure similar to (35), demonstrating how nonlinear homogeneous media and linear inhomogeneous media are, both display non-locality, and the creation of new spectral components in the spectral domain.

9. DISPERSIVE INHOMOGENEOUS NONLINEAR MEDIA

The generalization of the above results to inhomogeneous nonlinear media is straightforward and completes the consistent definition of constitutive relations. Similarly to the dispersive homogeneous nonlinear case (45), and inspired by the dispersive inhomogeneous linear case (29), we now define

$$\mathbf{D}^{(m)}(\mathbf{X}) = \tilde{\varepsilon}^{(m)}(\mathbf{X}, \partial_{\mathbf{X}_1}, \ldots, \partial_{\mathbf{X}_m}) \because \mathbf{E}(\mathbf{X}_1) \ldots \mathbf{E}(\mathbf{X}_m)\Big|_{\mathbf{X}_1,\ldots,\mathbf{X}_m \Rightarrow \mathbf{X}} \tag{51}$$

Or, in terms of the delta-expansions, instead of (46) we now have in (51)

$$\begin{aligned} \tilde{\varepsilon}^{(m)}(\mathbf{X}, \partial_{\mathbf{X}_1}, \ldots, \partial_{\mathbf{X}_m}) &= \left(\tilde{\mathbf{F}}_{n_1}(\mathbf{X}) \because \partial^{(n_1)}_{\mathbf{X}_1}\right) \ldots \left(\tilde{\mathbf{F}}_{n_m}(\mathbf{X}) \because \partial^{(n_m)}_{\mathbf{X}_m}\right) \\ &= \tilde{\mathbf{G}}_{n_1 \ldots n_m}(\mathbf{X}) \because \partial^{(n_1)}_{\mathbf{X}_1} \ldots \partial^{(n_m)}_{\mathbf{X}_m} \end{aligned} \tag{52}$$

By inspection of the linear inhomogeneous case (34)–(36) and the nonlinear homogeneous case (41), it becomes evident that inhomogeneity introduces an additional integration in $\mathbf{K}$ space.

Thus by exploiting (3) in (51), similarly to (34) and (48) we now find

$$\begin{aligned} \mathbf{D}^{(m)}(\mathbf{X}) &= \alpha^m \int (d^4\mathbf{K}_1) \ldots \int (d^4\mathbf{K}_m) \tilde{\varepsilon}^{(m)}(\mathbf{X}, \partial_{\mathbf{X}_1}, \ldots, \partial_{\mathbf{X}_m}) \\ &\quad \because \mathbf{E}(\mathbf{K}_1) \ldots \mathbf{E}(\mathbf{K}_m) e^{i\mathbf{K}_1\cdot\mathbf{X}_1 + \cdots + \mathbf{K}_m\cdot\mathbf{X}_m}\Big|_{\mathbf{X}_1,\ldots,\mathbf{X}_m \Rightarrow \mathbf{X}} \\ &= \alpha^{(m+1)} \int (d^4\mathbf{Q}) \int (d^4\mathbf{K}_1) \ldots \\ &\quad \int (d^4\mathbf{K}_m) \tilde{\varepsilon}^{\diamond(m)}(\mathbf{Q}, i\mathbf{K}_1, \ldots, i\mathbf{K}_m) \\ &\quad \because \mathbf{E}(\mathbf{K}_1) \ldots \mathbf{E}(\mathbf{K}_m) e^{i(\mathbf{Q}+\mathbf{K}_1+\cdots+\mathbf{K}_m)\cdot\mathbf{X}} \end{aligned} \tag{53}$$

More explicitly, in terms of the delta-expansion, (52) will be used in (53).

Similarly to (35)–(36) and (49) yields

$$\begin{aligned} \mathbf{D}^{(m)}(\mathbf{K}) = \alpha^m \int (d^4\mathbf{Q}) \int (d^4\mathbf{K}_1) \ldots \int (d^4\mathbf{K}_m) \tilde{\varepsilon}^{\diamond(m)}(\mathbf{Q}, i\mathbf{K}_1, \ldots, i\mathbf{K}_m) \\ \because \mathbf{E}(\mathbf{K}_1) \ldots \mathbf{E}(\mathbf{K}_m) \delta(\mathbf{Q} + \mathbf{K}_1 + \cdots + \mathbf{K}_m - \mathbf{K}) \end{aligned} \tag{54}$$

This completes our derivation.

10. DISCUSSION AND SUMMARY

The question and constitutive relations and locality is of for various areas of physics investigating yield theories, e.g., see [10]. The present study is a continuation of a previous project [1, 3], attempting to present a consistent framework for the constitutive relations associated with the Maxwell model for electromagnetism (1). The tool facilitating this approach is the Volterra functional series, in its present four-dimensional Minkowski space representation.

Is this a general representation for nonlinear constitutive relations? It is easy to argue that it is not: The Volterra series given by (40) is the functional analog of the Taylor expansion for functions. Both are useful if the series can be truncated after a few leading terms. In the nonlinear case, the Volterra series describes a hierarchy of field interactions. In processes where such a hierarchy cannot be displayed, e.g., when we deal with strong nonlinearity as in the case of phase conjugation effects, the formalism seems to be inapplicable.

The thrust of the present study was to show that the so-called Volterra differential operators approach is valid for functions which are highly localized in the spatiotemporal domain $\mathbf{X}$, which implies widely extended associated functions in the transform space $\mathbf{K}$.

We have extensively employed the delta-function expansions [4, 5], extended to the Minkowski spaces $\mathbf{X}$, $\mathbf{K}$. Although no attempt is made to introduce Special-Relativistic considerations, the four-dimensional Minkowski space proves to be a very powerful vehicle for our derivations, providing clear extensions from lower dimensionality, and facilitating compact notation.

One of the results demonstrated above is that both inhomogeneity and nonlinearity imply nonlocal behavior in the spectral domain. Thus spatial constitutive structures produce new wavenumbers, and time dependent constitutive parameters, including boundaries, create new frequencies. This property is evident in (34)–(36) and appears in the nonlinear case (54) as well. On the other hand, nonlinearity alone also shows effects of new spectral components, as in (41), (42).

Media possessing temporal dispersion (constitutive parameters depending on frequency) are often referred to as "media with memory", because, as demonstrated by (10), what happens at some time t is determined by all previous time instances. In the present case, this property exists in space as well as in time, or more concisely, at all previous spatiotemporal events within the past part of the light cone. In view of the integral representation, as in (10), this non-locality extends throughout the regime of integration. However, the present discussion demonstrates that for many physically interesting models, representable in terms of polynomials as in (24), highly localized functions are involved in the spatiotemporal domain, which can be efficiently be approximated by a finite number of leading terms.

To what extent is this a realistic perception of empirical reality as gleaned from theoretical arguments and experimentation? The answer to this question is in the regime of microscopic electromagnetic considerations for the properties of media, see for example Balanis' textbook [7] for introductory discussions, and von Hippel's comprehensive book [8]. Advanced chiral media are considered e.g., by Lindell, Sihvola, Tretyakov, and Viitanen [9]. Usually such materials are based on the equations governing electro-mechanical models for various charged particles facilitating polarization and magnetization effects, e.g., the Debye model [7–9]. The associated differential equations display a finite number of resonances, for which the constitutive parameters in the spectral domain display poles. Physically the resonances correspond to mechanisms such as electronic polarization, ionic or molecular polarization, or dipole orientational polarization, etc. When artificial particles are stipulated, e.g., for a resonating helix [9], the analysis becomes more complicated but the principle follows.

The polynomial representations described above, e.g., (24) are patently appropriate for such cases, justifying our derivation based on the delta-expansions discussed above. All this reflects once more on the locality question in electromagnetism: If the differential operator representation of constitutive parameters provides a physically sound model, then, in the sense that differential operations are local, in contradistinction to global integral operators, the claim that electromagnetic behavior is local, is vindicated.

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