Gaussian-beam propagation in generic anisotropic wave-number profiles

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The propagation characteristics of a scalar Gaussian beam in a homogeneous anisotropic medium are considered. The medium is described by a generic wave-number profile wherein the field is formulated by a Gaussian plane-wave distribution and the propagation is obtained by saddle-point asymptotics to extract the Gaussian beam phenomenology in the anisotropic environment. The resultant field is parameterized in terms of e.g., the spatial evolution of the Gaussian beam's curvature, beam width, which are mapped to local geometrical properties of the generic wave-number profile.

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Anisotropic materials are of interest for use as optical waveguides and microwave devices, in plasma science, and in various propagation environments. Comprehensive studies have been made of Gaussian-beam (GB) two- and three-dimensional propagation for specific wave-number profiles.[1–18] The generic profiles of the three-dimensional problem associated with a GB with complex wave-number spectral constituents are of fundamental significance. This is so because GB propagation is a practical issue as well as being theoretically significant, inasmuch as GBs form the basis propagators for the phase-space beam summation method, which is a general analytical framework for local analysis and modeling of radiation from extended source distributions.[19] With GBs used as basis wave objects for modeling, various anisotropic propagation and scattering problems were introduced in Refs. 20 and 21. The propagation of a GB over a generic anisotropy was studied[22] by the complex source point method. Applying the saddle-point asymptotic technique approximated over the z axis, Shin and Felsen arrived at a closed-form analytic solution for the GB field. In our view, the complex source point method cannot account for the astigmatic effects that are present in our analysis of the generic wave-number profiles; therefore the results in Ref. 22 may be applied only to a uniaxially anisotropic medium. Alternatively, by applying a plane-wave spectral representation to the propagation problem, we present an alternative rigorous solution for the GB field that is suitable for any generic wave-number profile [see the discussion following Eq. (8) below].

The current study is concerned with the effects of anisotropy on the propagation characteristics of the scalar Gaussian-beam field in a homogeneous medium described by the generic wave-number profile $k_z(\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is a wave-number-normalized coordinate. The field is formulated in the frequency domain by a plane-wave spectral integral and is evaluated asymptotically by the saddle-point technique. Given a field $u(\mathbf{r})$ in which an $\exp(-i\omega t)$ time dependence is assumed and suppressed, where $\mathbf{r} = (x_1, x_2, z)$ are conventional Cartesian coordinates, the wave-number spectral (plane wave) transform pairs on the $z = 0$ initial surface are given by

$$\tilde{u}_0(\boldsymbol{\xi}) = \int_{-\infty}^{\infty} \mathrm{d}^2 x u_0(\mathbf{x}) \exp(-ik\boldsymbol{\xi} \cdot \mathbf{x}), \tag{1a}$$

$$u_0(\mathbf{x}) = (k/2\pi)^2 \int \mathrm{d}^2 \xi \tilde{u}_0(\boldsymbol{\xi}) \exp(ik\boldsymbol{\xi} \cdot \mathbf{x}), \tag{1b}$$

where $\xi = (\xi_1, \xi_2)$ is the normalized spatial wave-number vector, $\mathbf{x} = (x_1, x_2)$, k is the homogeneous medium wave-number $k = \omega/c$, where c is the phase velocity of the homogeneous medium, and $\sim$ identifies a wave number's spectral function. The normalization with respect to wave number k renders $\boldsymbol{\xi}$ frequency independent. Therefore the GB field is formulated by means of the Gaussian distribution

$$u_0(\mathbf{x}) = \exp(-{}^1\!/\!{}_2\, k\mathbf{x}^2/\beta), \tag{2}$$

where $\beta = \beta_r + i\beta_i$, with $\beta_r > 0$ a parameter, and $\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x} = {x_1}^2 + {x_2}^2$. Substituting Eq. (2) into Eq. (1), we obtain the plane-wave spectral distribution that corresponds to Eq. (2):

$$\tilde{u}_0(\boldsymbol{\xi}) = (2\pi\beta/k) \exp(-{}^1\!/\!{}_2\, k\beta\boldsymbol{\xi}^2). \tag{3}$$

The plane-wave distribution in Eq. (3) can be propagated into the $z > 0$ half-space by use of the generic anisotropic propagator $\exp[ik\zeta(\xi)z]$, where, as in Eq. (1), we normalize longitudinal wave number ζ by k. With this propagator, the field propagating into the $z > 0$ half-space is given by

$$u(\mathbf{r}) = \beta \frac{k}{2\pi} \int \mathrm{d}^2 \xi \, \exp[ik\Phi(\boldsymbol{\xi}, \mathbf{r})],$$
$$\Phi(\boldsymbol{\xi}, \mathbf{r}) = \left[\boldsymbol{\xi} \cdot \mathbf{x} + \zeta(\boldsymbol{\xi}) z + \frac{i}{2} \beta \boldsymbol{\xi}^2 \right]. \tag{4}$$

The field in Eq. (4) cannot be evaluated in closed form. Next, we evaluate it asymptotically to obtain an analytic expression for the beam field in the high-frequency regime.

Stationary point $\boldsymbol{\xi}_s$ satisfies

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$$\nabla_\xi \Phi = \mathbf{x} + \nabla_\xi \zeta(\boldsymbol{\xi})|_{\boldsymbol{\xi}_s} z + i\beta \boldsymbol{\xi}_s = 0. \tag{5}$$

Equation (5) has a real solution only if $\boldsymbol{\xi}_s = 0$ and for observation points

$$\mathbf{x} + z\nabla_\xi \zeta_0 = 0, \tag{6}$$

where, here and henceforth, subscript 0 denotes sampling at $\boldsymbol{\xi} = 0$; i.e., $\nabla_\xi \zeta_0 \equiv \nabla_\xi \zeta|_{\boldsymbol{\xi}=0}$. The condition in Eq. (6) defines the beam axis as a tilted line in the configuration space, with an anisotropy-dependent tilt. For isotropic materials, where $\zeta = \sqrt{1 - \boldsymbol{\xi}^2}$, yielding $\nabla_\xi \zeta_0 = 0$, the beam axis coincides with the z axis. For the generic $\zeta(\boldsymbol{\xi})$ wave-number profile, the beam axis is directed along the unit vector

$$\hat{\kappa} = (\cos \vartheta_1,\ \cos \vartheta_2,\ \cos \vartheta_3), \tag{7}$$

where $\vartheta_{1,2,3}$, are the beam axis angles with respect to the (x_1, x_2, z) axes, repsectively. In view of Eq. (6), they are given by

$$\cos \vartheta_{1,2} = -\cos \vartheta_3 \partial_{\xi_{1,2}} \zeta_0,$$

$$\cos \vartheta_3 = \frac{1}{[(\partial_{\xi_1}\zeta_0)^2 + (\partial_{\xi_2}\zeta_0)^2 + 1]^{1/2}}. \tag{8}$$

Note that the beam axis direction as defined in Eq. (6) is different from the definition given in Ref. 22, in which the propagation of a Gaussian beam in a generic wave-number profile was investigated by the complex source point method. The beam axis as defined in Ref. 22 is directed along the complex source $\mathbf{b}$ parameter, which, for our problem, coincides with the z axis. Clearly, from Eq. (6), the z axis may serve as the beam axis only for symmetrical ζ, where $\nabla_\xi \zeta_0 = 0$, as in an isotropic or a uniaxially anisotropic medium. The condition in Eq. (6) may therefore serve as a more generalized definition of the beam axis that accounts for astigmatic effects of the medium's anisotropy (see Fig. 1).

For off-axis observation points, Eq. (5) could not be solved explicitly. Furthermore, the off-axis stationary point is complex and the solution requires analytic continuation of wave-number profile $\zeta = \zeta(\boldsymbol{\xi})$ for complex $\boldsymbol{\xi}$. To obtain a closed-form analytic solution for the beam field, we notice that the beam field decays away from the beam axis. Therefore we may apply a Taylor expansion of phase Φ about on-axis stationary point $\boldsymbol{\xi}_s = 0$:

$$\Phi \approx \Phi_0 + \boldsymbol{\Phi}_1 \cdot \xi + {}^1\!/_2\, \boldsymbol{\xi} \boldsymbol{\Phi}_2 \boldsymbol{\xi}, \tag{9}$$

with

$$\Phi_0 = \Phi|_{\boldsymbol{\xi}=0} = \zeta_0 z, \qquad \boldsymbol{\Phi}_1 = \nabla\Phi|_{\boldsymbol{\xi}=0} = \nabla_\xi \zeta_0 z + \mathbf{x}, \tag{10}$$

$$\boldsymbol{\Phi}_2 = \begin{bmatrix} i\beta + \partial_{\xi_1}{}^2 \zeta_0 z & \partial_{\xi_1\xi_2}{}^2 \zeta_0 z \\ \partial_{\xi_1\xi_2}{}^2 \zeta_0 z & i\beta + \partial_{\xi_2}{}^2 \zeta_0 z \end{bmatrix}, \tag{11}$$

where subscript 0 implies sampling at $\boldsymbol{\xi} = 0$. Using relation (9), we find that the saddle point for both on- and off-axis observation points is $\boldsymbol{\xi}_s = -\boldsymbol{\Phi}_2{}^{-1}\boldsymbol{\Phi}_1$, and the field in Eq. (4) may be evaluated asymptotically by

$$u(\mathbf{r}) = \frac{\beta}{\sqrt{-\det \boldsymbol{\Phi}_2}} \exp[ikS(\mathbf{r})],$$

$$S(\mathbf{r}) = \Phi_0 - {}^1\!/_2\, \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_2{}^{-1} \boldsymbol{\Phi}_1. \tag{12}$$

The beam field in Eq. (12) may be represented in terms of local beam coordinates over which the field exhibits a Gaussian decay away from the beam axis. The local beam coordinates, $\mathbf{r}_b = (x_{b1}, x_{b2}, z_b)$, are defined by the nonorthogonal transformation

$$\mathbf{r}_b = \mathbf{T}\mathbf{r}, \qquad \mathbf{T} = \begin{bmatrix} \cos\alpha & \sin\alpha & (-\cos\vartheta_2 \sin\alpha - \cos\vartheta_1 \cos\alpha)/\cos\vartheta_3 \\ -\sin\alpha & \cos\alpha & (-\cos\vartheta_2 \cos\alpha + \cos\vartheta_1 \sin\alpha)/\cos\vartheta_3 \\ 0 & 0 & 1/\cos\vartheta_3 \end{bmatrix}, \tag{13}$$

where $\cos \vartheta_{1,2,3}$ are defined in Eqs. (8) and angle α is given by

$$\tan 2\alpha = -2\partial_{\xi_1\xi_2}{}^2 \zeta_0 / (\partial_{\xi_2}{}^2 \zeta_0 - \partial_{\xi_1}{}^2 \zeta_0). \tag{14}$$

The transformation in Eq. (13) consists of a rotation transformation in the (x_1, x_2) plane by α, in which phase $S(\mathbf{r})$ in Eq. (12) exhibits Gaussian decay, followed by tilting of the z axis beam axis direction $\hat{\kappa}$ in Eq. (7) (Fig. 1). The inverse transform is given by

$$\mathbf{T}^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha & \cos\vartheta_1 \\ \sin\alpha & \cos\alpha & \cos\vartheta_2 \\ 0 & 0 & \cos\vartheta_3 \end{bmatrix}. \tag{15}$$

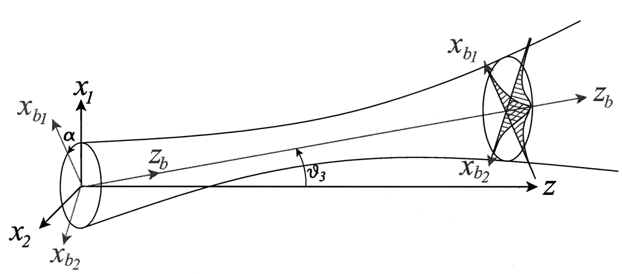

Fig. 1. Local beam coordinate frame for a Gaussian beam propagating in an anisotropic medium. The beam axis is directed along unit vector $\hat{\kappa}$. The local transverse coordinates $\mathbf{x}_b$ are given by the transformation in Eq. (13). The transverse local coordinates are located over the (x_1, x_2) plane, which is in general nonorthogonal to the beam axis. The rotation transformation of (x_1, x_2) into (x_{b1}, x_{b2}) by α is carried out such that the resultant field in Eq. (16) exhibits Gaussian decay in the local coordinates.

Substituting beam coordinate system (13) into Eq. (12), we may present the field in the Gaussian form

$$u(\mathbf{r}) = \frac{\beta}{\sqrt{-\Gamma_1\Gamma_2}} \times \exp\left\{ik\left[\zeta_0 z_b \cos\vartheta_3 + \frac{1}{2}\left(\frac{x_{b1}{}^2}{\Gamma_1} + \frac{x_{b2}{}^2}{\Gamma_2}\right)\right]\right\}, \quad (16)$$

where

$$\Gamma_{1,2} = -\frac{\partial_{\xi_1}{}^2\zeta_0 z + \partial_{\xi_2}{}^2\zeta_0 z + 2i\beta \mp [(\partial_{\xi_1}{}^2\zeta_0 z - \partial_{\xi_2}{}^2\zeta_0 z)^2 + 4(\partial_{\xi_1\xi_2}{}^2\zeta_0 z)^2]^{1/2}}{2}. \quad (17)$$

Substituting Eq. (14) into Eq. (17), we obtain

$$\Gamma_{1,2} = z_b a_{1,2} - i\beta,$$

$$a_{1,2} = \frac{\cos\vartheta_3}{\cos(2\alpha)}\left[\partial_{\xi_1}{}^2\zeta_0\begin{Bmatrix}-\cos^2\alpha\\ \sin^2\alpha\end{Bmatrix} + \partial_{\xi_2}{}^2\zeta_0\begin{Bmatrix}\sin^2\alpha\\ -\cos^2\alpha\end{Bmatrix}\right]. \quad (18)$$

Equation (16) has the form of a Gaussian beam propagating along beam axis z_b. The beam field exhibits a Gaussian decay in transverse local coordinate $\mathbf{x}_b$, which is, in general, tilted with respect to beam axis direction $\hat{\kappa}$. To parameterize the beam field we rewrite the element of $\Gamma_{1,2}$ in the form

$$\frac{1}{\Gamma_{1,2}} = \frac{1}{R_{1,2}} + \frac{i}{kD_{1,2}{}^2}, \quad (19)$$

where

$$D_{1,2} = \sqrt{F_{1,2}/k}\,[1 + a_{1,2}{}^2(z_b - Z_{1,2})^2/F_{1,2}{}^2]^{1/2}, \quad (20)$$

$$R_{1,2} = a_{1,2}(z_b - Z_{1,2}) + F_{1,2}{}^2/[a_{1,2}(z_b - Z_{1,2})], \quad (21)$$

with

$$Z_{1,2} = -\beta_i/a_{1,2}, \qquad F_{1,2} = \beta_r. \quad (22)$$

By substituting Eq. (19) into Eq. (16) we can readily identify $D_{1,2}$ as the beam width in the $(z, x_{b_{1,2}})$ plane, whereas $R_{1,2}$ is the phase front's radius of curvature. The resultant GB is therefore astigmatic; its waist in the $(z, x_{b_{1,2}})$ plane is located at $z_b = Z_{1,2}$, whereas $F_{1,2}$ is the corresponding collimation length. This astigmatism is caused by the beam tilt, which reduces the effective initial beam width in the $x_{b_{1,2}}$ directions.

The compact presentation in Eqs. (16)–(22) parameterizes the GB field in terms of local properties of the generic wave-number profile about stationary point $\boldsymbol{\xi} = 0$. This general parameterization can be compared to the isotropic profile in which $\zeta(\boldsymbol{\xi}) = (1 - \boldsymbol{\xi}\cdot\boldsymbol{\xi})^{1/2}$, in which case $\zeta_0 = 1$, $\partial_{\xi_{1,2}}\zeta_0 = 0$, and the beam axis in Eq. (7) coincides with the z axis. Furthermore, using $\partial_{\xi_1\xi_2}{}^2\zeta_0 = 0$ in Eq. (14), we obtain $\alpha = 0$, and therefore, from Eq. (18), $\Gamma_{1,2} = z - i\beta$. By substituting $\Gamma_{1,2}$ into Eq. (16) we obtain the well-known isotropic asymptotic GB field[19]

$$u(\mathbf{r}) = \frac{-i\beta}{z - i\beta}\exp\{ik[z + {}^1\!/_2\,\mathbf{x}^2/(z - i\beta)]\}. \quad (23)$$

In this Letter we have been concerned with parameterization of the effects of spectral anisotropy on the propagation characteristics of a paraxially approximated Gaussian beam in a medium with a generic wave-number profile. Various beam parameters have been systematically found to quantify the effect of anisotropy on various observables associated with the GB field. Introducing the anisotropy-dependent nonorthogonal local beam coordinate system enabled us to quantify the beam parametrization in terms of the local properties of the anisotropic surface $\zeta(\boldsymbol{\xi})$ at stationary on-axis point $\boldsymbol{\xi} = 0$.

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