Gaussian Beam Propagation in Generic Anisotropic Wavenumber Profiles

Igor Tinkelman and Timor Melamed

Abstract

The propagation characteristics of the scalar Gaussian Beam in a homogeneous anisotropic medium is considered. The medium is described by a generic wavenumber profile wherein the field is formulated by a Gaussian plane wave distribution and the propagation is obtained by saddle point asymptotics to extract the Gaussian Beam phenomenology in the anisotropic environment. The resulting field is parameterized in terms of the spatial evolution of the Gaussian beam curvature, beam width, etc., which are mapped to local geometrical properties of the generic wavenumber profile.

I. Introduction and Statement of the problem

Anisotropic materials are of interest for optical waveguides, microwave devices, plasma science, and different propagation environments. Comprehensive studies have been performed for the problem of Gaussian Beam (GB) 2D and 3D propagation for specific wavenumber profiles [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]. The generic profiles of the three-dimensional problem associated with a GB with complex wavenumber spectral constituents, are of fundamental significant, since GB propagation is a practical problem as well as theoretically significant, since GBs form the basis propagators for the "phase-space beam summation method", which is a general analytical framework for local analysis and modelling of radiation from extended source distributions [19]. Using GBs as basis waveobjects for modelling different anisotropic propagation and scattering problems were introduced in [20], [21]. The propagation of GB over a *generic* anisotropy has been studied in [22] using the Complex Source Point method. By applying the saddle point asymptotic technique, approximated over the z axis, the authors arrived at a close form analytic solution for the GB field. In our view, the Complex Source Point method cannot account for astigmatic effects which are present in our analysis of the generic wavenumber profiles, and therefore the results in [22] may be applied only to the case of uniaxially anisotropic medium. Alternatively, by applying a planewave spectral representation for the propagation problem, an alternative rigorous solution for the GB field is presented here, which is suitable for any generic wavenumber profile (see discussion following (8)) . The current study is concerned with the effects of anisotropy on the propagation characteristics of the scalar Gaussian beam field in a homogeneous medium described by the generic wavenumber profile $k_z(\boldsymbol{\xi})$ where $\boldsymbol{\xi}$ is wavenumber normalized coordinate. The field is formulated in the frequency-domain via a plane wave spectral integral and is evaluated asymptotically by saddle point technique. Given a field, $u(\mathbf{r})$, where a $\exp(-i\omega t)$ time dependence is assumed and suppressed, and $\mathbf{r} = (x_1, x_2, z)$ are conventional Cartesian coordinates, the wavenumber spectral (plane wave) transform pairs on the $z = 0$ initial surface are given by

$$\widetilde{u}_o(\boldsymbol{\xi}) = \int_{-\infty}^{\infty} d^2x \; u_o(\mathbf{x}) \; e^{-ik\boldsymbol{\xi}\cdot\mathbf{x}} \tag{1 a}$$

$$u_o(\mathbf{x}) = (k/2\pi)^2 \int d^2\xi \; \widetilde{u}_o(\boldsymbol{\xi}) \; e^{ik\,\boldsymbol{\xi}\cdot\mathbf{x}} \tag{1 b}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$ is the normalized spatial wavenumber vector, $\mathbf{x} = (x_1, x_2)$, k is the homogenous media wavenumber $k = \omega/c$, with c being the phase velocity of the homogeneous media, and $\widetilde{}$ identifies a wavenumber spectral function. The normalization with respect to the wavenumber k rendering $\boldsymbol{\xi}$ frequency-independent. Therefore, the GB field is formulated via the Gaussian distribution

$$u_0(\mathbf{x}) = \exp[-\frac{1}{2}k\mathbf{x}^2/\beta] \tag{2}$$

Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel.

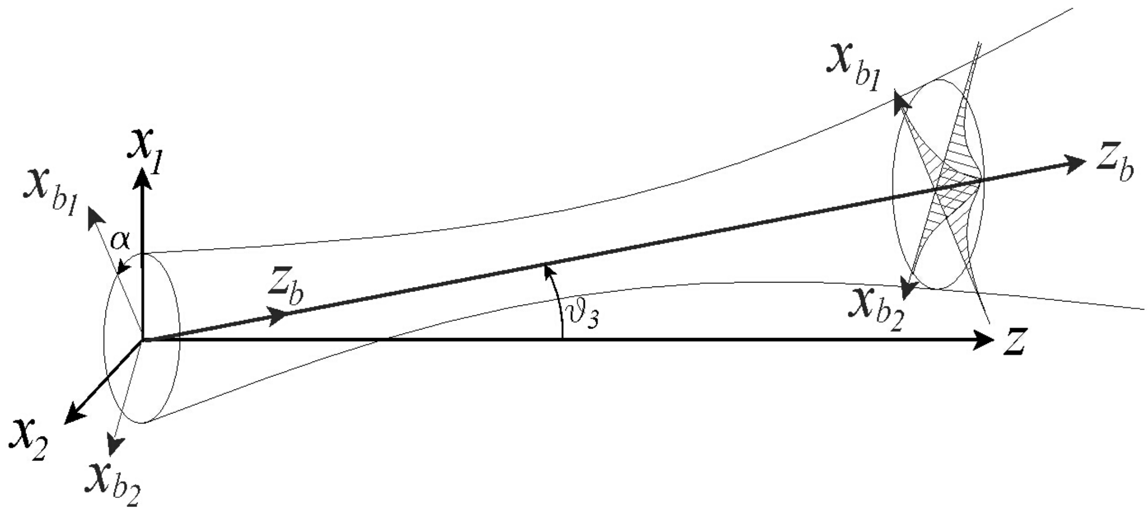

Fig. 1. Fig. 1. The local beam coordinate frame for the Gaussian beam propagating in the anisotropic medium. The beam axis is directed along the unit vector $\hat{\boldsymbol{\kappa}}$. The local transverse coordinates, $\mathbf{x}_b$, are given by the transformation in (13). The transverse local coordinates are located over the (x_1, x_2) plane which is in general non-orthogonal to the beam axis. The rotation transforation of (x_1, x_2) to (x_{b1}, x_{b2}) by α is carried out so that the resulting field in (16) exhibits Gaussian decay in the local coordinates.

were $\beta = \beta_r + i\beta_i$ with $\beta_r > 0$ is a parameter, and $\mathbf{x}^2 = \mathbf{x}\cdot\mathbf{x} = x_1^2 + x_2^2$. By inserting (2) into (1), we obtain the plane wave spectral distribution corresponding to (2)

$$\widetilde{u}_0(\boldsymbol{\xi}) = (2\pi\beta/k)\exp[-\frac{1}{2}k\beta\boldsymbol{\xi}^2]. \tag{3}$$

The plane wave distribution in (3) can be propagated into the $z > 0$ half space using the generic anisotropic propagator $\exp[ik\zeta(\boldsymbol{\xi})\ z]$, where, as in (1), we normalize the longitudinal wavenumber ζ by k. Using this propagator, the field propagating into the $z > 0$ half space is given by

$$u(\mathbf{r}) = \beta\frac{k}{2\pi}\int d^2\xi\ \exp[ik\Phi(\boldsymbol{\xi},\mathbf{r})], \quad \Phi(\boldsymbol{\xi},\mathbf{r}) = (\boldsymbol{\xi}\cdot\mathbf{x} + \zeta(\boldsymbol{\xi})z + \frac{i}{2}\beta\boldsymbol{\xi}^2). \tag{4}$$

The field in (4) cannot be evaluated in close form. In the next section we shall evaluate it asymptotically to obtain an analytic expression for the beam field in the high frequency regime.

II. Asymptotic evaluation and parameterization

The field representation in (4) may be evaluated asymptotically by the saddle point technique. The stationary point $\boldsymbol{\xi}_s$ satisfies

$$\nabla_\xi\Phi = \mathbf{x} + \nabla_\xi\zeta(\boldsymbol{\xi})|_{\boldsymbol{\xi}_s} z + i\beta\boldsymbol{\xi}_s = 0. \tag{5}$$

Equation (5) has a *real* solution only if $\boldsymbol{\xi}_s = 0$ and for observation points

$$\mathbf{x} + z\nabla_\xi\zeta_0 = 0, \tag{6}$$

where, here and henceforth, subscript 0 denotes sampling at $\boldsymbol{\xi} = 0$, i.e., $\nabla_\xi \zeta_0 \equiv \nabla_\xi \zeta|_{\boldsymbol{\xi}=0}$. The condition in (6) defines the *beam axis* being as a tilted line in the configuration space, with a anisotropy-dependent tilt. For isotropic materials, where $\zeta = \sqrt{1-\boldsymbol{\xi}^2}$, giving $\nabla_\xi \zeta_0 = 0$, the beam axis coincides with the z-axis. For the generic $\zeta(\boldsymbol{\xi})$ wavenumber profile, the beam axis is directed along the unit vector

$$\widehat{\boldsymbol{\kappa}} = (\cos\vartheta_1, \cos\vartheta_2, \cos\vartheta_3) \tag{7}$$

where $\vartheta_{1,2,3}$, are the beam axis angles with respects to the (x_1, x_2, z) axes, respectively. In view of (6), they are given by

$$\cos\vartheta_{1,2} = -\cos\vartheta_3 \partial_{\xi_{1,2}} \zeta_0, \quad \cos\vartheta_3 = \frac{1}{\sqrt{(\partial_{\xi_1}\zeta_0)^2 + (\partial_{\xi_2}\zeta_0)^2 + 1}}. \tag{8}$$

Note, that the beam axis direction, as defined in (6), is different from the definition given in [22] where the propagation of Gaussian beam in a generic wavenumber profile was investigated using the Complex Source Point method. The beam axis as defined in [22] is directed along the complex source $\mathbf{b}$ parameter which, for our problem, coincides with the z axis. Clearly, from (6), the z axis may serve as the beam axis only for symmetrical ζ, where $\nabla_\xi \zeta_0 = 0$, as in isotropic or uniaxially anisotropic medium. The condition in (6) may ,therefore, serves as a more generalized definition for the beam axis which accounts for astigmatic effects due to the medium anisotropy (see figure1).
For off-axis observation points, equation (5) could not be solved explicitly. Furthermore, the off-axis stationary point is complex and the solution requires analytic continuation of the wavenumber profile $\zeta = \zeta(\boldsymbol{\xi})$ for complex $\boldsymbol{\xi}$. In order to obtain a closed form analytic solution for the beam field, we notice that the beam field decays away from the beam axis. Therefore, we may apply a Taylor expansion of the phase Φ about the on-axis stationary point $\boldsymbol{\xi}_s = 0$.

$$\Phi \approx \Phi_0 + \mathbf{\Phi_1} \cdot \boldsymbol{\xi} + \tfrac{1}{2} \boldsymbol{\xi} \mathbf{\Phi_2} \boldsymbol{\xi} \tag{9}$$

with

$$\Phi_0 = \Phi|_{\boldsymbol{\xi}=\mathbf{0}} = \zeta_0 z, \quad \mathbf{\Phi_1} = \nabla\Phi|_{\boldsymbol{\xi}=\mathbf{0}} = \nabla_{\boldsymbol{\xi}} \zeta_0 z + \mathbf{x}, \tag{10}$$

and

$$\mathbf{\Phi_2} = \begin{bmatrix} i\beta + \partial^2_{\xi_1}\zeta_0 z & \partial^2_{\xi_1\xi_2}\zeta_0 z \\ \partial^2_{\xi_1\xi_2}\zeta_0 z & i\beta + \partial^2_{\xi_2}\zeta_0 z \end{bmatrix}, \tag{11}$$

where subscript 0 implies sampling at $\boldsymbol{\xi} = 0$. Using (9), one finds that the saddle point for both on- and off-axis observation points is $\boldsymbol{\xi}_s = -\mathbf{\Phi_2}^{-1}\mathbf{\Phi_1}$ and the field in (4) may be evaluated asymptotically by

$$u(\mathbf{r}) = \frac{\beta}{\sqrt{-\det \mathbf{\Phi_2}}} \exp[ikS(\mathbf{r})], \quad S(\mathbf{r}) = \Phi_0 - \frac{1}{2} \mathbf{\Phi_1} \mathbf{\Phi_2}^{-1} \mathbf{\Phi_1}. \tag{12}$$

The beam field in (12) may be presented in terms of *local beam coordinates* over which the field exhibits a Gaussian decay away from the beam axis. The local beam coordinates, $\mathbf{r}_b = (x_{b1}, x_{b2}, z_b)$, are defined by the non-orthogonal transformation

$$\mathbf{r}_b = \mathbf{T}\mathbf{r}, \quad \mathbf{T} = \begin{pmatrix} \cos\alpha & \sin\alpha & (-\cos\vartheta_2 \sin\alpha - \cos\vartheta_1 \cos\alpha)/\cos\vartheta_3 \\ -\sin\alpha & \cos\alpha & (-\cos\vartheta_2 \cos\alpha + \cos\vartheta_1 \sin\alpha)/\cos\vartheta_3 \\ 0 & 0 & 1/\cos\vartheta_3 \end{pmatrix} \tag{13}$$

where $\cos\vartheta_{1,2,3}$ are defined in (8), and the angle α is given by

$$\mathrm{tg}2\alpha = -2\partial^2_{\xi_1\xi_2}\zeta_0 / (\partial^2_{\xi_2}\zeta_0 - \partial^2_{\xi_1}\zeta_0). \tag{14}$$

The transformation in (13) is consist of a rotation transformation in the (x_1, x_2) plane by α, in which the phase $S(\mathbf{r})$ in (12) exhibit Gaussian decay, followed by tilting the z-axis to the beam-axis direction $\widehat{\boldsymbol{\kappa}}$ in (7) (see figure 1). The inverse transform is given by

$$\mathbf{T}^{-1} = \begin{pmatrix} \cos\alpha & -\sin\alpha & \cos\vartheta_1 \\ \sin\alpha & \cos\alpha & \cos\vartheta_2 \\ 0 & 0 & \cos\vartheta_3 \end{pmatrix}. \tag{15}$$

Using the beam coordinate system (13) in (12), the field may be presented in the Gaussian form

$$u(\mathbf{r}) = \frac{\beta}{\sqrt{-\Gamma_1\Gamma_2}} \exp\left\{ ik\left[\zeta_0 z_b \cos\vartheta_3 + \frac{1}{2}\left(\frac{x_{b1}^2}{\Gamma_1} + \frac{x_{b2}^2}{\Gamma_2} \right) \right] \right\} \tag{16}$$

where

$$\Gamma_{1,2} = -\frac{\partial^2_{\xi_1}\zeta_0 z + \partial^2_{\xi_2}\zeta_0 z + 2i\beta \mp \sqrt{(\partial^2_{\xi_1}\zeta_0 z - \partial^2_{\xi_2}\zeta_0 z)^2 + 4(\partial^2_{\xi_1\xi_2}\zeta_0 z)^2}}{2}. \tag{17}$$

Using (14) in (17), we obtain

$$\Gamma_{1,2} = z_b a_{1,2} - i\beta, \quad a_{1,2} = \frac{\cos\vartheta_3}{\cos(2\alpha)} \left[\partial^2_{\xi_1}\zeta_0 \left\{ \begin{array}{c} -\cos^2\alpha \\ \sin^2\alpha \end{array} \right\} + \partial^2_{\xi_2}\zeta_0 \left\{ \begin{array}{c} \sin^2\alpha \\ -\cos^2\alpha \end{array} \right\} \right] \tag{18}$$

Equation (16) has the form of a Gaussian beam, propagating along the beam axis z_b. The beam field exhibits a Gaussian decay in the transverse local coordinate $\mathbf{x}_b$ which are, in general, tilted with respect to the beam axis direction $\widehat{\boldsymbol{\kappa}}$. In order to parameterize the beam field, we rewrite the element of $\Gamma_{1,2}$ in the form

$$\frac{1}{\Gamma_{1,2}} = \frac{1}{R_{1,2}} + \frac{i}{kD^2_{1,2}} \tag{19}$$

where

$$D_{1,2} = \sqrt{F_{1,2}/k}\sqrt{1 + a^2_{1,2}(z_b - Z_{1,2})^2/F^2_{1,2}} \tag{20}$$

$$R_{1,2} = a_{1,2}(z_b - Z_{1,2}) + F^2_{1,2}/[a_{1,2}(z_b - Z_{1,2})]. \tag{21}$$

with

$$Z_{1,2} = -\beta_i/a_{1,2}, \quad F_{1,2} = \beta_r. \tag{22}$$

By substituting (19) into (16) one readily identifies $D_{1,2}$ as the *beam width* in the $(z, x_{b_{1,2}})$ plane, while $R_{1,2}$ is the *phase front radius of curvature*. The resulting GB is therefore *astigmatic*; its *waist* in the $(z, x_{b_{1,2}})$ plane, is located at $z_b = Z_{1,2}$, while $F_{1,2}$ is the corresponding *collimation length*. This astigmatism is caused by the beam tilt which reduces the effective initial beam width in the $x_{b_{1,2}}$ directions.
The compact presentation in equations (16)–(22), parameterizes the GB field in terms of local properties of the generic wavenumber profile about the stationary point $\boldsymbol{\xi} = 0$. This general parameterization can be compared to the isotropic profile where $\zeta(\boldsymbol{\xi}) = \sqrt{1 - \boldsymbol{\xi}\cdot\boldsymbol{\xi}}$. In which case $\zeta_0 = 1$, $\partial_{\xi_{1,2}}\zeta_0 = 0$, and the beam axis in (7) coincides with the z-axis. Furthermore, using $\partial^2_{\xi_1\xi_2}\zeta_0 = 0$ in (14), we obtain $\alpha = 0$, and therefore, from (18), $\Gamma_{1,2} = z - i\beta$. By inserting $\Gamma_{1,2}$ into (16) we obtain the well-known *isotropic* asymptotic GB field [19]

$$u(\mathbf{r}) = \frac{-i\beta}{z - i\beta} \exp[ik(z + \frac{1}{2}\mathbf{x}^2/(z - i\beta))]. \tag{23}$$

III. Conclusion

In this paper, we have been concerned with the parameterization of the effects of spectral anisotropy on the propagation characteristics of a paraxially approximated Gaussian beam in a medium with generic wavenumber profile. Various beam parameters have been systematically found to quantify the effect of anisotropy on various observables associated with the GB field. Introducing the anisotropy-dependent non-orthogonal *local beam coordinate system* enables the beam parameterization to be quantified in terms of local properties of the anisotropic surface $\zeta(\boldsymbol{\xi})$, at the stationary on-axis point $\boldsymbol{\xi} = 0$.

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