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EXACT GAUSSIAN BEAM EXPANSION OF TIME-HARMONIC ELECTROMAGNETIC WAVES

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Abstract—The present contribution is concern with an exact frame-based expansion of planar initial time-harmonic electromagnetic fields. The propagating field is described as a discrete superposition of tilted and shifted electromagnetic beam waveobjects over the frame spatial-spectral lattice. Explicit asymptotic expressions for the electromagnetic Gaussian beam propagators are obtained for the commonly used Gaussian windows.

1. INTRODUCTION

Beam-type (phase-space) spectral representations have been the subject of an intense research in the past decade, due to spectral and spatial localization, as well as the capability to propagate these waveobjects in complex environments [1–7]. In these expansion schemes, the propagating elements are Gaussian beams (or pulsed beams for time-dependent fields), which have been termed phase-space (spectral) Green's functions, as they link induced sources in the configuration-space to phase-space distributions of scattered fields, as well as phase-space distributions of incident fields to phase-space distributions of scattered fields [5, 8].

The beam locality feature yields the ability to obtain closed-form analytic expressions for propagation and scattering in complex media. Such *scalar* wave solutions have been obtained in homogeneous medium [1, 2, 4], anisotropic medium [9–11], dispersive medium [12–14] and inhomogeneous medium [15–17]. Several *electromagnetic* beam scattering and diffraction problems have been solved for rough surface scattering [18, 19], dielectric interfaces [20], PEC plates [21, 22], stratified [23] and negative isotropic media [24] , and more.

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The frame-based field expansion [4, 25] utilizes the key feature of the beam continues spectrum [1, 2] (i.e., being highly overcomplete), and hence may be discretized with no loss of essential data for reconstruction. For time-dependent fields, the overcompleteness was originally utilized in [26] by applying the Gabor series representation [27–30]. A recently developed theory for discrete field expansion, the frame-based beam summation method, overcomes inherent problems of Gabor representation by reducing the overcompleteness nature of beams spectrum [4]. Application of the theory for cylindrical apertures was presented in [31] using periodic frame-based decomposition.

In the frame-based spectral representation, the field is described by superposition of beams that emanate from a discrete set of points in the aperture and in a discreet set of directions (see Fig. 1). The excitation amplitudes of the beam spectral propagators, are the local (windowed) spectrum of the initial field distribution. Thus, only those beams that match the local radiation direction are significantly excited and should be accounted for the spectral (summation) representation. Thought beam-expansion schemes for *scalar* fields has been the subject of intense research, *electromagnetic* expansions in terms of Gaussian beams has been significantly less explored. In [32–34] Gaussian beams have been used for for analysis of large reflector antennas, in which the expansion coefficients are obtained by numerically matching Gaussian beams to the far zone field of the feed antenna. These methods, do not employ an *exact* field expansion schemes and therefore, cannot be applied for near-field analysis, or in exact field calculations. The present contribution is aimed at obtaining an exact beam-type electromagnetic waves expansion in the time harmonic regime which are define by their planar initial transverse field distribution.

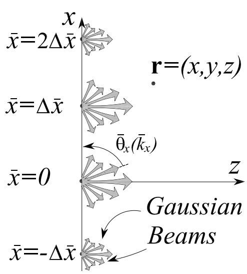

Figure 1. Discrete frame lattice. The fields in $z \geq 0$ are evaluated by superposition of tiled and shifted beams which are emanating from the initial distribution plane over the discrete frame spatial-spectral lattice in (6). Each beam propagator emanates from a lattice point $(\bar{x}, \bar{y}) = (m_x \Delta\bar{x}, m_y \Delta\bar{y})$, in a direction of $(\bar{\vartheta}_x, \bar{\vartheta}_y) = \cos^{-1}[(\bar{k}_x, \bar{k}_y)/k]$ with respect to the corresponding axis.

2. PLANE-WAVE EXPANSION

We are concerned with obtaining a discrete exact spectral representation for the time-harmonic electromagnetic (EM) field in $z \geq 0$ due to sources in $z < 0$, given the transverse electric field components over $z = 0$ plane

$$\mathbf{E}_0(x, y) = E_x(x, y)\hat{\mathbf{x}} + E_y(x, y)\hat{\mathbf{y}}. \tag{1}$$

The propagation medium is homogeneous with ϵ_0 and μ_0 denoting the free space permittivity and permeability, respectively, and a time-dependence of $\exp(j\omega t)$ is assumed for all field quantities. The spatial Fourier transform (plane-wave spectrum) of the initial field transverse components, which is denoted here by

$$\tilde{\mathbf{E}}_0(k_x, k_y) = \tilde{E}_x(k_x, k_y)\hat{\mathbf{x}} + \tilde{E}_y(k_x, k_y)\hat{\mathbf{y}}, \tag{2}$$

is defined as

$$\tilde{\mathbf{E}}_0(k_x, k_y) = \int dxdy \mathbf{E}_0(x, y)e^{j(k_x x + k_y y)}, \tag{3}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the convectional cartesian unit-vectors. Throughout this work, all integral limits of $-\infty$ to ∞ are omitted and plane-wave (wavenumber) spectral distributions, such as $\tilde{\mathbf{E}}_0(k_x, k_y)$, are denoted by an over $\sim$. By applying a standard plane-wave analysis, the longitudinal spectrum, $\tilde{E}_z$, is given by [35, 36]

$$\tilde{E}_z(k_x, k_y) = -(k_x\tilde{E}_x + k_y\tilde{E}_y)/k_z, \tag{4}$$

where $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$, $\mathrm{Im}k_z \leq 0$, $\mathrm{Re}k_z \geq 0$, is the longitudinal wavenumber with $k = \omega/c$ denoting the medium's wavenumber. Thus, the electric field in $z \geq 0$ is given by the plane-wave superposition

$$\mathbf{E}(\mathbf{r}) = \frac{1}{(2\pi)^2}\int dk_x dk_y[\tilde{\mathbf{E}}_0(k_x, k_y) + \hat{\mathbf{z}}\tilde{E}_z(k_x, k_y)]e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{5}$$

where $\mathbf{k} = (k_x, k_y, k_z)$, $\mathbf{r} = (x, y, z)$, and $\tilde{E}_z$ is given in (4). The plane-wave representation in (5) describes the electric field in term of EM plane-wave propagators each emanates from $z = 0$ plane in direction of unit-vector $\hat{\boldsymbol{\kappa}} = \mathbf{k}/k$.

3. SCALAR FRAME-BASED BEAM EXPANSION

In order to establish the EM frame-based beam expansion, we shall briefly review here the main results of the *scalar* frame-based beam decomposition which was introduced in [4]. The later is constructed on a discrete frame spatial-spectral lattice

$$(\bar{x}, \bar{y}, \bar{k}_x, \bar{k}_y) = (m_x \Delta\bar{x}, m_y \Delta\bar{y}, n_x \Delta\bar{k}_x, n_y \Delta\bar{k}_y), \tag{6}$$

where $(\Delta\bar{x}, \Delta\bar{y})$ and $(\Delta\bar{k}_x, \Delta\bar{k}_y)$, are the unit-cell dimensions in the (x, y) and (k_x, k_y) coordinates, respectively, and the index $\boldsymbol{\mu} = (m_x, m_y, n_x, n_y)$ is used to tag the lattice points (see Fig. 1). These unit-cell dimensions satisfy

$$\Delta\bar{x}\Delta\bar{k}_x = 2\pi\nu_x, \quad \Delta\bar{y}\Delta\bar{k}_y = 2\pi\nu_y, \tag{7}$$

where $0 \leq \nu_{x,y} \leq 1$ are the overcompleteness (or oversampling) parameters in x and y axes, respectively. The lattice is overcomplete for $\nu_{x,y} < 1$, critically complete in the Gabor limit $\nu_{x,y} \uparrow 1$ [26, 28, 37], and for $\nu_{x,y} \downarrow 0$ the discrete parametrization attains the continuity limit as in [1, 2].

Upon setting the frame lattice (see considerations in [4]), one proceeds by choosing a proper "mother" window, $\psi(x, y)$, and construct the windowed Fourier transform frame in the space of all the square integrable functions $\mathbb{L}_2(\mathbb{R})$ on the frame lattice. The 2D frame window, $\psi(x, y)$, is attained by a Cartesian multiplication of two 1D windows, each one yielding a proper frame in $\mathbb{L}_2(\mathbb{R})$. The frame representation of some initial scalar field distribution over $z = 0$ plane, $u_0(x, y)$, is given by

$$u_0(x, y) = \sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}} \psi(x, y; \boldsymbol{\mu}), \tag{8}$$

where the expansion functions, $\psi(x, y; \boldsymbol{\mu})$, are obtained from the "mother" window via

$$\psi(x, y; \boldsymbol{\mu}) = \psi(x - \bar{x}, y - \bar{y}) e^{-j[\bar{k}_x(x-\bar{x}) + \bar{k}_y(y-\bar{y})]}, \tag{9}$$

and the expansion coefficients, $a_{\boldsymbol{\mu}}$, are given by the inner product of the initial distribution with the so-called dual frame, $\varphi(x, y)$, namely

$$a_{\boldsymbol{\mu}} = \int dxdy \; u_0(x, y) \varphi^*(x, y; \boldsymbol{\mu}), \tag{10}$$

where, similarly to (9),

$$\varphi(x, y; \boldsymbol{\mu}) = \varphi(x - \bar{x}, y - \bar{y}) e^{-j[\bar{k}_x(x-\bar{x}) + \bar{k}_y(y-\bar{y})]}. \tag{11}$$

The dual frame may be evaluated by several ways which are listed in [4]. Here, we shall make use of high-oversampling approximation

$$\varphi(x, y) \cong \frac{\nu_x \nu_y}{\|\psi\|^2} \psi(x, y), \tag{12}$$

which, for Gaussian windows, is valid for $\nu_{x,y} < 0.4$ (for further details please refer to [4]).

The scalar field in $z > 0$, due to sources in $z < 0$, is obtained by propagating each $\psi(x, y; \boldsymbol{\mu})$ window element in summation (8), into $z > 0$ half-space. Therefore, the frame-based representation of the field is given by

$$u(\mathbf{r}) = \sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}} B(\mathbf{r}; \boldsymbol{\mu}), \tag{13}$$

where each beam field, $B(\mathbf{r}; \boldsymbol{\mu})$, satisfies the scalar Helmholtz equation

$$\left[\nabla^2 + k^2\right] B(\mathbf{r}; \boldsymbol{\mu}) = 0, \tag{14}$$

and may be evaluated via several spectral representations, such as plane-wave or Green's function (Kirchhoff's) integration (see details in [2, 4]). The spectral representation in (13) describes the field as a discrete superposition of beam waveobjects which emanate from points $(\bar{x}, \bar{y})$ on the frame lattice, in a discrete set of directions which are determine by the spectral wavenumbers $(\bar{k}_x, \bar{k}_y)$ over the frame lattice (see Fig. 1). In the next section, the vectorial EM analogue of this representation is obtained, in which the electric and magnetic fields are described by a similar superposition of *EM beam propagators* (see (21) and (24)).

In order to obtain a discrete *vectorial* frame-based representation, we shall relate the plane-wave spectrum of $u_0(x, y)$ to its frame representation in (8). The (scalar) plane-wave spectrum is obtained applying the Fourier integral in (3) to $u_0(x, y)$. By inserting (8) into the later and inverting the order of integration and summation, we obtain

$$\widetilde{u}_0(k_x, k_y) = \sum_{\boldsymbol{\mu}} a^{\boldsymbol{\mu}} \, \tilde{\psi}(k_x, k_y; \boldsymbol{\mu}), \tag{15}$$

where

$$\tilde{\psi}(k_x, k_y; \boldsymbol{\mu}) = \tilde{\psi}(k_x - \bar{k}_x, k_y - \bar{k}_y) e^{j(k_x \bar{x} + k_y \bar{y})}, \tag{16}$$

with $\tilde{\psi}(k_x, k_y)$ denoting the plane-wave spectral distribution of $\psi(x, y)$, which is obtained using the Fourier operator in (3).

4. VECTORIAL EM FIELD EXPANSION

In order to obtain a frame-based representation of the electric field, $\mathbf{E}(\mathbf{r})$, we introduce the coefficients vector

$$\mathbf{a}^{\boldsymbol{\mu}} = a_x^{\boldsymbol{\mu}} \hat{\mathbf{x}} + a_y^{\boldsymbol{\mu}} \hat{\mathbf{y}} = \int dx dy \; \mathbf{E}_0(x, y) \varphi^*(x, y; \boldsymbol{\mu}), \tag{17}$$

where $\varphi(x, y; \boldsymbol{\mu})$ is given in (11). Using (15) for each electric field transverse component, we may write

$$\tilde{\mathbf{E}}_0(k_x, k_y) = \sum_{\boldsymbol{\mu}} \mathbf{a}^{\boldsymbol{\mu}}\, \tilde{\psi}(k_x, k_y; \boldsymbol{\mu}), \tag{18}$$

where $\tilde{\psi}(k_x, k_y; \boldsymbol{\mu})$ is given in (16). By inserting (18) with (4) into (5) and inverting the order of integration and summation, we obtain

$$\mathbf{E}(\mathbf{r}) = \sum_{\boldsymbol{\mu}} \frac{1}{(2\pi)^2} \int dk_x dk_y [a_x^{\boldsymbol{\mu}} \tilde{\mathbf{V}}_x + a_y^{\boldsymbol{\mu}} \tilde{\mathbf{V}}_y] \tilde{\psi}(k_x, k_y; \boldsymbol{\mu}) e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{19}$$

where $a_x^{\boldsymbol{\mu}}$ and $a_y^{\boldsymbol{\mu}}$ are given in (17), and

$$\tilde{\mathbf{V}}_x = \hat{\mathbf{x}} - k_z^{-1} k_x \hat{\mathbf{z}}, \quad \tilde{\mathbf{V}}_y = \hat{\mathbf{y}} - k_z^{-1} k_y \hat{\mathbf{z}}. \tag{20}$$

By using $k_x \exp(-j\mathbf{k}\cdot\mathbf{r}) = j\partial_x \exp(-j\mathbf{k}\cdot\mathbf{r})$, and so forth, we rewrite (19) in the form

$$\mathbf{E}(\mathbf{r}) = \sum_{\boldsymbol{\mu}} a_x^{\boldsymbol{\mu}} \mathbf{E}_x^B(\mathbf{r}; \boldsymbol{\mu}) + a_y^{\boldsymbol{\mu}} \mathbf{E}_y^B(\mathbf{r}; \boldsymbol{\mu}), \tag{21}$$

where the electric fields of the *EM beam propagators*, $\mathbf{E}_x^B(\mathbf{r}; \boldsymbol{\mu})$ and $\mathbf{E}_y^B(\mathbf{r}; \boldsymbol{\mu})$, are obtained from the *scalar* beam propagator, which is defined by

$$B(\mathbf{r}; \boldsymbol{\mu}) = \frac{1}{(2\pi)^2} \int dk_x dk_y \ (-jk_z)^{-1} \tilde{\psi}(k_x, k_y; \boldsymbol{\mu}) e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{22}$$

via

$$\begin{aligned} \mathbf{E}_x^B(\mathbf{r}; \boldsymbol{\mu}) &= (\hat{\mathbf{x}}\partial_z - \hat{\mathbf{z}}\partial_x) B(\mathbf{r}; \boldsymbol{\mu}), \\ \mathbf{E}_y^B(\mathbf{r}; \boldsymbol{\mu}) &= (\hat{\mathbf{y}}\partial_z - \hat{\mathbf{z}}\partial_y) B(\mathbf{r}; \boldsymbol{\mu}), \end{aligned} \tag{23}$$

where, we denote, $\partial_x = \partial/\partial x$, and so forth. Equation (21) with (22) and (23) represent the electric field, $\mathbf{E}(\mathbf{r})$, as a discrete superposition of EM beam waveobjects, $\mathbf{E}_x^B$ and $\mathbf{E}_y^B$, which are the electric field propagators due to the initial electric field x and y components over $z = 0$ plane, respectively. The excitation amplitudes of these EM waveobjects are obtained from the initial field distribution, $\mathbf{E}_0$, via (17). The spectral summation in (21), represents the electric field in terms of a discrete superposition of localized beam propagators which emanate from each processing (spectral) point, $(\bar{x}, \bar{y})$, in a processing-dependent (spectral), $(\bar{k}_x, \bar{k}_y)$ direction. The beam propagators are characterized by transversal localization and high directivity (see specific example for Gaussian windows in (32)–(35)).

By applying Faraday's low, $\mathbf{H} = (-j\omega\mu_0)^{-1}\nabla \times \mathbf{E}$, to (21), we obtain the corresponding magnetic field in $z \geq 0$

$$\mathbf{H}(\mathbf{r}) = \sum_{\boldsymbol{\mu}} a_x^{\boldsymbol{\mu}} \mathbf{H}_x^B(\mathbf{r};\boldsymbol{\mu}) + a_y^{\boldsymbol{\mu}} \mathbf{H}_y^B(\mathbf{r};\boldsymbol{\mu}), \tag{24}$$

where the magnetic field of the *EM beam propagators*, $\mathbf{H}_x^B$ and $\mathbf{H}_y^B$, are given by

$$\begin{aligned} \mathbf{H}_x^B(\mathbf{r};\boldsymbol{\mu}) &= \frac{1}{j\omega\mu_o}[\hat{\mathbf{x}}\partial_{xy}^2 - \hat{\mathbf{y}}(\partial_x^2 + \partial_z^2) + \hat{\mathbf{z}}\partial_{yz}^2]B(\mathbf{r};\boldsymbol{\mu}) \\ \mathbf{H}_y^B(\mathbf{r};\boldsymbol{\mu}) &= \frac{1}{j\omega\mu_o}[\hat{\mathbf{x}}(\partial_y^2 + \partial_z^2) - \hat{\mathbf{y}}\partial_{xy}^2 - \hat{\mathbf{z}}\partial_{xz}^2]B(\mathbf{r};\boldsymbol{\mu}), \end{aligned} \tag{25}$$

where the *scalar* beam propagator, $B(\mathbf{r};\boldsymbol{\mu})$, is given in (22).

5. GAUSSIAN FRAMES AND ASYMPTOTIC EVALUATION

The general frame representation in (8) is applied here for the special case of Gaussian windows which have been used extensively for modeling beam propagation, since they maximize the localization as implied by the uncertainty principle, and yield analytically trackable beam-type propagators [1–6, 15, 16]. The Gaussian "mother" frame spatial and spectral distributions are

$$\begin{aligned} \psi(x,y) &= e^{-jk\Gamma(x^2+y^2)/2}, \\ \tilde{\psi}(k_x,k_y) &= -2\pi j(k\Gamma)^{-1} e^{j(k_x^2+k_y^2)/(2k\Gamma)}, \end{aligned} \tag{26}$$

where $\Gamma = \Gamma_r + j\Gamma_j$ is the window complex *frequency independent* parameter with $\Gamma_j < 0$. Thus, by inserting $\|\psi\|^2 = -\pi/(k\Gamma_j)$ into (12), we may approximate the corresponding dual frame by

$$\varphi(x,y) = \left(-\nu^2 k\Gamma_j/\pi\right) e^{-jk\Gamma(x^2+y^2)/2}, \tag{27}$$

where $\nu = \nu_x = \nu_y$ is the oversampling parameter in (7). This type of windows gives rise to Gaussian beam waveobjects which exhibits frequency independent collimation (Rayleigh) distance and therefore are termed iso-diffracting [38]. The iso-diffracting nature makes these waveobjects highly suitable for UWB radiation representations [2, 4, 10, 25, 39, 40].

The scalar beam propagators are obtained by inserting (26) into (22). The resulting plane-wave spectral integral was evaluated asymptotically in [2]. The resulting paraxial scalar Gaussian beam waveobjects are obtained by utilizing the local beam coordinates, $\mathbf{r}_b = (x_b, y_b, z_b)$, which are defined, for a given spectral point $(\bar{x}, \bar{y}, \bar{k}_x, \bar{k}_y)$ on the frame lattice, by the transformation

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos\bar{\vartheta}\cos\bar{\varphi} & \cos\bar{\vartheta}\sin\bar{\varphi} & -\sin\bar{\vartheta} \\ -\sin\bar{\varphi} & \cos\bar{\varphi} & 0 \\ \sin\bar{\vartheta}\cos\bar{\varphi} & \sin\bar{\vartheta}\sin\bar{\varphi} & \cos\bar{\vartheta} \end{bmatrix} \begin{bmatrix} x - \bar{x}_x \\ y - \bar{x}_y \\ z \end{bmatrix} \tag{28}$$

where $(\bar{\vartheta}, \bar{\varphi})$ are the spherical angles that define the plane-wave unit-vector $(\bar{k}_x, \bar{k}_y, \bar{k}_z)/k$, where $\bar{k}_z = \sqrt{k^2 - \bar{k}_x^2 - \bar{k}_x^2}$, i.e.,

$$\cos\bar{\vartheta} = \bar{k}_z/k, \quad \cos\bar{\varphi} = \bar{k}_x/\sqrt{\bar{k}_x^2 + \bar{k}_y^2}, \quad \sin\bar{\varphi} = \bar{k}_y/\sqrt{\bar{k}_x^2 + \bar{k}_y^2}. \tag{29}$$

By utilizing the beam coordinates, the beam waveobject may be evaluated asymptotically by

$$\begin{aligned} B(\mathbf{r}_b; \boldsymbol{\mu}) &\sim \frac{j}{k\cos\bar{\vartheta}} \sqrt{\frac{\Gamma_x(z_b)}{\Gamma_x(0)}} \sqrt{\frac{\Gamma_y(z_b)}{\Gamma_y(0)}} \exp[-jk\Psi(\mathbf{r}_b; \boldsymbol{\mu})] \\ \Psi(\mathbf{r}_b; \boldsymbol{\mu}) &= z_b + \frac{1}{2}\left[\Gamma_x(z_b)x_b^2 + \Gamma_y(z_b)y_b^2\right], \end{aligned} \tag{30}$$

where

$$\Gamma_x(z_b) = 1/(z_b + \cos^2\bar{\vartheta}\Gamma^{-1}), \quad \Gamma_y(z_b) = 1/(z_b + \Gamma^{-1}), \tag{31}$$

are the so-called complex curvatures of the Gaussian beams. Parameterization of the waveobjects in (30) may be found in [2].

Next, the electric field Gaussian propagators are evaluated by inserting (30) into (23) and collecting the higher asymptotic order. Alternatively, the asymptotic electric field may be evaluated directly from (19). Note, that the difference between the electric field spectral integral (19) and the scalar field representation in (22), is only in the *amplitude* elements, so we can sample the amplitude at the on-axis stationary point $(k_x, k_y) = (\bar{k}_x, \bar{k}_y)$. Thus, using (29), the electric field $\mathbf{E}_x^B$ is given by

$$\mathbf{E}_x^B(\mathbf{r}; \boldsymbol{\mu}) \sim \left(\hat{\mathbf{x}} - \hat{\mathbf{z}}\tan\bar{\vartheta}\cos\bar{\varphi}\right) \sqrt{\frac{\Gamma_x(z_b)}{\Gamma_x(0)}} \sqrt{\frac{\Gamma_y(z_b)}{\Gamma_y(0)}} \exp[-jk\Psi(\mathbf{r}_b; \boldsymbol{\mu})], \tag{32}$$

where $\Gamma_{x,y}(z_b)$ and $\Psi(\mathbf{r}_b; \boldsymbol{\mu})$ are given in (31) and (30), respectively. The corresponding magnetic field spectral integral may be easily obtained from the electric field by using the well-known relation for plane-waves $\tilde{\mathbf{H}} = \eta_0^{-1}\hat{\boldsymbol{\kappa}} \times \tilde{\mathbf{E}}$. Evaluating the resulting spectral integral asymptotically, one obtains

$$\mathbf{H}_x^B(\mathbf{r};\boldsymbol{\mu}) \sim \left(\frac{\sin^2\bar{\vartheta}\sin 2\bar{\varphi}}{2\cos\bar{\vartheta}}\hat{\mathbf{x}} + \frac{\sin^2\bar{\vartheta}\sin^2\bar{\varphi}-1}{\cos\bar{\vartheta}}\hat{\mathbf{y}} + \sin\bar{\vartheta}\sin\bar{\varphi}\hat{\mathbf{z}}\right) \times \frac{-1}{\eta_0}\sqrt{\frac{\Gamma_x(z_b)}{\Gamma_x(0)}}\sqrt{\frac{\Gamma_y(z_b)}{\Gamma_y(0)}}\exp[-jk\Psi(\mathbf{r}_b;\boldsymbol{\mu})], \tag{33}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0} = 120\pi\,\Omega$ is the (vacuum) wave impedance. The EM wave components, $\mathbf{E}_y^B$ and $\mathbf{H}_y^B$, may be obtained in a similar manner. The result is

$$\mathbf{E}_y^B(\mathbf{r};\boldsymbol{\mu}) \sim \left(\hat{\mathbf{y}} - \hat{\mathbf{z}}\tan\bar{\vartheta}\sin\bar{\varphi}\right)\sqrt{\frac{\Gamma_x(z_b)}{\Gamma_x(0)}}\sqrt{\frac{\Gamma_y(z_b)}{\Gamma_y(0)}}\exp[-jk\Psi(\mathbf{r}_b;\boldsymbol{\mu})], \tag{34}$$

and

$$\mathbf{H}_y^B(\mathbf{r};\boldsymbol{\mu}) \sim \left(\frac{\sin^2\bar{\vartheta}\cos^2\bar{\varphi}-1}{\cos\bar{\vartheta}}\hat{\mathbf{x}} + \frac{\sin^2\bar{\vartheta}\sin 2\bar{\varphi}}{2\cos\bar{\vartheta}}\hat{\mathbf{y}} + \sin\bar{\vartheta}\cos\bar{\varphi}\hat{\mathbf{z}}\right) \times \frac{1}{\eta_0}\sqrt{\frac{\Gamma_x(z_b)}{\Gamma_x(0)}}\sqrt{\frac{\Gamma_y(z_b)}{\Gamma_y(0)}}\exp[-jk\Psi(\mathbf{r}_b;\boldsymbol{\mu})]. \tag{35}$$

The EM field frame-based Gaussian beam expansion is, therefore, given by (21) and (24) with the expansion coefficients in (17), and the EM beam propagators in (32)–(35).

6. CONCLUSIONS

An exact expansion scheme for EM wave in terms of Gaussian beams has been introduced, in which the fields are defined by the transverse electric field components. By applying a frame-based expansion, the electric and magnetic fields are described as a discrete summation of tilted and shifted Gaussian beams which emanate from the initial distribution plane over the spectral frame lattice. The EM propagators are obtained from a scalar one via differential operators which, in the high-frequency regime, give rise to simple closed form expressions. This EM field expansion is suitable for analyzing EM wave propagation and scattering in complex environments, both in the near and far field regions.

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