On Localization Aspects of Frequency-Domain Scattering From Low-Contrast Objects

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***Abstract*—The (phase-space) local spectrum of the field scattered by a low-contrast object due to Gaussian beam incidence, is evaluated in the high-frequency regime. The scattering problem is linearized using the Born approximation for which the incident field and local transform domain Green's function can be evaluated asymptotically. The scattering phenomenology is described in terms of local samplings of the object function which are localized in the object domain according to the incidence and processing parameters. Application in the field of inverse scattering are expected to yield fast and efficient algorithms due to the availability of analytic solutions for both the incident wave and the local domain Green's function.**

***Index Terms*—Born approximation, frequency domain scattering, inverse patterns, local transform, phase-space analysis.**

I. Introduction and Statement of the Problem

As STATED in the abstract, the present study is concerned with extending the previously formulated pulsed plane-wave based time-domain global [1] and local-processing [2], [3] diffraction tomography for forward and inverse scattering, to a highly localized frequency domain Gaussian beam (GB)-based tomography. The GB-based tomography links the scattering medium, illuminated by a GB, to the (phase-space) local spectrum of the scattered field. The local spectrum has been the subject of intense research over the past few years with application to propagation [4]–[7], scattering [2], and inverse scattering [3], [8]. Localization aspect of time-domain scattering from low contrast objects was introduced in [2], but local scattering phenomena, associated with local pre- and post- processing in the *frequency domain* does not seem to exist in the literature even for Born-type media.

We are concern with the field scattered by a low-contrast object characterized by a wave velocity of $v(\mathbf{r})$, embedded in a homogeneous background of v_o (see Fig. 1). The total field $u(\mathbf{r})$, with $e^{-i\omega t}$ time-dependence assumed and suppressed, satisfies the scalar Helmholtz equation

$$\left[\nabla^2 + k^2\left(\frac{v_o}{v2(\mathbf{r})}\right)^2\right]u(\mathbf{r}) = 0, \qquad k = \omega/v_o \tag{1}$$

where $\mathbf{r} = (\mathbf{x}, z)$, $\mathbf{x} = (x_1, x_2)$ are the conventional Cartesian coordinate frame. The scattered field may be approximated by the Born approximation, which linearizes the scattering due to small deviations of the scattering medium from the background v_o, which, for convenience, are described by the *object function*

$$O(\mathbf{r}) = [v_o/v(\mathbf{r})]^2 - 1. \tag{2}$$

Following the strategy outlined in the introduction, we are considering a Gaussian-beam incidence propagating in the direction of the unit vector $\hat{\boldsymbol{\kappa}}_i = (\vartheta_i, \varphi_i)$, which determines the incident beam axis. Denoting $\mathbf{r}_i = (x_{i1}, x_{i2}, z_i)$ as the incident beam coordinates, obtained via a conventional rotation transform $\mathbf{r}_i = \mathbf{r}_0 + \mathbf{T}_i\mathbf{r}$, where $\mathbf{T}_i$ is a 3 × 3 matrix, the Gaussian beam field can be modeled asymptotically within the paraxial approximation by [6]

$$u^i(\mathbf{r}_i) = \frac{-\beta_i}{z^i - i\beta_i}\exp\left[ik\left(z_i + \frac{1}{2}(\rho_i)^2\Big/(z_i - i\beta_i)\right)\right] \tag{3}$$

where $\rho_i = \sqrt{x_{i_1}^2 + x_{i_2}^2}$ and the parameter $\beta_i = \mathrm{Re}\beta_i + i\mathrm{Im}\beta_i$, with $\mathrm{Re}\beta_i > 0$. Here and henceforth, subscript i denotes incident beam constituents. In order to parameterized the Gaussian beam, we write

$$(z_i - i\beta)^{-1} = 1/R_i + i/kD_i^2 \tag{4}$$

where

$$D_i = \sqrt{\mathrm{Re}\beta_i/k}\sqrt{1 + ((z_i + \mathrm{Im}\beta_i)/\mathrm{Re}\beta_i)^2} \tag{5}$$

$$R_i = (z_i + \mathrm{Im}\beta_i) + (\mathrm{Re}\beta_i)^2/(z_i + \mathrm{Im}\beta_i). \tag{6}$$

By substituting (4) into (3), one readily identifies D_i as the *beamwidth*, while R_i is the *phase-front radius of curvature*. The GB *waist* is located at $z_i = -\mathrm{Im}\beta_i$, while $\mathrm{Re}\beta_i$ is the corresponding *collimation length*.

The scattered field is evaluated over planar aperture that, without loss of generality, is assumed to be the $z = 0$ plane. For low-contrast objects, the data field $u^d(\mathbf{x}) \equiv \{u(\mathbf{r}) - u^i(\mathbf{r})\}|_{z=0}$ is given by the Born approximation

$$u^d(\mathbf{x}) = k^2\int d^3r' O(\mathbf{r}')u^i(\mathbf{r}'_i)G(\mathbf{r}, \mathbf{r}')|_{z=0}, \qquad \mathbf{r}'_i = \mathbf{r}_0 + \mathbf{T}_i\mathbf{r}' \tag{7}$$

where $G(\mathbf{r}, \mathbf{r}') = e^{ik|\mathbf{r}-\mathbf{r}'|}/4\pi|\mathbf{r} - \mathbf{r}'|$ being the free-space Green's function. The conditions under which this approximation is valid are discussed in [9]. Equation (7) describes the scattered field in term of induced sources $O(\mathbf{r}')u^i(\mathbf{r}'_i)$, which are radiating in the background media v_o. Free-space Green's function $G(\mathbf{r}, \mathbf{r}')$ propagates these induced sources to the data plane $z = 0$ via the spatial convolution integral in (7).

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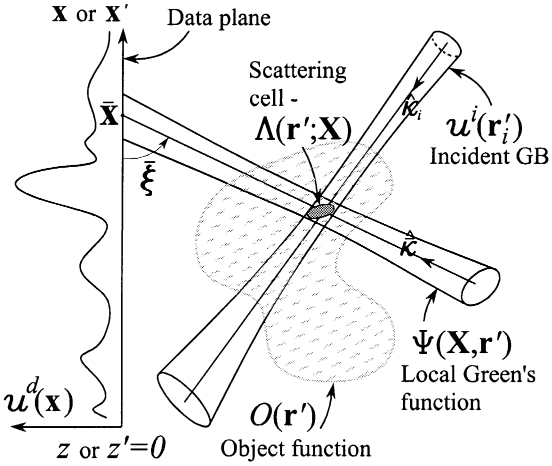

Fig. 1. Physical configuration and scattering phenomena. The object function $O(\mathbf{r})$ is illuminated by a Gaussian beam $u^i(\mathbf{r}_i)$ propagating in the direction $\hat{\boldsymbol{\kappa}}_i$ and the scattered field $u^d(\mathbf{x})$ is measured on the plane $z = 0$. The local spectrum of the data is obtained by spatial samplings (13) of the object function. The sampling window Λ is oriented so that its normal bisects the angle between the direction of incidence $\hat{\boldsymbol{\kappa}}_i$ and the spectral (processed) scattering direction $\hat{\bar{\boldsymbol{\kappa}}}$.

II. Local Processing of the Data

In this section, we summarize the phase-space analysis and synthesis formalisms that parameterize the field on the $z = 0$ observation plane. For the desired *local* spectral analysis of the data, we generate the frequency-domain local plane wave spectrum via a windowed Fourier transform of the data in *configuration* space [6]

$$U(\mathbf{X}) = \int d^2x u^d(\mathbf{x})\hat{w}^*(\mathbf{x} - \overline{\mathbf{x}})e^{-ik\overline{\boldsymbol{\xi}}\cdot\mathbf{x}} \tag{8}$$

where the asterisk denotes the complex conjugate and $\mathbf{X} = (\overline{\mathbf{x}}, \overline{\boldsymbol{\xi}})$. In (8), $w(\mathbf{x} - \overline{\mathbf{x}})$ is a spatial window function centered at $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2)$, with linear phasing specified by $\overline{\boldsymbol{\xi}} = (\overline{\xi}_1, \overline{\xi}_2)$. The vector $\mathbf{X}$ incorporates the configuration-spectrum *phase-space coordinates* $(\overline{\mathbf{x}}, \overline{\boldsymbol{\xi}})$, where $U(\mathbf{X})$ is referred to as a *phase-space distribution* of the data $u^d(\mathbf{x})$. The transform (8) extracts from u^d the local spectrum around the $\overline{\boldsymbol{\xi}}$-directed propagation at the window center $\overline{\mathbf{x}}$. In typical scattering problems, the spectrum at a given $\overline{\mathbf{x}}$ is localized about a preferred spectral direction $\overline{\boldsymbol{\xi}}(\overline{\mathbf{x}})$ that describes the direction of arrival of the scattered field at this point. The local spectrum may be used for forward propagation of the field away from the initial distribution plane. The field at $z > 0$ is synthesized via a phase-space superposition of beams emerging from all points $\overline{\mathbf{x}}$ on the initial surface in all directions $\overline{\boldsymbol{\xi}}$ (for detailed expressions see [4] and [6]).

It is advantageous to use Gaussian windows since they yield analytically trackable beam-type propagators. In the frequency domain, we use a Gaussian window whose spatial and spectral distributions are

$$w(\mathbf{x}) = e^{-(1/2)k\beta^{-1}\rho^2}, \quad k = \omega/v_o, \quad \rho = \sqrt{x_1^2 + x_2^2} \tag{9}$$

where the parameter $\beta = \mathrm{Re}\beta + i\mathrm{Im}\beta$, with $\mathrm{Re}\beta > 0$. The Gaussian window in (9) is localized around $\mathbf{x} = 0$. Recalling that $\mathrm{Re}\beta > 0$, the argument of the exponent function generating a smooth Gaussian window, which is strongest for $\rho = 0$ and weakens as ρ increases. The degree of spatial and spectral localization achieved, can be quantified in terms of the spatial Δ_x and spectral Δ_ξ root mean square (rms) widths of the window [6]

$$\Delta_x = |\beta| \Big/ \sqrt{k\mathrm{Re}\beta} = \Delta_\xi |\beta|. \tag{10}$$

Note the uncertainty principle $\Delta_x \Delta_\xi = |\beta|/k\mathrm{Re}\beta \geq 1/k$ with an equality for $\mathrm{Im}\beta = 0$. Further properties of this window, in particular those pertaining to numerical implementation for analysis of space-time data, have been explored in [6, Sec. 4.2].

III. Local Spectrum Analysis Under the Born Approximation

Next, we explore weak scattering by applying the spectral analysis tools described in Section II to the Born-approximated data gathered on the $z = 0$ observation plane of Fig. 1. By inserting (7) into (8), we obtain

$$U(\mathbf{X}) = k^2 \int d^3r' O(\mathbf{r}') u^i(\mathbf{r}_i') \Psi(\mathbf{X}, \mathbf{r}')$$
$$\mathbf{r}_i' = \mathbf{r}_0 + \mathbf{T}_i \mathbf{r}' \tag{11}$$

with

$$\Psi(\mathbf{X}, \mathbf{r}') = \int d^2x w^*(\mathbf{x} - \overline{\mathbf{x}}) e^{-ik\overline{\boldsymbol{\xi}}\cdot\mathbf{x}} G(\mathbf{r}, \mathbf{r}')|_{z=0} \tag{12}$$

where $G(\mathbf{r}, \mathbf{r}')$ is the free space Green's function [see (7)]. Equation (11) describes the local spectrum of the data in terms of a spatial convolution integral of the induced sources $O(\mathbf{r}')u^i(\mathbf{r}_i')$ with $\Psi(\mathbf{X}, \mathbf{r}')$. Comparing (11) with the Born approximation in (7), one finds that the two has essentially the same form. Therefore, $\Psi(\mathbf{X}, \mathbf{r}')$ may be regarded as Born approximated *Green's function in the local transform domain*. The local Green's function, $\Psi(\mathbf{X}, \mathbf{r}')$, propagates the contribution of the induced sources in the configurational space, $\mathbf{r}'$, to the local transform of the data over the observation plane, parameterized by the phase-space coordinate $\mathbf{X} = (\overline{\mathbf{x}}, \overline{\boldsymbol{\xi}})$. Also, from (12), we note that the local Green's function satisfies the Helmholtz in the v_o medium and, therefore, for a proper choice of window function w, may be evaluated asymptotically (see Section IV) and even extends the present Born-approximated scatterings to a more general framework for scattering from high-contrast objects.

In order to gain insight into the scattering phenomena, we rewrite (11) in the form

$$U(\mathbf{X}) = \int d^3r' O(\mathbf{r}') \Lambda(\mathbf{r}'; \mathbf{X}) \tag{13}$$

where $\Lambda(\mathbf{r}'; \mathbf{X})$ is a sampling window in the $\mathbf{r}'$ object domain

$$\Lambda(\mathbf{r}'; \mathbf{X}) = k^2 u^i(\mathbf{r}_i') \Psi(\mathbf{X}, \mathbf{r}'). \tag{14}$$

Equation (13) represents the local spectrum of the time harmonic data in terms of local samples of the object function $O(\mathbf{r}')$. Since both $u^i(\mathbf{r}_i')$ and $\Psi(\mathbf{X}, \mathbf{r}')$ are beam-like waveobjects, the multiplication in (14) results in a *local scattering cell* which exhibits a Gaussian decay away from its center over the intersection of the beam axes. Therefore, $\Lambda(\mathbf{r}'; \mathbf{X})$ provides windowing of the object function along the beam axis as determined by the phase-space parameter $\mathbf{X}$. The above results imply that the interaction of the incident GB with the object domain, when parameterized in terms of scattered Gaussian beams propagators (i.e., the local spectrum), occurs as if each scattered beam were *specularly reflected* from the local medium inhomogeneities (Fig. 1). This interpretation is a localized version of the "pseudoreflection law" discussed in connection with the transient plane-wave incidence in [1] and in [2].

IV. Asymptotic Evaluation of the Propagators

In this section, the general formulation for the scattering process is evaluated for the special case of the Gaussian windows in (9). These windows enable close form asymptotic evaluation of the local Green's function Ψ and the scattering cell Λ. The formal integral representation of the local Green's function for Gaussian windows is obtained by inserting (9) into (12). The resulting expression has been evaluated asymptotically in [2] with connection to the local processing of pulsed plane-wave excited scattering. It was found there that if the window is "large" on a wavelength scale $\Psi(\mathbf{X}, \mathbf{r}')$ in (12), yields collimated beam fields in the $\mathbf{r}'$-domain. Via asymptotic evaluation and paraxial approximation, one obtains

$$\Psi(\mathbf{X}, \mathbf{r}') \sim \frac{i}{2k\overline{\zeta}} \sqrt{\frac{\det \mathbf{Q}(z_b')}{\det \mathbf{Q}(0)}} \cdot \exp\left[ik\left(\overline{\boldsymbol{\xi}} \cdot \overline{\mathbf{x}} - z_b' + \frac{1}{2}\mathbf{x}_b' \cdot \mathbf{Q}(z_b') \cdot \mathbf{x}_b'\right)\right] \quad (15)$$

where $\overline{\zeta} = \sqrt{1 - \overline{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{\xi}}}$ and

$$\mathbf{Q}(z_b') = \begin{bmatrix} (-z_b' - i\beta^* \overline{\zeta}^2)^{-1} & 0 \\ 0 & (-z_b' - i\beta^*)^{-1} \end{bmatrix}. \quad (16)$$

In (15), we utilize the beam coordinates $(x_{b_1}', x_{b_2}', z_b')$ defined, for a given phase-space point $\mathbf{X}$ by the transformation

$$\begin{bmatrix} x_{b_1}' \\ x_{b_2}' \\ z_b' \end{bmatrix} = \begin{bmatrix} \cos\overline{\vartheta}\cos\overline{\varphi} & \cos\overline{\vartheta}\sin\overline{\varphi} & -\sin\overline{\vartheta} \\ -\sin\overline{\varphi} & \cos\overline{\varphi} & 0 \\ \sin\overline{\vartheta}\cos\overline{\varphi}, & \sin\overline{\vartheta}\sin\overline{\varphi} & \cos\overline{\vartheta} \end{bmatrix} \begin{bmatrix} x_1' - \overline{x}_1 \\ x_2' - \overline{x}_2 \\ z' \end{bmatrix} \quad (17)$$

where $(\overline{\vartheta}, \overline{\varphi})$ are the spherical angles associated with the unit vector $\hat{\overline{\boldsymbol{\kappa}}} = (\overline{\boldsymbol{\xi}}, -\overline{\zeta}) = (\sin\overline{\vartheta}\cos\overline{\varphi}, \sin\overline{\vartheta}\sin\overline{\varphi}, \cos\overline{\vartheta})$. Thus, the z_b' axes coincide with the beam axes in the positive (outward) $\hat{\overline{\boldsymbol{\kappa}}}$ direction; the transverse coordinates $\mathbf{x}_b' = (x_{b_1}', x_{b_2}')$ are rotated such that x_{b_2}' is parallel to the z plane while x_{b_1}' lies in the plane $(\overline{\boldsymbol{\xi}}, \hat{\overline{\boldsymbol{\kappa}}})$.The parameters of this astigmatic beam field, may be obtained by the method described in [2].

Next, we consider the scattering kernel under Gaussian windows processing. By inserting $\Psi(\mathbf{X}, \mathbf{r}')$ in (15), with (3) into (14), we obtain the asymptotic expression of $\Lambda(\mathbf{r}'; \mathbf{X})$. The result implies the following: The local scattering cell exhibits Gaussian decay normal to both the incident and scattering axes directions. Therefore, the window center is located at the intersection of the incident beam and the local Green's function axes. The location of the scattering cell is determine by the phase-space processing variable $\mathbf{X}$, which determine, via (17), the beam axis of $\Psi(\mathbf{X}, \mathbf{r}')$. Furthermore, the exponential decay of Λ, is determine by the sum of the rotation transformation $\mathbf{T}_i$ and (17). Since the sum of two rotation transformations may be represented as a rotation transformation in the direction of the bisector between the two unit vectors associated with each transformation, one finds that the sampling window is oriented so that its normal bisects the angle between the direction of incidence $\hat{\boldsymbol{\kappa}}_i = (\vartheta_i, \varphi_i)$ and the spectral scattering direction $\hat{\overline{\boldsymbol{\kappa}}} = (\overline{\vartheta}, \overline{\varphi})$.

V. Concluding Remarks

In this paper, a previously developed pulsed plane-wave- and pulsed-beam-based time domain diffraction tomography for forward Born-type scattering, was extended to a more localized Gaussian beam pre- and post-processing version which confines interrogation of the scattering domain to scattering cells centered along the incident beam axis. Operating in the configuration-spectrum phase-space accessed by windowed transforms, the mathematical methodology for forward Gaussian beam scattering was developed, and the results were explained in terms of physically meaningful wave phenomena, thereby laying the foundation for an inversion procedure. In the inverse scattering scenario, the phase-space variable $\mathbf{X}$ should be chosen so as to ensure probing of the object function along the incident-wave axis. The location of the scattering cell over the indecent-wave axis, implies that several beam-incidence angles are required for probing of the entire object domain.

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