Near Optimal Multicriteria Spanner Constructions in Wireless Ad-Hoc Networks

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***Abstract*—In this paper we study asymmetric power assignment which induce a low energy k-strongly connected communication graph with spanner properties. We address two spanner models: energy and distance. The former serves as an indicator for the energy consumed in a message propagation between two nodes, and the latter reflects the latency overhead in the induced communication graph. We consider a random wireless ad-hoc network with $|V| = n$ nodes distributed uniformly and independently in a unit square.**

For $k \in \{1, 2\}$ we propose several power assignments which obtain a good bicriteria approximation on the total cost and stretch factor under the two models. For $k > 2$ we analyze a power assignment developed in [1], and derive some interesting bounds on the stretch factor for both models as well. We also describe how to compute all the power assignments distributively, and provide some simulation results. To the best of our knowledge, these are the first provable theoretical bounds for low cost spanners in wireless ad-hoc networks.

***Index Terms*—Communication systems, Network fault tolerance, Point processes, Stochastic approximation.**

I. Introduction

A wireless ad-hoc network consists of several transceivers (nodes) located in the plane, communicating by radio. Unlike wired networks, in which the link topology is fixed at the time the network is deployed, wireless ad-hoc networks have no fixed underlying topology. In addition, the relational disposition of wireless nodes is constantly changing. The temporary physical topology of the network is determined by the distribution of the wireless nodes, as well as the transmission range of each node. The ranges determine a directed communication graph, in which the nodes correspond to the transceivers and the edges correspond to the communication links.

The key difference between wireless ad-hoc networks and "conventional" communication structures, from the designer's point of view, is in the *power assignment model*. Each node decides on a transmission power level, and a transmission from node u can be received at node v if the transmission power of u is at least $d_{u,v}^c$, where $d_{u,v}$ is the Euclidean distance between u and v, and c the *distance-power gradient*, usually taken to be in the interval $[2, 4]$ (see [2]).

Producing a *strongly connected*[1] communication graph for wireless ad-hoc networks, through power assignments, was introduced by Chen and Huang [3] and has been studied since for the past 20 years. This not surprising since many applications in civilian, industrial and military areas require a strongly connected underlying topology to carry out different networking tasks [4]. Unlike nodes in wired networks, wireless devices are typically equipped with limited energy supplies making *energy efficiency* one the primary objectives in network design [5]. Energy efficiency is especially important for networks where battery replacement is infeasible.

One of the most studied topology control problems in the context of energy preservation in wireless networks is the MinSC problem: given a set of nodes in the plane, find a power assignment so that the induced communication graph is strongly connected and the *total energy consumption* (also referred to as *cost*) is minimized. The problem appears to be NP-hard [6] for the plane and polynomially solvable in the linear case (a special case when all the nodes are placed along a line segment). Thus, the majority of existing works produce approximation algorithms that induce a strongly connected graph with an upper guarantee on the total energy consumption ([3], [7], [8], [9], [10], [11], [12]).

In many scenarios wireless ad-hoc networks are deployed in hostile environments where node failures are very likely to happen. Developing *fault resistant*[2] topology control algorithms can play a crucial factor in keeping the network in an operable state. Naturally, the fault resistant version of the MinSC problem is also NP-hard, so the focus was developing power assignment algorithms which approximate the total power consumption ([1], [13], [14], [15], [16], [17], [18], [19], [20], [21]).

Low total energy consumption and fault resistance are fundamental for successful network deployment. However, there are additional factors which need to be taken into account. A key component in the overall network performance is the *efficiency of routing algorithms* [22]. There are many possible metrics to measure the efficiency of a routing algorithm, such as power, hop-count and residual energy [23]. Ultimately, each node has a link to any other node in the system, so that the routing possibilities are unlimited and any routing graph is feasible. Unfortunately, this assumption is far from being

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[1] A graph is strongly connected if for any pair of nodes, u, v, there is a path from u to v in the graph.

[2] A graph is k-strongly connected if it remains strongly connected even after $k - 1$ node failures.

realistic; it is impractical and usually impossible to allow each node to have a transmission range sufficient to reach all the other nodes. Instead, each node is assigned with enough power to reach only a relatively small subset of nodes. As a result, the topology of the induced communication graph has a strong effect on the routing algorithms efficiency.

In this paper we focus on two parameters which have an impact on the efficiency of several routing algorithms. More specifically, we study the stretch factor of the induced communication graph under two *spanner* models: energy and distance as described below.

- **Energy spanner model**: Let $\gamma_{u,v}$ be the minimum energy required to send a message from u to v by using relay nodes (if necessary). Then, the energy spanner is aimed to minimize the *energy stretch factor* t_E of the induced communication graph, that is for any pair of nodes, the energy required to propagate a message from u to v is at most $t_E \cdot \gamma_{u,v}$. The energy model reflects the power efficiency metric of routing protocols, which is essential for prolonging the network lifetime due to the constrained energy resource. A very good survey of power-aware routing protocols in wireless network can be found in [24].
- **Distance spanner model**: The distance spanner minimizes the *distance stretch factor* t_D of the induced communication graph, that is for any pair of nodes, the minimum distance path from u to v is at most $t_D \cdot d_{u,v}$. The distance stretch factor has a strong effect on the quality of *geographic routing protocols* [25]. These protocols use greedy forwarding decisions based on the geographic progress towards the destination, thus having a low distance stretch factor in the underlying topology graph is essential for efficient and successful geographic routing. For existing protocols using the geographic scheme see the survey in [23].

Remark: For both spanner models we choose to measure the stretch factor by comparing the efficiency of the best possible, in terms of either energy or distance, path in the induced communication graph to the best one in the *complete graph*, where every node can reach any other node in a single hop. As already stated, the use of complete communication graph, in terms of radio interference and energy consumption, is unpractical and inefficient. However, the complete graph is an optimal underlying topology for routing algorithms as it makes no restrictions on available routes; thus, by providing performance guarantees compared to the complete graph we make a stronger statement about performance guarantees in real network settings.

The majority of routing (and other) network protocols were traditionally developed for undirected graphs with symmetric (bidirectional) communication links. However, in wireless ad-hoc networks it is not uncommon to have asymmetric (unidirectional) links due to non-uniform background noise, non-uniform external interference and energy efficiency considerations. Some recent research addressed this phenomena by providing several approaches for various network tasks (e.g. [26], [27], [28], [29], [30]). We choose not to enforce symmetry over communication links, thus allowing unidirectional links to exist, which addresses a more general and realistic model of wireless ad-hoc networks.

As stated above, obtaining a minimum energy power assignment that induces a k-strongly connected, $k \geq 1$, graph is NP-hard. Adding an additional optimization objective, namely the stretch factor (in either of the models), makes the problem even harder. Almost all previous spanner constructions for ad-hoc wireless networks assume the unit disk graph (UDG) as the underlying topology ([31], [32], [33], [34], [35]). The research efforts were generally targeted at constructing a subgraph of UDG which holds the spanner property, and additional criteria, such as planarity and bounded degree. To the best of our knowledge, [36] is the only paper to address the spanner problem with the optimization objective of *minimizing the total energy consumption*. However, the authors only provided heuristics for the problem, without provable theoretical bounds.

One possible reason that all the current research efforts provide no guarantee on the total energy consumption is that the classical approaches for energy spanners, which involves using relative neighborhood graphs (RNG), Gabriel graphs (GG), and Yao graphs (YG), might result in an unbounded cost, as stated in [31]. What makes it even more complicated for the distance model is that the tradeoff between the total energy consumption and distance stretch factor can be as high as $n^{1/3}$ times the total squared weight of the minimum spanning tree as shown in Fig. 5.

We make the first step towards energy efficient spanner construction in wireless settings by considering a random wireless ad-hoc network whose nodes are uniformly and independently distributed in a unit square. For these networks we were able to develop low cost fault resistant spanners with provable theoretical bounds. Throughout the paper we indicate the probabilistic nature of our developments by stating that the specific result is *with high probability*, or in short *w.h.p.*, if the probability of the result converges to one as the number of network nodes, n, increases.

This paper is organized as follows. In the rest of this section, we present our system settings, discuss previous work and state our results. In Section II we discuss probability related results which we use in this paper. Followed by Section III, where we describe k-fault resistant power assignments used in later parts. Then, in Sections IV and V we show low cost spanner constructions under two spanner models, energy and distance, respectively. In Section VI we describe how the power assignments discussed in previous sections can be computed distributively. Finally, in Section VII we conclude and discuss possible future developments.

A. System settings

Let $G_V = (V, E_V)$ be a complete directed graph of the wireless nodes V positioned in the plane. We define a weight function on the edge set E_V as $w(u,v) = d^c_{u,v}$, for any $u, v \in V$. We also use the notation $w(e)$ to indicate the weight of an edge $e = (u,v)$. Note that the weight of an edge matches the amount of energy required for transmission

between its endpoints. Let $MST_V = (V, E_{MST})$ be the minimum spanning tree of G_V (with a weight function w).

A power assignment is a function $p : V \to \mathbb{R}^+$, which assigns each node $v \in V$ a transmission range $r_v = \sqrt[c]{p(v)}$. The transmission possibilities resulting from a power assignment induce a directed communication graph $H_p = (V, E_p)$, where $E_p = \{(u, v) : r_u \geq d_{u,v}\}$ is a set of directed edges. The cost of the power assignment is defined as

$$c(p) = \sum_{v \in V} p(v) = \sum_{v \in V} \max_{(v,u) \in E_p} w(v, u).$$

We assume the use of *frame-based* MAC protocols which divide the time into frames, containing a fixed number of slots. The main difference from the classic TDMA is that instead of having one access point which controls transmission slot assignments, there is a localized distributed protocol mimicking the behavior of TDMA. The advantage of a frame-based (TDMA-like) approach compared to the traditional IEEE 802.11 (CSMA/CA) protocol for a Wireless LAN is that collisions do not occur, and that idle listening and overhearing can be drastically reduced. When scheduling communication links, that is, specifying the sender-receiver pair per slot, nodes only need to listen to those slots in which they are the intended receiver – eliminating all overhearing. When scheduling senders only, nodes must listen in to all occupied slots, but can still avoid most overhearing by shutting down the radio after the MAC (slot) header has been received. In both variants (link and sender-based scheduling) idle listening can be reduced to a simple check if the slot is used or not. Several MAC protocols have been developed that take classical TDMA solutions using an access point to ad-hoc settings without any infrastructure by employing a distributed slot-selection mechanism that self-organizes a multi-hop network into a conflict-free schedule (see [37], [38]).

Let $G = (V, E)$ be some spanning subgraph of G_V, and P be any path in G. We use the notations $h(P)$, $d(P)$, and $w(P)$ to denote the total number, length, and weight of the edges in P, respectively. For any two nodes $u, v \in V$, if there is a path from u to v in G, we define the hop, Euclidean and energy distances, respectively, from u to v as follows,

$$\chi_{u,v}(G) = \min\{h(P) : P \text{ is a path from } u \text{ to } v \text{ in } G\},$$

$$\delta_{u,v}(G) = \min\{d(P) : P \text{ is a path from } u \text{ to } v \text{ in } G\},$$

$$\gamma_{u,v}(G) = \min\{w(P) : P \text{ is a path from } u \text{ to } v \text{ in } G\},$$

Otherwise, we define $\chi_{u,v}(G) = \delta_{u,v}(G) = \gamma_{u,v}(G) = \infty$. The *hop-diameter* of G is the maximum of the hop-distance between any pair of nodes, and is denoted $h(G)$. A path P, so that $h(G) = h(P)$ is called the *hop-diameter path* of G. The total weight of G is given by $w(G) = \sum_{e \in E} w(e)$. Note that these definitions are applicable to undirected graphs as well.

In this paper we use two spanner models, energy and distance. Some spanner related definitions follow. The *stretch factor* of G is the maximum deviation in one of the measures comparing to G_V. The energy and distance stretch factors of G are denoted by $t_E(G)$ and $t_D(G)$, respectively, such that

$$t_E(G) = \max_{u,v \in V} \frac{\gamma_{u,v}(G)}{\gamma_{u,v}(G_V)},$$

$$t_D(G) = \max_{u,v \in V} \frac{\delta_{u,v}(G)}{\delta_{u,v}(G_V)}.$$

Graph G is an energy t-spanner of G_V if $t_E(G) \leq t$. Similarly, G is a distance t-spanner of G_V if $t_D(G) \leq t$.

Graph G is *strongly connected* if for any two nodes $u, v \in V$, there exists a directed path from u to v in G. The wireless nodes are unreliable by their nature and may fail. For this purpose fault resistance is used in wireless networks. A graph G is said to be k-strongly connected if the removal of any set of at most $k - 1$ nodes does not disconnect the graph.[3] The famous Menger theorem states that a graph is k-strongly connected iff for any pair of nodes $u, v \in V$ such that $(u, v) \notin E$, there are k node-disjoint paths from u to v.

An interesting combination of fault resistance and the spanner property (in either of the models defined above), is the k-fault resistant spanner. Graph G is an energy (resp. distance) k-fault resistant t-spanner of G_V, if it remains an energy (resp. distance) t-spanner of G after the removal of at most $k - 1$ nodes. Therefore, in order to show that G is an energy (resp. distance) k-fault resistant t-spanner of G_V, we need that one of the following conditions hold for any $u, v \in V$:

a) There exist k node-disjoint paths $P_1, P_2, \ldots, P_k$ so that $w(P_i) \leq t \cdot \gamma_{u,v}(G_V)$, $1 \leq i \leq k$.
b) The edge (u, v) is in G, and $w(u, v) \leq t \cdot \gamma_{u,v}(G_V)$.

We denote by p_k^{OPT} the minimum cost power assignment so that $H_{p_k^{OPT}}$ is k-strongly connected.

This paper addresses the minimum power k-fault resistant spanner problem (MPkS) under the energy and distance models. Formally, we consider the following two problems.

Problem 1 (MPkES):
Input: A Euclidean graph $G_V = (V, E)$, and parameter $k \geq 1$.
Output: A power assignment p so that H_p is an energy k-fault resistant t-spanner of G_V.
Objective: Minimize $c(p)$ and t.

Problem 2 (MPkDS):
Input: A Euclidean graph $G_V = (V, E)$, and parameter $k \geq 1$.
Output: A power assignment p so that H_p is a distance k-fault resistant t-spanner of G_V.
Objective: Minimize $c(p)$ and t.

In this work, for simplicity, we assume that the distance-power gradient, c is $c = 2$, although our results can be easily extended to any constant c.

B. Previous work

The idea of spanners was first introduced by Chew [39], who claimed that for a set of points in the plane, there is a triangulation of the Euclidean geometric graph so that the path within the triangulation between any two points is bounded by a constant times the Euclidean distance between the points.

This initiated the study of distance spanners, mainly within the computational geometry society. In addition to the distance spanner property, distance spanners are usually required to have additional properties, such as planarity [39], [40], [41], low number of edges [41], [42], [43], [44], bounded degree

[3]The deletion of a node removes all edges adjacent to it as well.

[45], [46], [47], and low total edge length [48], [49], [50]. The NP-hardness of various aspects of the distance spanner problem was addressed in [51], [52], [53], [54]. A very good survey can be found in [55].

Recently, there has been an increased interest in both, distance and power spanners in the context of wireless ad-hoc networks. One of the first works to address the spanner property in wireless settings were Li et al. [31]. They model the network as a unit disk graph (UDG) and analyze the power stretch factor of several common subgraphs of UDG, the relative neighborhood graph (RNG), the Gabriel graph (GG), and the Yao graph (YG): $n-1$ for RNG, 1 for GG, and $O(1)$ for YG. They also propose a local construction of a sparse spanner that has both constant degree and constant power stretch factor. These three basic structures (RNG, GG, YG) have been used in subsequent works. In [35] a distributed algorithm for a power spanner is proposed with a power stretch of 2 and a constant bounded degree. Alzoubli et al. [32] develop a planar distance spanner; it has a constant degree and stretch factors (distance and hop). Li et al. [34] presented a $\frac{\sqrt{2}^c}{1-(2\sqrt{2}\sin\pi/k)^c}$ power spanner, where $k \geq 9$ is a customizable parameter; in addition, their spanner is planar, has a bounded degree, and the total edge length is within a constant factor of the total edge length of MST_V. Kanj and Perkovic [33] improved the power stretch factor by showing a localized distributed power spanner with a stretch factor of $1+(2\sin\pi/k)^c$ and a bounded degree of $k+5$, $k \geq 10$. They also claimed that this stretch factor is near-optimal. Schindelhauer et al. [56] consider a different power model; the stretch factor is compared against $d^c_{u,v}$ instead of the cost of the minimum energy path from u to v. The authors investigate the relations between various spanner models and apply their results to the sparsified Yao graphs. Levcopoulos et al. [57] incorporate fault resistance into spanner construction. They transform an arbitrary spanner into a k-fault resistant spanner. Additional interesting results for fault resistant spanners can be found in [58], [59].

None of the results above addresses the *total energy consumption*. Furthermore, the total energy might be unbounded, as stated in [31], since it is easy to give examples that the RNG, GG, and YG could consume arbitrarily more total energy than the minimum total energy necessary to maintain the connectivity of the network. What makes it even more complicated for the distance model is that the tradeoff between the total energy consumption and distance stretch factor can be as high as $n^{1/3} \cdot w(MST_V)$ as shown in Section V-A.

Unfortunately, the classic algorithms developed for general graphs do not work for power spanners, as they were developed for undirected graphs, while the asymmetric model is directed. Existing works for spanners in directed graphs [60], [61], [62], [63], do not address the weight property, which effects the total energy consumption.

To the best of our knowledge, [36] is the only paper to address the spanner problem with the optimization objective of minimizing the total energy consumption. However, the authors only provided heuristics for the problem, without provable theoretical bounds.

C. Our contribution

We study the minimum cost power assignment for random wireless ad-hoc networks so that the induced communication graph is a k-fault resistant spanner of the complete graph $G_V = (V, E_V)$. We assume that the nodes V are uniformly and independently distributed in a unit square. Hence, all of our results are with high probability. In particular, our main contributions are:

- For $k = 1$, we construct two power assignments, $\tilde{p}_1^{e,m}$ in $O(mn^2)$ time and p_1^d in $O(n^4 \log n)$ time for the energy and distance models, respectively, so that:
  1) $H_{\tilde{p}_1^{e,m}}$ is an energy $O\left(\alpha \cdot \left(\frac{n-m}{m} \cdot \frac{\log n}{n^\varepsilon} + 1\right)\right)$-spanner of G_V and $c(\tilde{p}_1^{e,m}) = O(\beta \cdot m + n^\varepsilon) \cdot c(p_1^{OPT})$, for any $\alpha > 1$, $\beta \geq 1 + \frac{2}{\alpha - 1}$, $0 \leq \varepsilon \leq 1$, and any positive integer $m \leq n$.
  2) $H_{p_1^d}$ is a distance $\sqrt{2}$-spanner of G_V and $c(p_1^d) = O(\log n) \cdot c(p_1^{OPT})$.
- For $k = 2$, we develop a power assignment p_2 for the distance model that can be computed in $O(1)$ time, which is then converted into $\tilde{p}_2$ (in $O(1)$ time) for the energy model, so that:
  1) $H_{\tilde{p}_2}$ is an energy 2-fault resistant $O\left(\sqrt{\frac{n^{1-2\varepsilon}}{\log n}}\right)$-spanner of G_V and $c(\tilde{p}_2) = O(n^\varepsilon \log n) \cdot c(p_2^{OPT})$, for any $0 \leq \varepsilon < 1/2$.
  2) H_{p_2} is a distance 2-fault resistant $O(1)$-spanner of G_V, and $c(p)O(\log n) \cdot c(p_2^{OPT})$.
- For any other k, $k > 2$, we analyze the energy and distance stretch factors of the power assignment $\tilde{p}_k$, which can be computed in $O(n \log n)$ time, based on a power assignment presented in [1], so that $c(\tilde{p}_k) = O(k) \cdot c(p_k^{OPT})$. We obtained the following:
  1) $H_{\tilde{p}_k}$ is an energy k-fault resistant $O\left(h(MST_V) \cdot k^2 \log n\right)$-spanner of G_V.
  2) $H_{\tilde{p}_k}$ is a distance k-fault resistant $O\left(h(MST_V) \cdot \sqrt{k \log n}\right)$-spanner of G_V.

To the best of our knowledge, these are the first provable theoretic results for low cost distance spanners in wireless ad-hoc networks. Although our results for the 1-fault resistant energy spanner is not a strict improvement over the Chandra et al. [48] result for arbitrary weighted spanners, it is nevertheless the attempt to make any progress in this new open question.

II. Preliminaries

Chen and Huang [3], and later Kirousis et al. [7] made the following statement, which already became a common folklore in the study of wireless networks.

Theorem 2.1 ([3]): $c(p_1^{OPT}) \geq w(MST_V)$.

Recall that the weight function is defined as $w(u,v) = d^2_{u,v}$ for any $u, v \in V$. The triangle inequality in the Euclidean plane states that for any $x, y, z \in V$ $d_{x,y} \leq d_{x,z} + d_{y,z}$. We can easily derive the *weak triangle inequality* for the weight function w. For any $x, y, z \in V$,

$$w(x,y) \leq 2 \cdot (w(x,z) + w(z,y)).$$

By using the well-known Cauchy-Schwarz inequality we obtain that for any $x_1, x_2, \ldots, x_m \in V$,

$$w(x_1, x_m) \leq (m-1) \cdot \sum_{i=1}^{m-1} w(x_i, x_{i+1}). \quad (1)$$

In this paper we consider a wireless ad-hoc network with nodes distributed uniformly and independently in a unit square. We make use of several relevant theoretical results, which apply to the random distribution. The probability of all the statements below converges to one as the number of network nodes, n, increases.

Zhang and Hou in [64] derived a lower bound on the cost of a power assignment required to induce a k-strongly connected communication graph under the assumption that the nodes form a homogeneous Poisson point process with density λ. They also mentioned that according to [65] it is well accepted that n nodes whose locations are independent random variables, each with a uniform distribution over the unit square, are essentially a Poisson process with $\lambda = n$, for large values of n. In the next theorem we bring the main result of [64] adapted to our model.

Theorem 2.2 ([64]): For any $k \geq 1$, $c(p_k^{OPT}) = \Omega(1)$.

For some node v, let $d_v(k)$ be the distance from v to its k-th nearest neighbor. Berend et al. [66] bounded the distance to the k-th nearest neighbor from any node.

Theorem 2.3 ([66]): For every node $v \in V$ and any positive integer k, $1 \leq k \leq \frac{n}{(1+\varphi)\log n}$, where φ is any positive constant, it holds

$$d_v(k) \leq 2\sqrt{\frac{(k+1)\log n}{\pi(n-1)}}.$$

Let e^* be the maximum length edge in MST_V. Penrose [67] stated the following theorem, which compares the length of e^* and the maximum distance to the nearest neighbor.

Theorem 2.4 ([67]): $w(e^*) = \max_{v \in V} d_v(1)$.

Using the upper bound of Theorem 2.3 we derive the next corollary.

Corollary 2.5: $w(e^*) = O\left(\sqrt{\frac{\log n}{n}}\right)$.

III. k-STRONG CONNECTIVITY

In this section we address the issue of fault resistance in the context of strong connectivity. We first describe an elegant low cost power assignment p_2, which w.h.p. induces a 2-strongly connected graph. Then we present a power assignment p_k, developed in [1] for k-strong connectivity, for any $k > 1$. In later sections we show that these power assignments also induce good spanners under both, energy and distance, models. We propose a different power assignment, p_2, for the case of biconnectivity since it has better theoretical bounds on the stretch factor (in both models) than general assignment p_k with $k = 2$.

A. Strong biconnectivity

The power assignment p_2 is based on the following technical lemma.

Lemma 3.1: For n nodes uniformly distributed in a unit square U, let D^* be a maximum radius disk, which can be placed inside the unit square, so that there are no nodes in D^*. Let ε be the radius of D^*. Then,

$$\lim_{n\to\infty} \Pr\left[\varepsilon \geq \sqrt{\frac{2\log n}{n}}\right] = 0.$$

Proof: Let D be a disk with radius $\varepsilon = \sqrt{\frac{2\log n}{n}}$, and S be a square of size $\sqrt{2}\varepsilon$. Let $\Pr[D]$ and $\Pr[S]$ be the probabilities that given a uniform distribution of n nodes in the unit square, disk D and square S, respectively, can be put inside the unit square without covering any node. Clearly $\Pr[D] \leq \Pr[S]$, as S fits entirely inside of D (see Fig. 1). To bound $\Pr[S]$ we divide the unit square into grid cells, each of size $\varepsilon/\sqrt{2} \times \varepsilon/\sqrt{2}$. Let $\Pr[G]$ be the probability that one of the grid cells is empty. If S can be placed inside the unit square without covering any node, then there exists an empty grid cell (see Fig. 1), therefore $\Pr[S] \leq Pr[G]$. To prove the lemma it is now sufficient to show that $\lim_{n\to\infty} \Pr[G] = 0$ as follows,

$$\Pr[G] \leq \frac{2}{\varepsilon^2} \cdot \Pr[\text{a specific cell is empty}] = \frac{2}{\varepsilon^2}\left(1 - \frac{\varepsilon^2}{2}\right)^n$$
$$= \frac{n}{\log n}\left(1 - \frac{\log n}{n}\right)^n \approx \frac{n}{\log n \cdot e^{\log n}} \leq \frac{1}{\log n}.$$

Therefore, $\lim_{n\to\infty} \Pr[D] = 0$ and the lemma holds. ■

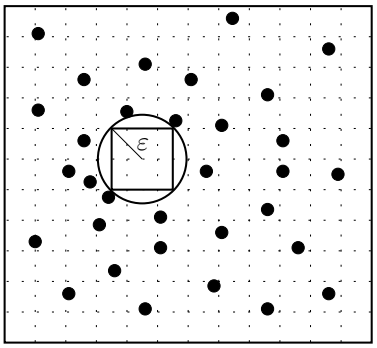


Fig. 1. Square S lies within disk D with radius ε, and cell size $\frac{\varepsilon}{\sqrt{2}} \times \frac{\varepsilon}{\sqrt{2}}$

Following the proof of the lemma above we can see that if we divide the unit square into grid cells, each of size $\sqrt{\frac{\log n}{n}} \times \sqrt{\frac{\log n}{n}}$, then w.h.p. each cell contains at least one node.[4] Let $g(1,1)$ and $g(\sqrt{n/\log n}, \sqrt{n/\log n})$ be the leftmost top and rightmost bottom grid cells, respectively. The rest of the cells are indexed as depicted in Fig. 2. Let $N(i,j)$ be the set of nodes in a grid cell $g(i,j)$. To make the notation simpler we define $N(0,j) = N(i,0) = \emptyset$, for $1 \leq i,j \leq \sqrt{n/\log n}$.

To induce a 2-strongly connected graph, we would like each node in a grid cell $g(i,j)$ to (a) reach all nodes in $N(i,j)$, and (b) reach all the nodes in vertically or horizontally adjacent cells, $A(i,j)$, where

$$A(i,j) = N(i-1,j) \cup N(i+1,j) \cup N(i,j-1) \cup N(i,j+1).$$

[4]To simplify the discussion we avoid the use of floors and ceilings.

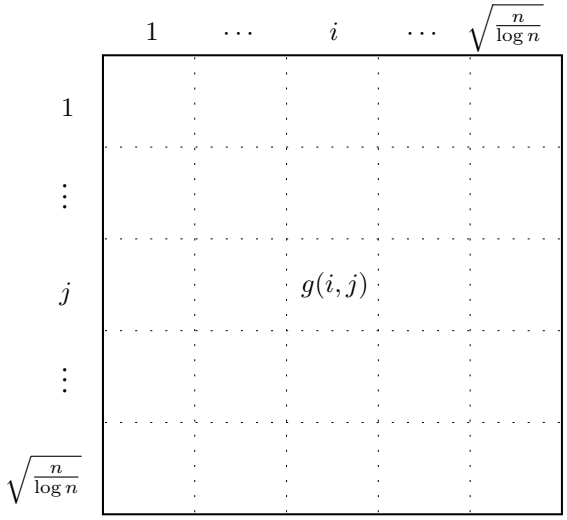


Fig. 2. The grid cells in a unit square

We can easily obtain the following observation.

Observation 3.2: For any $1 \leq i,j \leq \sqrt{\frac{n}{\log n}}$ and $u \in N(i,j)$, $v \in N(i,j) \cup A(i,j)$, $d_{u,v} \leq \sqrt{\frac{8\log n}{n}}$.

The power assignment p_2 is then defined as follows. For every $v \in V$,

$$p_2(v) = \frac{8\log n}{n}.$$

Note that following the observation, both conditions (a) and (b) hold. The next theorem shows the properties of p_2.

Theorem 3.3: With high probability, the graph H_{p_2} is 2-strongly connected and $c(p_2) = O(\log n) \cdot c(p_2^{OPT})$.

Proof: The cost bound is immediate due to Theorem 2.2. To prove that the graph H_{p_2} is 2-strongly connected we need to show that for any pair of nodes $u, v \in V$ there are either 2 node-disjoint paths from u to v, or $p_2(u) \geq w(u,v)$. Suppose $u \in N(i,j)$ and $v \in N(k,l)$, $1 \leq i,j,k,l \leq \sqrt{n/\log n}$. If $i = k$ and $j = l$, then $p_2(u) \geq w(u,v)$ by definition. Otherwise, there are two cases to consider.

Case 1: The two grid cells $g(i,j)$ and $g(k,l)$ are either in the same column or row. Without the loss of generality let $i < k$ and $1 < j = l$. Then the first path can be described as

$$P_1(u,v) = \langle u, z_{i+1,j}, z_{i+2,j}, \ldots, z_{k-1,l}, v\rangle,$$

where $z_{m,j}$ is an arbitrary node in $N(m,j)$, $i+1 \leq m \leq k-1$. That is, we hop to the right along the grid row from $g(i,j)$ to $g(k,l)$ – one hop per cell. The second path would be to do the same, only one row above,

$$P_2(u,v) = \langle u, z_{i,j-1}, z_{i+1,j-1}, z_{i+2,j-1}, \ldots, z_{k,j-1}, v\rangle,$$

where $z_{m,j-1}$ is an arbitrary node in $N(m,j-1)$, $i \leq m \leq k$ (see Fig. 3(a)).

Case 2: The nodes u, v are not on the same row or column. Then the paths are constructed in a similar fashion (see Fig. 3(b)). ∎

Based on the previous theorem, we can easily derive the following two corollaries.

Corollary 3.4: With high probability, for any pair of nodes $u, v \in V$, which are not in the same grid cell, there exist two node-disjoint paths, P_1, P_2, from u to v in H_{p_2}, so that $\max\{h(P_1), h(P_2)\} \leq 2\sqrt{\frac{n}{\log n}}$.

Corollary 3.5: With high probability, for any two nodes $u, v \in V$ such that $u \in N(i,j)$, $v \in N(l,m)$, and $v \notin N(i,j) \cup A(i,j)$, it holds $|l-i| + |m-j| \leq 2d_{s,t} \cdot \sqrt{\frac{n}{\log n}}$.

B. General k-strong connectivity

In [1] Carmi et al. developed a power assignment p_k, so that H_{p_k} is k-strongly connected, for any positive integer k, and $c(p_k) = O(k) \cdot c(p_k^{OPT})$. The power assignment p_k is defined as follows.

For every $v \in V$, initialize $p_k(v) = d_v(k)^2$, where $d_v(k)$ is the distance from v to its k-th nearest neighbor. Let $N_k(v) \subseteq V$ be the set of k nearest neighbors of v in G_v. For every edge $e = (u,v)$ in MST_V, increase the range of the nodes in $N_k(u) \cup N_k(v)$ (if necessary), so that each node $w \in N_k(u)$ reaches all nodes in $N_k(v) \cup \{v\}$, and vice versa. The cost of the power assignment has an approximation ratio of $O(k)$ times the optimum and can be computed in $O(n \log n)$ time.

It is rather simple to show that H_{p_k} is k-strongly connected. There k node-disjoint paths along the edges of MST_V. Every $N_k(v)$ can be viewed as a large intersection, which contains k intersection points, and for every edge (u,v) in MST_V, all nodes in $N_k(v) \cup \{v\}$ are reachable within one hop from the nodes in $N_k(u)$. The following observation is straightforward.

Observation 3.6: For any pair of nodes, $u, v \in V$, either the edge (u,v) is in H_{p_k}, or there exist k node-disjoint paths $P_1, P_2, \ldots, P_k$, from u to v in H_{p_k} so that $h(P_i) < h(MST_V) + 2$, $1 \leq i \leq k$.

For technical reasons, we want to ensure that $p_k(v) \geq 1/n$. We define for every $v \in V$,

$$\tilde{p}_k(v) = \max\left\{p_k(v), \frac{1}{n}\right\}.$$

Then w.h.p., $c(\tilde{p}_k) \leq c(p_k) + 1$, and due to Theorem 2.2, $c(\tilde{p}_k) = O(k) \cdot c(p_k^{OPT})$. Finally we can present a theorem which summarizes the properties of $\tilde{p}_k$. It is a combination of the results obtained in [1] and [66].

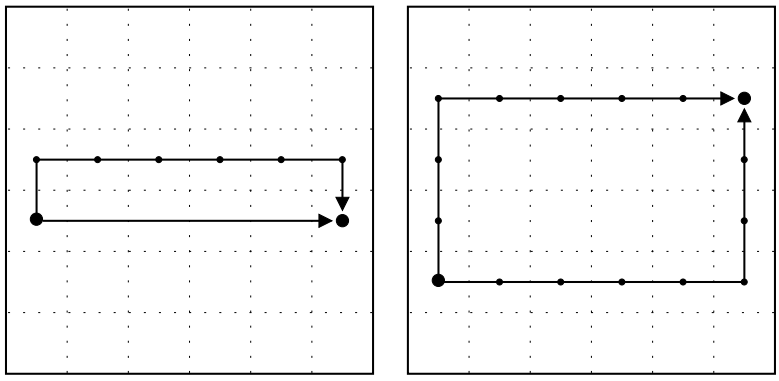
Theorem 3.7 ([1],[66]): With high probability, $c(\tilde{p}_k) = O(k) \cdot c(p_k^{OPT})$ and for every $v \in V$, $\tilde{p}_k(v) = O\left(\frac{k\log n}{n}\right)$.

IV. Energy spanner

In this section we address the MPkES problem. We start with the case $k = 1$ and construct several power assignments based on multiple Light Approximate Shortest-path Trees (LASTs) [68]. Then, for $k = 2$, we analyze the energy stretch factor of the power assignment described in Section III-A. Finally, we derive some interesting results for the general case.

A. 1-strong connectivity

The first power assignment is constructed on top of n LASTs - we call it the basic construction. Then, we generalize the idea and produce, for any given integer m $(1 \leq m \leq n)$, a power assignment on top of m LASTs. Finally, we show how the energy stretch factor can be improved at the expense of an additive factor to the power assignment cost.



(a) Case 1: w.l.g., $i < k$ and $1 < j = l$ (b) Case 2: w.l.g., $i < k$ and $j > l$

Fig. 3. Two node-disjoint paths from u to v in H_{p_2}, where $u \in N(i,j)$ and $v \in N(k,l)$, so that u and v are not in the same grid cell

A Light Approximate Shortest-path Tree (LAST) is a combination of a minimum spanning tree and a shortest path tree. Given an instance $\langle G, w, s\rangle$, where G is an undirected graph with a weight function on edges w, and s is a source node, Khuller et al. [68] presented a linear time algorithm, which computes a spanning tree T of G, so that its weight is at most β times the weight of a minimum spanning tree of G, and for every node v, $\gamma_{s,v}(T) \leq \alpha \cdot \gamma_{s,v}(G)$, where $\alpha > 1$ and $\beta \geq 1 + \frac{2}{\alpha - 1}$. A spanning tree which complies with the α, β bounds is called an (α, β)-LAST. The LAST can be constructed in $O(n^2)$ time for complete graphs.

Since the LASTs are computed for undirected graphs, we define G'_V to be a simple complete undirected graph, which is obtained from G_V by omitting the edge directions. Note that since the weight function w is symmetric, it can be applied to G'_V as well. Also, the minimum spanning tree of G'_V is exactly MST_V.

For every $u \in V$, let T_u be an arborescence[5] of an (α, β)-LAST computed for an instance $\langle G'_V, w, u\rangle$. Denote by e_v^u and e_v^* the maximum weight outgoing edge from v in T_u and MST_V, respectively.

1) Basic construction: We start with a construction based on n LASTs. This construction also serves as a good exposition of the general case, where the number of LASTs is arbitrary. We would like to define a power assignment $p_1^{e,n}$, so that each of the n arborescenses, $\{T_u\}_{u \in V}$, is a subgraph in $H_{p_1^{e,n}}$. Denote by e_v^u the maximum weight outgoing edge from v in T_u. Then for every $v \in V$

$$p_1^{e,n}(v) = \max_{u \in V} w(e_v^u).$$

The algorithm computes n LASTs. We can maintain the maximum range requirement for each node during the whole process, and update it only if some LAST requires a bigger range. Therefore the total construction time is $O(n^3)$.

Theorem 4.1: $H_{p_1^{e,n}}$ is an energy α-spanner of G_V and $c(p_1^{e,n}) \leq \beta \cdot n \cdot c(p_1^{OPT})$, for any $\alpha > 1$ and $\beta \geq 1 + \frac{2}{\alpha - 1}$.

Proof: Clearly, for every $u \in V$, T_u is a subgraph of $H_{p_1^{e,n}}$, by definition. Therefore, for every $u, v \in V$, $\gamma_{u,v}(H_{p_1^{e,n}}) \leq \alpha \cdot \gamma_{u,v}(G_V)$. Hence we conclude that $H_{p_1^{e,n}}$ is an energy α-spanner of G_V. Also,

$$\begin{aligned} c(p_1^{e,n}) &= \sum_{v \in V} \max_{u \in V} w(e_v^u) \leq \sum_{v \in V} \sum_{u \in V} w(e_v^u) \\ &\leq \sum_{u \in V} w(T_u) \leq \beta \cdot n \cdot w(MST_V). \end{aligned}$$

From Theorem 2.1, $c(p_1^{e,n}) \leq \beta \cdot n \cdot c(p_1^{OPT})$. ■

2) General construction: We consider a wireless network with uniform and independent random node distribution in a unit square. Different from the previous construction, this time we base our power assignment on m LASTs, where m is any positive integer between 1 and n. The difficulty is to find m nodes to be the roots of LASTs, so that each of the remaining nodes is relatively close to one of them. The rest is similar to the previous construction. We call this set of m root nodes U, and use the following graph-theoretical lemma to compute it.

Lemma 4.2: For any tree $T = (V, E)$, and a positive integer $m \leq |V| = n$, there exists a set of m nodes, $1 \leq m \leq n$, $U \subseteq V$, so that for every $v \in V \setminus U$, there exists $u \in U$, with a hop-distance of at most $\left\lceil \frac{n-m}{m} \right\rceil$ between v and u.

Proof: Let $x = \left\lceil \frac{n-m}{m} \right\rceil$. The construction of the set U is recurssive. We show the first step, in which we choose u_1 in T.

Let $P = \langle z_1, z_2, \ldots, z_{h(T)+1}\rangle$ be the hop-diameter path in T. Choose $u_1 = z_{\max\{x, h(T)\}+1}$. We consider two cases.

Case 1: If $x \geq h(T)$, then we are finished. Easy to see that all nodes in V reach z_x in less than x hops since z_x is the leaf on the hop-diameter path of T.

Case 2: If $x < h(T)$, then take the cut (V', V'') induced by the edge (z_{x+1}, z_{x+2}), so that $z_{x+1} \in V'$ and $z_{x+2} \in V''$. Let T' and T'' be the subtrees of T induced by the node sets V' and V'', respectively. Repeat the process to find $u_2, u_3, \ldots, u_m$ in T''.

We argue that all the nodes in V' are within x hops from z_{x+1} in T. Let z_{x+1} be the root of T'. The height of T' is at most x, otherwise $h(T) \geq h(T') + h(T) - x > h(T)$; a contradiction.

Clearly, the process ends after at most m steps, since in each step we decrease the tree size by at least $x + 1 = n/m$ nodes. Also for every node $v \in V$, there exists a node $u \in U$ so that the hop-distance from v to u is at most x. ■

[5]An *arborescence* is a directed, rooted tree in which all edges point away from the root.

Next, we describe the power assignment $p_1^{e,m}$. According to the lemma above, there exists a subset of nodes, $U = \{u_1, \ldots, u_m\}$, so that for every $v \in V \setminus U$, there exists $u \in U$, with a hop-distance of at most $\lceil \frac{n-m}{m} \rceil$ between v and u in MST_V. Note that the construction of U takes $O(n^2)$ time.

Similarly to the basic construction, we would like the communication graph $H_{p_1^{e,m}}$ to contain all the edges of all the arborescences. In addition, to guarantee there is a path from every node to one of the nodes in U, the induced communication graph will include the MST_V paths. Finally, for technical reasons we would like to ensure that each node is assigned a range of at least $\sqrt{1/n}$. To achieve all of the above, we define for every $v \in V$,

$$p_1^{e,m}(v) = \max\left\{ \max_{1 \le i \le m} w(e_v^{u_i}), w(e_v^*), \frac{1}{n} \right\}.$$

The algorithm works in two phases. First, it computes an MST_V and the set of root nodes U, and then the m LASTs. Again, throughout the algorithm we can maintain the maximum range required from each nodes, and as a result the total running time is $O(mn^2)$.

Theorem 4.3: With high probability, $H_{p_1^{e,m}}$ is an energy $O\left(\alpha \cdot \left(\frac{n-m}{m} \cdot \log n + 1\right)\right)$-spanner of G_V, such that $c(p_1^{e,m}) = O(\beta \cdot m) \cdot c(p_1^{OPT})$, for any $\alpha > 1$, $\beta \ge 1 + \frac{2}{\alpha - 1}$, and any positive integer $m \le n$.

Proof: The power assignment cost, $c(p_1^{e,m})$, is bounded in a similar way as in Theorem 4.1.

$$\begin{aligned} c(p_1^{e,m}) &= \sum_{v \in V} \max\left\{ \max_{1 \le i \le m} w(e_v^{u_i}), w(e_v^*), \frac{1}{n} \right\} \\ &\le \sum_{v \in V} \sum_{i=1}^{m} w(e_v^{u_i}) + \sum_{v \in V} e_v^* + 1 \\ &\le (\beta \cdot m + 2) \cdot w(MST_V) + 1. \end{aligned}$$

From Theorems 2.1 and 2.2 if follows,

$$c(p_1^{e,m}) = O(\beta \cdot m) \cdot c(p_1^{OPT}).$$

We next focus on analyzing the energy stretch factor of $H_{p_1^{e,m}}$, and prove that for any pair of nodes $s, t \in V$, $\gamma_{s,t}(H_{p_1^{e,m}}) = O\left(\alpha \cdot \left(\frac{n-m}{m} \cdot \log n + 1\right)\right) \cdot \gamma_{s,t}(G_V)$.

Let P^* be a minimum weight path from s to t in G_V, so that $\gamma_{s,t}(G_V) = w(P^*)$. We consider two cases.

Case 1: For every edge e in P^*, $w(e) < 1/n$. Then, from the definition of $p_1^{e,m}$, P^* is also a path in $H_{p_1^{e,m}}$, and therefore $\gamma_{s,t}(H_{p_1^{e,m}}) = \gamma_{s,t}(G_V)$.

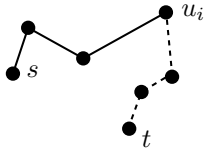


Fig. 4. The path in $H_{p_1^{e,m}}$ for $\gamma_{s,t}G_V(s,t) = \Omega(1/n)$, where solid lines are MST_V edges and dashed lines are T_i edges

Case 2: There is at least one edge e in P^* so that $w(e) \ge 1/n$. As a result, $\gamma_{s,t}(G_V) = \Omega(1/n)$. There exists a path P from s to t in $H_{p_1^{e,m}}$, which first arrives at some LAST origin node $u_i \in U$ (closest to s in terms of hop-distance in MST_V), using the edges of MST_V, and then to t using the edges of the LAST rooted at u_i (see Fig. 4). Note that P is unique as MST_V and T_i are trees. Therefore,

$$w(P) = \gamma_{s,u_i}(MST_V) + \gamma_{u_i,t}(T_i). \tag{2}$$

From the definition of T_i, $\gamma_{u_i,t}(T_i) \le \alpha \cdot \gamma_{u_i,t}(G_V)$. Let P' be a path in G_V from u_i to t which first travels along the edges of MST_V to s and then coincides with P^*. Clearly, $\gamma_{u_i,t}(G_V) \le w(P') \le \gamma_{u_i,s}(MST_V) + \gamma_{s,t}(G_V)$ and $\gamma_{s,u_i}(MST_V) = \gamma_{u_i,s}(MST_V)$. Combining with (2) we derive,

$$w(P) \le (1+\alpha) \cdot \gamma_{s,u_i}(MST_V) + \alpha \gamma_{s,t}(G_V). \tag{3}$$

Next we analyze $\gamma_{s,u_i}(MST_V)$. Following Lemma 4.2, $\chi_{s,u_i}(MST_V) \le \lceil \frac{n-m}{m} \rceil$. By using Theorem 2.5, w.h.p.

$$\gamma_{s,u_i}(MST_V) = O\left(\frac{n-m}{m} \cdot \frac{\log n}{n} \right). \tag{4}$$

Finally, from (3), (4), and the assumption $w(P^*) = \Omega(1/n)$ we conclude,

$$\gamma_{s,t}(H_{p_1^{e,m}}) = O\left(\alpha \cdot \frac{n-m}{m} \cdot \log n + \alpha \right) \gamma_{s,t}(G_V).$$

Therefore, $t_E(H_{p_1^{e,m}}) = O\left(\alpha \cdot \left(\frac{n-m}{m} \cdot \log n + 1\right)\right)$. ∎

3) Improving the energy stretch factor: The analysis of the energy stretch factor in Theorem 4.3 was divided into two cases; in the first case, the minimum weight path P^* in G_V consisted of low weight edges only (below $1/n$), while the second case was based on a lower bound $f(n) = \Omega(1/n)$ on the weight of P^*. In this section we improve the energy stretch factor of $p_1^{e,m}$ by raising the lower bound $f(n)$ at the expense of an additive factor to the power assignment cost.

Based on $p_1^{e,m}$ we define the power assignment $\tilde{p}_1^{e,m}$ as follows. For every $v \in V$,

$$\tilde{p}_1^{e,m}(v) = \max\{p_m(v), n^{\varepsilon - 1}\},$$

where $0 \le \varepsilon \le 1$. The next theorem analyzes the cost and the energy stretch factor of $\tilde{p}_1^{e,m}$.

Theorem 4.4: With high probability, $H_{\tilde{p}_1^{e,m}}$ is an energy $O\left(\alpha \cdot \left(\frac{n-m}{m} \cdot \frac{\log n}{n^\varepsilon} + 1\right)\right)$-spanner of G_V and $c(\tilde{p}_1^{e,m}) = O(\beta \cdot m + n^\varepsilon) \cdot c(p_1^{OPT})$, for any $\alpha > 1$, $\beta \ge 1 + \frac{2}{\alpha - 1}$, and any positive integer $m \le n$.

Proof: From theorems 2.2 and 4.3 it follows,

$$c(\tilde{p}_1^{e,m}) \le c(p_1^{e,m}) + n \cdot n^{\varepsilon - 1} = O(\beta \cdot m + n^\varepsilon) \cdot c(p_1^{OPT}).$$

The energy stretch factor of $H_{\tilde{p}_1^{e,m}}$ is analyzed similarly to the proof of Theorem 4.3. For any two nodes $s, t \in V$, let P^* be the minimum weight path from s to t in G_V. Again, we consider two cases.

Case 1: Every edge e in P^* has a weight $w(e) \le n^{\varepsilon - 1}$. Then, P^* is a path in $H_{\tilde{p}_1^{e,m}}$, and $\gamma_{s,t}(H_{\tilde{p}_1^{e,m}}) = \gamma_{s,t}(G_V)$.

Case 2: There exists an edge e in P^*, so that $w(e) > n^{\varepsilon - 1}$.

Then, $\gamma_{s,t}(G_V) = \Omega(n^{\varepsilon-1})$. Following the same reasoning as in the proof of Theorem 4.3 we conclude

$$\begin{aligned}\gamma_{s,t}(H_{\tilde{p}_1^{e,m}}) \leq & \gamma_{s,u_i}(MST_V) + \gamma_{u_i,t}(T_i) \\ \leq & (1+\alpha)\cdot\gamma_{s,u_i}(MST_V) + \alpha\cdot\gamma_{s,t}(G_V) \\ = & O\left((1+\alpha)\cdot\frac{n-m}{m}\cdot\frac{\log n}{n}\right) + \alpha\gamma_{s,t}(G_V) \\ = & O\left(\alpha\cdot\frac{n-m}{m}\cdot\frac{\log n}{n^{1-\varepsilon}} + \alpha\right)\gamma_{s,t}(G_V).\end{aligned}$$

Therefore, $t_E(H_{\tilde{p}_1^{e,m}}) = O\left(\alpha\cdot\left(\frac{n-m}{m}\cdot\frac{\log n}{n^{\varepsilon}}+1\right)\right)$. ■

B. 2-strong connectivity

In Section III-A we showed that the power assignment p_2 induces a 2-strongly connected graph with high probability and has a cost $O(\log n)\cdot c(p_2^{OPT})$. In this section we first analyze the energy stretch factor of H_{p_2}, and then show a modified power assignment $\tilde{p}_2$ with an improved energy stretch factor at the expense of a multiplicative factor to the cost.

Theorem 4.5: With high probability, H_{p_2} is an energy 2-fault resistant $O\left(\sqrt{\frac{n}{\log n}}\right)$-spanner of G_V.

Proof: For any two nodes $s,t\in V$, let Let

$$P^* = \langle s = z_0, z_1, \ldots, z_{l-1}, z_l = t\rangle,$$

be the minimum energy path from s to t in G_V, so that $\gamma_{s,t}(G_V) = w(P^*)$. Similar to the proof of Theorem 4.3 we consider two cases for any pair of nodes $s,t\in V$.

Case 1: For every i, $0\leq i\leq l-1$, $w(u_i,u_{i+1})\leq\frac{\log n}{n}$. Then, due to the weak triangle inequality, $w(z_i,z_{i+2})\leq 4\frac{\log n}{n}$. We can conclude that the edges $\{(z_i,z_{i+2})\}_{i=0}^{l-2}$ and $\{(z_i,z_{i+1})\}_{i=0}^{l-1}$ are in H_{p_2}. For $l\leq 2$, clearly the edge (s,t) is in H_{p_2} and $w(s,t) = O(1)\cdot w(P^*) = O(1)\cdot\gamma_{s,t}(G_V)$. So we assume without the loss of generality that l is an odd integer and $l>2$. Let,

$$P_1 = \langle z_0, z_2, z_4, \ldots, z_{l-1}, z_l\rangle,$$

be a path which uses even indexed intermediate nodes, and

$$P_2 = \langle z_0, z_1, z_3, \ldots, z_{l-2}, z_l\rangle,$$

to be a path which uses odd indexed intermediate nodes. Since the only edges in P_1 and P_2 are either $\{(z_i,z_{i+2})\}_{i=0}^{l-2}$ or $\{(z_i,z_{i+1})\}_{i=0}^{l-1}$, both paths are in H_{p_2}. Also, it is easy to see that the weight of each of the paths is at most $O(1)\cdot w(P^*) = O(1)\cdot\gamma_{s,t}(G_V)$.

Case 2: There exists an edge (z_i,z_{i+1}), $0\leq i\leq i-1$, so that $w(z_i,z_{i+1}) > \frac{\log n}{n}$ and hence $\gamma_{s,t}(G_V) = \Omega\left(\frac{\log n}{n}\right)$. If s and t are in the same grid cell, then (s,t) is in H_{p_2} and $w(s,t) = O(1)\cdot\gamma_{s,t}(G_V)$. Otherwise, let P_1 and P_2 be the two node-disjoint paths constructed in the proof of Theorem 3.3. Due to symmetry it is sufficient to analyze the weight of P_1. From Corrolary 3.4, $h(P_1)\leq 2\sqrt{\frac{n}{\log n}}$. Combined with the fact that $p_2(v) = 8\frac{\log n}{n}$ we obtain,

$$w(P_1) = O\left(\sqrt{\frac{n}{\log n}}\cdot\frac{\log n}{n}\right) = O\left(\sqrt{\frac{n}{\log n}}\right)\gamma_{s,t}(G_V).$$

Therefore, $t_E(H_{p_1}) = O\left(\sqrt{\frac{n}{\log n}}\right)$. ■

As in Theorem 4.4 we now show how it is possible to gain a decrease in an energy stretch factor by increasing the cost of the power assignment p_2. We define the power assignment $\tilde{p}_2$ as follows. For every $v\in V$,

$$\tilde{p}_2(v) = n^{\varepsilon}p_2(v),$$

where $0\leq\varepsilon<1/2$. The next theorem analyzes the cost and energy stretch factor of $\tilde{p}_2$.

Theorem 4.6: With high probability, $H_{\tilde{p}_2}$ is an energy 2-fault resistant $O\left(\sqrt{\frac{n^{1-2\varepsilon}}{\log n}}\right)$-spanner of G_V and $c(\tilde{p}_2) = O(n^{\varepsilon}\log n)\cdot c(p_2^{OPT})$.

Proof: Clearly, $c(\tilde{p}_2) = O(n^{\varepsilon}\log n)\cdot c(p_2^{OPT})$. We now concentrate on the energy stretch factor of H_{p_2}. For any pair of nodes $s,t\in V$, let P^* be the minimum energy path in G_V. We adopt the approach used in the proof of Theorem 4.5, where we considered two cases: either all the edges in P^* have a bounded weight of $\frac{\log b}{n}$, or $\gamma_{s,t}(G_V) = \Omega\left(\frac{\log n}{n}\right)$. This time if for every e in P^*, $w(e)\leq n^{\varepsilon-1}\log n$, then the rest is analyzed in exactly the same fashion as the first case in Theorem 4.5. Otherwise, $\gamma_{s,t}(G_V) = \Omega(n^{\varepsilon-1}\log n)$. Again, we follow the same reasoning as in the second case of Theorem 4.5 and construct two paths P_1, P_2. This time the evaluation of the energy stretch factor is slightly different.

$$w(P_1) = O\left(\sqrt{\frac{n}{\log n}}\cdot\frac{\log n}{n}\right) = O\left(\sqrt{\frac{n^{1-2\varepsilon}}{\log n}}\right)\gamma_{s,t}(G_V).$$

Therefore, $t_E(H_{p_1}) = O\left(\sqrt{\frac{n^{1-2\varepsilon}}{\log n}}\right)$. ■

C. k-strong connectivity

The power assignment $\tilde{p}_k$ was introduced in Section III-B. We now analyze its energy stretch factor using the same technique as in Theorem 4.5.

Theorem 4.7: With high probability, $H_{\tilde{p}_k}$ is an energy k-fault resistant $O\left(h(MST_V)\cdot k^2\log n\right)$-spanner of G_V.

Proof: Following the steps of the proof of Theorem 4.5, let

$$P^* = \langle s = z_0, z_1, \ldots, z_{l-1}, z_l = t\rangle,$$

be the minimum energy path from s to t in G_V, so that $\gamma_{s,t}(G_V) = w(P^*)$. The two cases we consider this time are as follows.

Case 1: For every edge e in P^*, $w(e)\leq\frac{1}{k\cdot n}$. If $l\leq k$, then due to (1),

$$w(s,t)\leq k\sum_{j=0}^{l-1}w(z_j,z_{j+1})\leq\frac{1}{n},$$

and therefore (s,t) is in $H_{\tilde{p}_k}$ and $w(s,t) = O(k)\cdot\gamma_{s,t}(G_V)$. Otherwise $l>k$ and we define

$$P_i = \langle z_0, z_i, z_{i+k}, z_{i+2\cdot k}, \ldots, z_{i+\lfloor\frac{l-i}{k}\rfloor\cdot k}, z_l\rangle,$$

for $1\leq i\leq k$. Again due to (1) for every $1\leq i\leq k$ and $0\leq j\leq l-k$,

$$w(z_j,z_{j+i})\leq k\cdot\sum_{q=j}^{j+i-1}w(z_q,z_{q+1})\leq\frac{1}{n},$$

and therefore the edge (z_j, z_{j+i}) is in $H_{\tilde{p}_k}$. Also, $w(P_i) \leq k \cdot w(P^*)$. We conclude that for every $1 \leq i \leq k$, P_i is a path in $H_{\tilde{p}_k}$, and $w(P_i) \leq k \cdot \gamma_{s,t}(G_V)$.

Case 2: There exists an edge e in P^* so that $w(e) > \frac{1}{k \cdot n}$, and therefore $\gamma_{s,t}(G_V) = \Omega\left(\frac{1}{k \cdot n}\right)$. From Theorem 3.7, $p(s) = O\left(\frac{k \log n}{n}\right)$. Due to Observation 3.6 there are two possibilities: (a) if the edge (s,t) is in $H_{\tilde{p}_k}$, then

$$w(s,t) \leq p(s) = O\left(\frac{k \log n}{n}\right) = O(k^2 \log n) \cdot \gamma_{s,t}(G_V);$$

(b) there exist k node-disjoint paths $P_1, \ldots, P_k$, so that $h(P_i) \leq h(MST_V) + 2$, $1 \leq i \leq k$. As a result,

$$\begin{aligned} w(P_i) &= O\left(h(MST_V) \cdot \frac{k \log n}{n}\right) \\ &= O\left(h(MST_V) \cdot k^2 \log n\right) \cdot \gamma_{s,t}(G_V). \end{aligned}$$

Therefore, $t_E(H_{\tilde{p}_k}) = O\left(h(MST_V) \cdot k^2 \log n\right)$. ■

V. Distance spanner

Here we consider the MPkDS problem for any value of k. We construct a wireless network, which complies with a more common spanner definition – the distance spanner. We start by showing a tradeoff between the total energy consumption and the distance stretch factor for arbitrarily placed nodes. Then, we first consider the cases $k = 1, 2$ for which we obtain better results than in the general case. Note that for any pair of nodes $u, v \in V$, $\delta_{u,v}(G_V) = d_{u,v}$.

A. Worst case tradeoff analysis

It turns out that the tradeoff between the total energy consumption and the distance stretch factor for arbitrarily placed nodes can be as high as $n^{1/3} \cdot w(MST_V)$. We construct an example that demonstrates this. In Fig. 5, the nodes are placed along the sides AD, AB, and BC of a rectangle $ABCD$, with $|AD| = |BC| = 1$, and $|AB| = |CD| = \frac{1}{n^{1/3}}$. The nodes are placed in a continuous manner at a fixed distance of $\frac{2+1/n^{1/3}}{n-1}$ from each other. Let u and v be two nodes positioned at D and C, respectively. The following theorem proves the lower bound of the tradeoff.

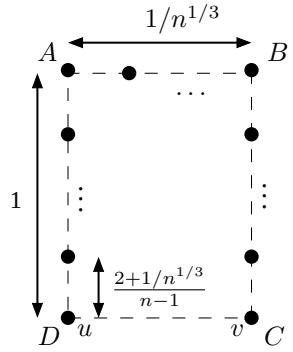


Fig. 5. Worst case tradeoff example for a distance spanner

Theorem 5.1: Given node placements as in Fig. 5, for any power assignment p, $t_D(H_p) \cdot c(p) = \Omega(n^{1/3} \cdot w(MST_V))$.

Proof: Let p be a power assignment so that H_p is strongly connected. We consider two cases:

Case 1: If for every $u \in V$, $p(u) < n^{2/3}$, there are no edges between the nodes on the side AD and the nodes on the side BC. As a result, $t_D(H_p) \geq \delta_{u,v}(H_p) = \Omega(n^{1/3})$. Combining with Theorem 2.1, $t_D(H_p) \cdot c(p) = \Omega(n^{1/3} \cdot w(MST_V))$.

Case 2: Otherwise, $c(p) \geq 1/n^{2/3}$. It is easy to verify that $w(MST_V) = \Theta(1/n)$. Therefore, $t_D(H_p) \cdot c(p) \geq c(p) = \Omega(n^{1/3} \cdot w(MST_V))$. ■

B. 1-strong connectivity

We construct a power assignment p_1^d so that $H_{p_1^d}$ is a distance 2-spanner of G_V and $c(p_1^d) = O(\log n) \cdot c(p_1^{OPT})$. We make use of Lemma 3.1.

1) Power assignment: For a given fixed integer $i \geq 0$, let $D(u,v,i) = \{D_1^i, D_2^i, \ldots, D_{2^i}^i\}$, be a set of adjacent, non-intersecting disks with diameter $d_{u,v}/2^i$, centered along the edge (u,v) in G_V. For example, in Fig. 6(a) the set $D(u,v,2)$ of 4 disks is shown.

Start with a zero power assignment function p_1^d. For every pair of nodes $u, v \in V$, increase the power assignment of certain nodes, as described in algorithm Construct-Path, to induce a directed path with low distance stretch factor from u to v in $H_{p_1^d}$.

Construct-Path(u,v)

1 $i \leftarrow 0$
2 **if** *there are no nodes in* $D_1^0 \in D(u,v,0)$ **then**
3 $\quad p_1^d(u) \leftarrow \max\{p_1^d(u), w(u,v)\}$
4 $\quad$ **return**
5 **while** *there exists a node in every* $D_j^{i+1} \in D(u,v,i+1)$ **do**
6 $\quad i \leftarrow i + 1$
7 Let $z_1, z_2, \ldots, z_{2^i}$ be arbitrary nodes, so that $w_j \in D_j^i$, for $D_j^i \in D(u,v,i)$
8 $z_0 \leftarrow u$; $z_{2^i+1} \leftarrow v$
9 **for** $j \leftarrow 0$ **to** 2^i **do**
10 $\quad p_1^d(z_j) \leftarrow \max\{p_1^d(z_j), w(z_j, z_{j+1})\}$
11 **return**

To analyze the running time of the algorithm, we make the following simple observation. The condition verification in the main loop of the algorithm Construct-Path (line 5) takes at most $O(n^2)$ time, and the number of iterations (line 6) is at most $\lfloor \log n \rfloor$. This is because the number of non-empty disks can be at most n, and the number of disks doubles with each iteration. The path is then constructed (line 10) in linear time.

Therefore, the running time of one execution of the Construct-Path algorithm is $O(n^2 \log n)$, and the total running time for all pairs is $O(n^4 \log n)$.

Clearly, the communication graph $H_{p_1^d}$ is strongly connected, as there is a directed path from any $u \in V$ to any $v \in V$. Next we analyze the cost of p_1^d and the distance stretch factor of $H_{p_1^d}$.

2) Analysis: We base our analysis on the following lemma.

Lemma 5.2: With high probability, for every $v \in V$, $p_1^d(v) \leq \frac{128 \log n}{n}$.

Proof: The power assignment p_1^d is first initialized to be a zero function. Then, it is modified in consecutive applications of the algorithm CONSTRUCT-PATH. We follow a single application of the algorithm on the pair u, v. The update of p_1^d can occur in two places – lines 3 and 10. We show that when one of these lines is executed, the values $w(u, v)$ (line 3) and $w(z_j, z_{j+1})$ (line 10) are at most $O(\log n/n)$ with high probability.

Line 3 is executed only in the case where there are no nodes in the disk D_1^0. Therefore, according to Lemma 3.1, w.h.p. $d_{u,v} \leq \sqrt{\frac{8 \log n}{n}}$.

Line 10 is executed once i is increased up to the point where there are no nodes in some disk D_j^{i+1}. Recall, that the diameter of any disk in $D(u, v, i+1)$ is $d_{u,v}/2^{i+1}$. Then by Lemma 3.1, w.h.p. $d_{u,v}/2^{i+1} \leq 2\sqrt{\frac{2 \log n}{n}}$. It is easy to verify that $d_{z_j, z_{j+1}} \leq 2d_{u,v}/2^i$, for $0 \leq j \leq 2^i$. Combining the above, $d_{z_j, z_{j+1}} \leq 8\sqrt{\frac{2 \log n}{n}}$ w.h.p., and therefore $p_1^d(v) \leq \frac{128 \log n}{n}$. ∎

To prove the following technical lemma we need some definitions. For any two points x, y, we denote by $||x, y||$ the Euclidean distance between x and y, and for any three points, we denote by $\angle xyz$ the angle between the two line segments xy and yz.

Lemma 5.3: For any two points in the plane A and B, let D be a disk which has the line segment AB as its diameter. Then, for any point C in D, $||A, C|| + ||C, B|| \leq \sqrt{2} \cdot ||A, B||$.

Proof: Clearly the expression $||A, C|| + ||C, B||$ is maximized when C is on the circumference of D and $\angle ACB = \pi/2$. Therefore,

$$||A, C|| + ||C, B|| \leq ||A, B|| \sin \angle ABC + ||A, B|| \cos \angle ABC \leq \sqrt{2}||A, B||.$$

This completes our proof. ∎

We are ready to prove our main theorem of this section.

Theorem 5.4: With high probability, $H_{p_1^d}$ is a distance $\sqrt{2}$-spanner of G_V and $c(p_1^d) = O(\log n) \cdot c(p_1^{OPT})$.

Proof: We start from the cost of the power assignment. From Lemma 5.2 it follows directly that w.h.p. $c(p_1^d) = O(\log n)$. From Theorem 2.2, w.h.p. $c(p_1^d) = O(\log n) \cdot c(p_1^{OPT})$.

To analyze the distance stretch factor between any two nodes u and v, we focus on a directed path,

$$P = \langle u = z_0, z_1, z_2, \ldots, z_{2^i+1} = v \rangle,$$

induced in lines 7 – 10 (e.g., in Fig. 6(b) there is a path for $i = 2$).

Let $u = z_{0,1}, z_{1,2}, \ldots, z_{2^i,2^i+1} = v$ be $2^i + 1$ points placed at equal distances along the edge (u, v). These are the tangent points between disks in addition to u and v. Note that $d_{z_{i,i+1}, z_{i+1,i+2}} = d_{u,v}/2^i$, for any i, $0 \leq i \leq 2^i - 1$. We define a path P', from u to v which uses the nodes V and the just added points $z_{i,i+1}$ as follows (see Fig. 6(c)),

$$P' = \langle z_{0,1}, z_1, z_{1,2}, z_2, \ldots, z_{2^i}, z_{2^i,2^i+1} \rangle.$$

It is easy to see that $d(P) \leq d(P')$ and therefore,

$$d(P) \leq d(P') = \sum_{i=0}^{2^i-1} (d_{z_{i,i+1}, z_{i+1}} + d_{z_{i+1}, z_{i+1,i+2}}).$$

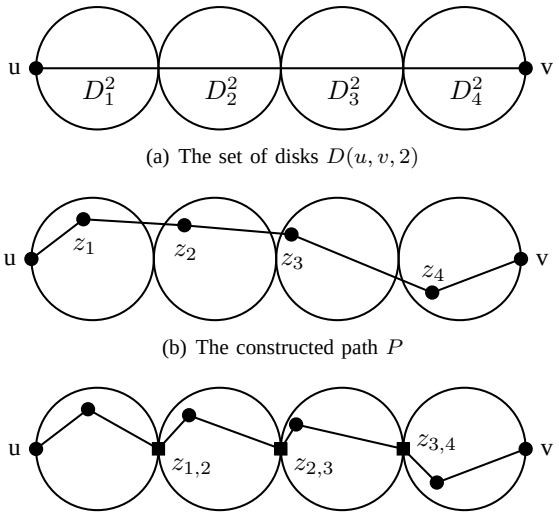
Due to Lemma 5.3 for any i, $0 \leq i \leq 2^i - 1$,

$$d_{z_{i,i+1}, z_{i+1}} + d_{z_{i+1}, z_{i+1,i+2}} \leq \sqrt{2} d_{u,v}/2^i.$$

We conclude,

$$\delta_{s,t}(H_{p_1^d}) = d(P') \leq 2^i \sqrt{2} d_{u,v}/2^i = \sqrt{2} d_{u,v}.$$

Therefore, $t_D(H_{p_1^d}) = \sqrt{2}$. ∎



(a) The set of disks $D(u, v, 2)$

(b) The constructed path P

(c) The path P' (the squares represent nodes $z_{1,2}, z_{2,3}, z_{3,4}$)

Fig. 6. The construction of a path from u to v

C. 2-strong connectivity

We will now show that the power assignment p_2, described in Section III-A is a distance 2-fault resistant $O(1)$-spanner of G_V. Recall that $N(i, j)$ denotes the set of nodes in a grid cell $g(i, j)$, and $A(i, j)$ denotes the nodes in adjacent cells of $g(i, j)$, $1 \leq i, j \leq \sqrt{\frac{n}{\log n}}$. The distance stretch factor of H_{p_2} is derived in the following Theorem.

Theorem 5.5: With high probability, H_{p_2} is a distance 2-fault resistant $O(1)$-spanner of G_V.

Proof: For any pair of nodes, $s, t \in V$, let $s \in N(i, j)$ and $t \in N(l, m)$, $1 \leq i, j, l, m \leq \sqrt{\frac{n}{\log n}}$. If $t \in N(i, j) \cup A(i, j)$, then from the definition of p_2, $p_2(s) \geq w(s, t)$, and $\delta_{s,t}(H_{p_2}) = 1$. Otherwise, $v \notin N(i, j) \cup A(i, j)$. Let P_1 and P_2 be the two paths constructed in Theorem 3.3. From Corollary 3.5, the hop-distance of each of the paths is at most $2d_{s,t} \cdot \sqrt{\frac{n}{\log n}}$. Combining it with the fact that $p_2(v) = 8\frac{\log n}{n}$, for every $v \in V$, we conclude

$$d(P_i) \leq 2d_{s,t} \cdot \sqrt{\frac{n}{\log n}} \cdot \sqrt{\frac{8 \log n}{n}} = O(1) \cdot d_{s,t},$$

where $i \in \{1, 2\}$. Therefore, $t_D(H_{p_2}) = O(1)$. ∎

D. k-strong connectivity

We now analyze the distance stretch factor of $\tilde{p}_k$ defined in Section III-B, and show that $H_{\tilde{p}_k}$ is a distance k-fault resistant $O\left(h(MST_V)\cdot\sqrt{k\log n}\right)$-spanner of G_V.

Theorem 5.6: With high probability, H_{p_k} is a distance k-fault resistant $O\left(h(MST_V)\cdot\sqrt{k\log n}\right)$-spanner of G_V.

Proof: For any two nodes $s,t\in V$, we consider two cases. If $d_{s,t}\leq\sqrt{1/n}$, then from the definition of $\tilde{p}_k$, the edge (s,t) is in $H_{\tilde{p}_k}$ and therefore $\delta_{s,t}(H_{\tilde{p}_k})=1$. Otherwise, $\delta_{s,t}(G_V)=d_{s,t}>\sqrt{1/n}$. From Observation 3.6 there exist k node-disjoint paths $P_1,P_2,\ldots,P_k$, from s to t in H_{p_k} so that $h(P_i)<h(MST_V)+2$, $1\leq i\leq k$. The power assignment $\tilde{p}_k$ is bounded in Theorem 3.7, so we can deduce that for any edge (u,v) in $H_{\tilde{p}_k}$, $d_{u,v}=O\left(\sqrt{\frac{k\log n}{n}}\right)$. As a result for every i, $1\leq i\leq k$,

$$\begin{aligned}d(P_i)&=O\left(h(MST_V)\cdot\sqrt{\frac{k\log n}{n}}\right)\\&=O\left(h(MST_V)\cdot\sqrt{k\log n}\right).\end{aligned}$$

Therefore, $t_D(H_{\tilde{p}_k})=O\left(h(MST_V)\cdot\sqrt{k\log n}\right)$. ∎

VI. Distributed implementation

Usually, the deployment of wireless nodes is not controlled by some centralized entity which coordinates network activities. As a result, the nodes are required to carry on their tasks independently, communicating only with a small set of close nodes. In this section we describe how the power assignments discussed in Sections V and IV can be computed distributively. Our only requirement is that each node has a unique ID and knows the size of the deployment area, the total number of nodes, n, and various parameters (if needed), such as ε and m.

The power assignments described in this paper can be roughly divided into two types: power assignments a bound on the maximum assigned power and those which do not have this bound. The first type can be computed easily as each node has the information about the total number of nodes and the size of the deployment area, so maximum power can be assigned without effecting the theoretical bounds. The second type of power assignments do not have a bound on the maximum power, and hence require an execution of the algorithm to compute each individual power assignment. In what follows we show that p_1^d, p_2, and $\tilde{p}_2$ fall into the first category. Then we describe how to compute distributively the power assignments $\tilde{p}_1^{e,m}$ and $\tilde{p}_k$.

A. Power assignments p_1^d, p_2, and $\tilde{p}_2$

From Lemma 5.2, $p_1^d\leq 8\sqrt{\frac{2\log n}{n}}$. Clearly, if we assign $p_1^d(u)=8\sqrt{\frac{2\log n}{n}}$, for every $u\in V$, the asymptotic bounds are not effected. By definition, $p_2=\frac{8\log n}{n}$, and $\tilde{p}_2$ can be easily derived from p_2.

The three power assignments, p_1^d, p_2, and $\tilde{p}_2$, can be computed *locally* without engaging in any kind of communication, based solely on the number of nodes n, the size of the deployment area, S, and the parameter ε. Note that the power assignment needs to be scaled by $|S|$, as it is initially computed for a unit square.

B. Power assignments $\tilde{p}_1^{e,m}$ and $\tilde{p}_k$

We now outline how to construct the power assignments $p_1^{e,m}$ and p_k distributively. This will allow us to compute $\tilde{p}_1^{e,m}$ and $\tilde{p}_k$.

1) Distributed computation of $p_1^{e,m}$: The algorithm in Section IV-A first constructs the MST_V, followed by the set of nodes U, and finally the computation of LASTs. We address each of these steps in detail.

The construction of MST_V: First, each node $u\in V$ transmits its ID using power level of $p'(u)=\frac{8\log n}{\pi(n-1)}$ (scaled according to the size of the deployment area). After some timeout, u receives all the transmissions of close nodes and obtains a list of all nodes $N(u)$ which are within a distance of $2\sqrt{\frac{2\log n}{\pi(n-1)}}$ from it. Using the standard methods described in [69] it is possible to compute all distances $d_{u,v}$, $v\in S(u)$ (we skip the discussion about the correct reception of all these transmissions). Due to the upper bound of Theorem 2.3, MST_V is a subgraph of H_p. Recall that a weight of an edge $(u,v)\in E_p$ is $d_{u,v}^2$, and therefore can be computed. As all the links in H_p are bidirectional, we can now use the Distributed Algorithm for Minimum-Weight Spanning Trees [70] to compute the minimum weight spanning tree of H_p, which is exactly MST_V.

The construction of the set U: Recall that U is constructed by recursively selecting the x-th node, $x=\lceil\frac{n-m}{m}\rceil$ on the diameter, and then removing the subtree rooted at that node. The diameter can be computed by executing the distributed BFS algorithm [71] twice; first from the node with the smallest ID (denote by u), and then from the node at the maximum hop-distance from v (denote by v). This involves two simple leader election procedures to find u and v. The x-th node (denote z) on the diameter is easily selected by following a path from v. Finally, all nodes in a subtree rooted at z (with v as a leaf) are notified not to participate in further BFS queries. The process continues until all nodes are notified.

The computation of LASTs: The m nodes which were identified as the sources for LASTs execute the distributed LAST algorithm as described in [72].

It is important to note that the second and third steps can be executed in parallel.

2) Distributed computation of p_k: As described in Section III-B, the power assignment p_k is based on MST_V, which is then used to connect the k-closest neighborhoods along MST_V edges. The distributed implementation consists of three steps.

Querying the k-closest nodes: Similar to the technique used in the first step of $p_1^{e,m}$ construction, the information about the k-closest neighborhood, $N_k(u)$, of node $u\in V$ can be gathered using a transmission range of $2\sqrt{\frac{k+1\log n}{\pi(n-1)}}$ which, according to Theorem 2.3, is an upper bound on $d_k(u)$.

The construction of MST_V: Similar to the first step in the construction of $p_1^{e,m}$.

Power assignment: Once MST_V is obtained, the endpoints of every MST_V edge $(u, v) \in E_{MST}$ exchange information about their k-neighborhoods, $N_k(u)$ and $N_k(v)$. Then each node propagates the received information to its k-neighborhood. Recall that the power assignment of each node w only depends on the distance to the k-closest node, $d_k(w)$, and the received information, $N_k(v) \cup \{v\}$, where $(u, v) \in E_{MST}$ and $w \in N_k(u)$. Thus, the power assignment can be computed after this step.

Note that the first two steps can be executed in parallel.

VII. Conclusions and future work

In this paper we studied asymmetric power assignments of low cost for which the induced communication graph is a good fault resistant spanner of G_V. We addressed two spanner models, energy and distance, under the requirement that the stretch factor (in both models) remains unchanged if the number of node failures is at most $k-1$, where k is any positive integer.

We assume that the nodes are uniformly and independently distributed in a unit square. The probability of all our results converges to one as the number of network nodes, n, increases.

For $k \in \{1, 2\}$ we propose several power assignments which obtain a good bicriteria approximation on the total cost and stretch factor under the two models. For $k > 2$ we analyze a power assignment developed in [1], and derive some interesting bounds on the stretch factor for both models as well. To the best of our knowledge, these are the first provable theoretic results for low cost spanners in wireless ad-hoc networks.

To the best of our knowledge, these are the first provable theoretic results for low cost distance spanners in wireless ad-hoc networks. Although our results for the 1-fault resistant energy spanner is not a strict improvement over the Chandra et al. [48] result for arbitrary weighted spanners, is is nevertheless the attempt to make any progress in this new open question.

One of the possible future directions would be to perform finer analysis of the power assignment $\tilde{p}_1^{e,m}$ in Section IV. It would be also interesting to perform simulations to measure the performance of the power assignment p_1 developed in Section V for large scaled networks.

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References

[1] P. Carmi, M. J. Katz, M. Segal, and H. Shpungin, "Fault-tolerant power assignment and backbone in wireless networks," in *FAWN'06*, 2006, pp. 80–84.

[2] K. Pahlavan and A. H. Levesque, *Wireless information networks*. Wiley-Interscience, 1995.

[3] W.-T. Chen and N.-F. Huang, "The strongly connecting problem on multihop packet radio networks," *IEEE Transactions on Communications*, vol. 37, no. 3, pp. 293–295, 1989.

[4] K. Römer and F. Mattern, "The design space of wireless sensor networks," *IEEE Wireless Communications*, vol. 11, no. 6, pp. 54–61, 2004.

[5] A. Chandrakasan, R. Amirtharajah, S. Cho, J. Goodman, G. Konduri, J. Kulik, W. Rabiner, and A. Wang, "Design considerations for distributed microsensor systems," in *CICC'99*, 1999, pp. 279–286.

[6] A. E. F. Clementi, P. Penna, and R. Silvestri, "Hardness results for the power range assignmet problem in packet radio networks," in *RANDOM-APPROX'99*, 1999, pp. 197–208.

[7] L. M. Kirousis, E. Kranakis, D. Krizanc, and A. Pelc, "Power consumption in packet radio networks," *Theoretical Computer Science*, vol. 243, no. 1–2, pp. 289–305, 2000.

[8] C. Ambuhl, A. Clementi, P. Penna, G. Rossi, and R. Silvestri, "Energy consumption in radio networks: Selfish agents and rewarding mechanisms," *Theoretical Computer Science*, vol. 343(1-2), pp. 27–41, 2004.

[9] C. Ambuhl, A. Clementi, M. D. Ianni, A. Monti, G. Rossi, and R. Silvestri, "The range assignment problem in non-homogeneous static ad-hoc networks," in *IPDPS'04*, 2004.

[10] G. Calinescu, S. Kapoor, A. Olshevsky, and A. Zelikovsky, "Network lifetime and power assignment in ad hoc wireless networks," in *ESA'03*, 2003, pp. 114–126.

[11] A. E. F. Clementi, G. Huiban, P. Penna, G. Rossi, and Y. Verhoeven, "Some recent theoretical advances and open questions on energy consumption in ad-hoc wireless networks," in *ARACNE'02*, 2002, pp. 23–38.

[12] R. Ramanathan and R. Hain, "Topology control of multihop wireless networks using transmit power adjustment," in *INFOCOM'00*, 2000, pp. 404–413.

[13] G. Calinescu and P.-J. Wan, "Range assignment for high connectivity in wireless ad hoc networks," in *AdHoc-NOW'03*, 2003, pp. 235–246.

[14] M. T. Hajiaghayi, N. Immorlica, and V. S. Mirrokni, "Power optimization in fault-tolerant topology control algorithms for wireless multi-hop networks," in *MobiCom'03*, 2003, pp. 300–312.

[15] X. Jia, D. Kim, S. Makki, P.-J. Wan, and C.-W. Yi, "Power assignment for k-connectivity in wireless ad hoc networks," in *INFOCOM'05*, 2005, pp. 2206–2211.

[16] E. Althaus, G. Calinescu, I. Mandoiu, S. Prasad, N. Tchervenski, and A. Zelikovsky, "Power efficient range assignment in ad-hoc wireless networks," in *WCNC'03*, 2003, pp. 1889–1894.

[17] D. M. Blough, M. Leoncini, G. Resta, and P. Santi, "On the symmetric range assignment problem in wireless ad hoc networks," in *IFIP TCS'02*, 2002, pp. 71–82.

[18] G. Calinescu, I. I. Mandoiu, and A. Zelikovsky, "Symmetric connectivity with minimum power consumption in radio networks," in *IFIP TCS'02*, 2002, pp. 119–130.

[19] A. E. F. Clementi, P. Penna, and R. Silvestri, "On the power assignment problem in radio networks," *Electronic Colloquium on Computational Complexity*, vol. 7, no. 54, 2000.

[20] M. T. Hajiaghayi, G. Kortsarz, V. S. Mirrokni, and Z. Nutov, "Power optimization for connectivity problems," in *IPCO'05*, 2005, pp. 349–361.

[21] E. L. Lloyd, R. Liu, M. V. Marathe, R. Ramanathan, and S. S. Ravi, "Algorithmic aspects of topology control problems for ad hoc networks," *MONET*, vol. 10, no. 1-2, pp. 19–34, 2005.

[22] J. P. Macker and M. S. Corson, "Mobile ad hoc networking and the ietf," *ACM Mobile Computing and Communications Review*, vol. 2, pp. 1–14, 1998.

[23] S. Giordano, I. Stojmenovic, and L. Blazevic, "Position based routing algorithms for ad hoc networks: A taxonomy," in *Ad Hoc Wireless Networking*, 2001, pp. 103–136.

[24] S. Lindsey, K. M. Sivalingam, and C. S. Raghavendra, "Power optimization in routing protocols for wireless and mobile networks," pp. 407–423, 2002.

[25] J. Gao, L. J. Guibas, J. Hershberger, L. Zhang, and A. Zhu, "Geometric spanner for routing in mobile networks," in *MobiHoc'01*, 2001, pp. 45–55.

[26] M. Gerla, Y.-Z. Lee, J.-S. Park, and Y. Yi, "On demand multicast routing with unidirectional links," in *WCNC'05*, 2005, pp. 2162–2167.

[27] C.-H. Lin, B.-H. Liu, H.-Y. Yang, C.-Y. Kao, and M.-J. Tsai, "Virtual-coordinate-based delivery-guaranteed routing protocol in wireless sensor networks with unidirectional links," in *INFOCOM'08*, 2008, pp. 351–355.

[28] L. Bao and J. J. Garcia-Luna-Aceves, "Channel access scheduling in ad hoc networks with unidirectional links," in *DIALM'01*, 2001, pp. 9–18.

[29] V. Ramasubramanian, R. Chandra, and D. Mossé, "Providing a bidirectional abstraction for unidirectional adhoc networks," in *INFOCOM*, 2002.

[30] Y. Zhang, D. Gu, and B. Preneel, "Reliable key establishment scheme exploiting unidirectional links in wireless sensor networks," in *EUC'08*, 2008, pp. 272–279.

[31] X.-Y. Li, P.-J. Wan, and Y. Wang, "Power efficient and sparse spanner for wireless ad hoc networks," in *ICCCN'01*, 2001, pp. 564–567.

[32] K. Alzoubi, X.-Y. Li, Y. Wang, P.-J. Wan, and O. Frieder, "Geometric spanners for wireless ad hoc networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 14, no. 4, pp. 408–421, 2003.

[33] I. A. Kanj, L. Perković, and G. Xia, "Local construction of near-optimal power spanners for wireless ad hoc networks," *IEEE Transactions on Mobile Computing*, vol. 8, no. 4, pp. 460–474, 2009.

[34] X.-Y. Li, W.-Z. Song, and W. Wang, "A unified energy-efficient topology for unicast and broadcast," in *MobiCom'05*, 2005, pp. 1–15.

[35] Y. Wang and X. yang Li, "Distributed spanner with bounded degree for wireless ad hoc networks," in *IPDPS'02*, 2002.

[36] Y. Wang and X.-Y. Li, "Minimum power assignment in wireless ad hoc networks with spanner property," *Journal of Combinatorial Optimization*, vol. 11, no. 1, pp. 99–112, 2006.

[37] V. Rajendran, K. Obraczka, and J. J. Garcia-Luna-Aceves, "Energy-efficient collision-free medium access control for wireless sensor networks," in *SenSys'03*, 2003, pp. 181–192.

[38] L. van Hoesel and P. Havinga, "A lightweight medium access protocol (lmac) for wireless sensor networks:reducing preamble transmissions and transceiver state switches," in *INSS'04*, 2004, pp. 205–208.

[39] L. P. Chew, "There is a planar graph almost as good as the complete graph," in *SCG'86*, 1986, pp. 169–177.

[40] ——, "There are planar graphs almost as good as the complete graph," *Journal Compute and System Sciences*, vol. 39, no. 2, pp. 205–219, 1989.

[41] J. M. Keil and C. A. Gutwin, "Classes of graphs which approximate the complete euclidean graph," *Discrete Computational Geometry*, vol. 7, no. 1, pp. 13–28, 1992.

[42] J. S. Salowe, "Constructing multidimensional spanner graphs," *International Journal of Computational Geometry and Applications*, vol. 1, no. 2, pp. 99–107, 1991.

[43] P. M. Vaidya, "A sparse graph almost as good as the complete graph on points in k dimensions," *Discrete Computational Geometry*, vol. 6, no. 4, pp. 369–381, 1991.

[44] P. B. Callahan and S. R. Kosaraju, "Faster algorithms for some geometric graph problems in higher dimensions," in *SODA'93*, 1993, pp. 291–300.

[45] P. Bose, J. Gudmundsson, and M. H. M. Smid, "Constructing plane spanners of bounded degree and low weight," *Algorithmica*, vol. 42, no. 3-4, pp. 249–264, 2005.

[46] G. Das and P. J. Heffernan, "Constructing degree-3 spanners with other sparseness properties," *International Journal of Foundations of Computer Science*, vol. 7, no. 2, pp. 121–136, 1996.

[47] I. A. Kanj and L. Perkovic, "On geometric spanners of euclidean and unit disk graphs," in *STACS'08*, 2008, pp. 409–420.

[48] B. Chandra, G. Das, G. Narasimhan, and J. Soares, "New sparseness results on graph spanners," in *SCG '92*, 1992, pp. 192–201.

[49] J. Gudmundsson, C. Levcopoulos, and G. Narasimhan, "Fast greedy algorithms for constructing sparse geometric spanners," *SIAM Journal on Computing*, vol. 31, no. 5, pp. 1479–1500, 2002.

[50] I. A. Kanj, L. Perković, and G. Xia, "Computing lightweight spanners locally," in *DISC '08*, 2008, pp. 365–378.

[51] J. Gudmundsson and M. H. M. Smid, "On spanners of geometric graphs," in *SWAT'06*, 2006, pp. 388–399.

[52] R. Klein and M. Kutz, "Computing geometric minimum-dilation graphs is np-hard," in *Graph Drawing*, 2006, pp. 196–207.

[53] O. Cheong, H. J. Haverkort, and M. Lee, "Computing a minimum-dilation spanning tree is np-hard," in *CATS'08*, 2007, pp. 15–24.

[54] E. L. Lloyd, "On triangulations of a set of points in the plane," in *FOCS'77*, 1977, pp. 228–240.

[55] P. Bose and M. Smid, "On plane geometric spanners: A survey and open problems," *http://people.scs.carleton.ca/ michiel/surveyplanespanners.pdf*.

[56] C. Schindelhauer, K. Volbert, and M. Ziegler, "Geometric spanners with applications in wireless networks," *Computational Geometry: Theory and Applications*, vol. 36, no. 3, pp. 197–214, 2007.

[57] C. Levcopoulos, G. Narasimhan, and M. H. M. Smid, "Improved algorithms for constructing fault-tolerant spanners," *Algorithmica*, vol. 32, no. 1, pp. 144–156, 2002.

[58] A. C. Czumaj and H. Zhao, "Fault-tolerant geometric spanners," *Discrete and Computational Geometry*, vol. 32, no. 2, 2004.

[59] S. Chechik, M. Langberg, D. Peleg, and L. Roditty, "Fault-tolerant spanners for general graphs," in *STOC'09*, 2009, pp. 435–444.

[60] L. Roditty, M. Thorup, and U. Zwick, "Roundtrip spanners and roundtrip routing in directed graphs," in *SODA'02*, 2002, pp. 844–851.

[61] L. J. Cowen and C. G. Wagner, "Compact roundtrip routing for digraphs," in *SODA'99*, 1999, pp. 885–886.

[62] ——, "Compact roundtrip routing in directed networks (extended abstract)," in *PODC'00*, 2000, pp. 51–59.

[63] ——, "Compact roundtrip routing in directed networks," *Journal of Algorithms*, vol. 50, no. 1, pp. 79–95, 2004.

[64] H. Zhang and J. C. Hou, "Asymptotic critical total power for k-connectivity of wireless networks," *IEEE/ACM Transactions on Networking*, vol. 16, no. 2, pp. 347–358, 2008.

[65] P. Hall, *Introduction to the Theory of Coverage Processes*. John Wiley and Sons, 1988.

[66] D. Berend, M. Segal, and H. Shpungin, "Power efficient resilience and lifetime in wireless ad-hoc networks," in *FOWANC'08*, 2008, pp. 17–24.

[67] M. D. Penrose, "The longest edge of the random minimal spanning tree," *Annals of Applied Probability*, vol. 7, no. 2, pp. 340–361, 1997.

[68] S. Khuller, B. Raghavachari, and N. Young, "Balancing minimum spanning and shortest path trees," in *SODA'93*, 1993, pp. 243–250.

[69] N. Bulusu, J. Heidemann, and D. Estrin, "Gps-less low cost outdoor localization for very small devices," *IEEE Personal Communications*, vol. 7, no. 5, pp. 28–34, 2000.

[70] R. G. Gallager, P. A. Humblet, and P. M. Spira, "A distributed algorithm for minimum-weight spanning trees," *ACM Transactions on Programming Languages and Systems*, vol. 5, no. 1, pp. 66–77, 1983.

[71] B. Awerbuch and R. G. Gallager, "Distributed bfs algorithms," in *SFCS'85*, 1985, pp. 250–256.

[72] Y. Ben-Shimol, A. Dvir, and M. Segal, "Splast: a novel approach for multicasting in mobile wireless ad hoc networks," *PIMRC'04*, vol. 2, pp. 1011–1015.