

Competitive Algorithms for Maintaining a Mobile Center *

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Abstract

In this paper we investigate the problem of locating a mobile facility at (or near) the center of a set of clients that move independently, continuously, and with bounded velocity. It is shown that the Euclidean 1-center of the clients may move with arbitrarily high velocity relative to the maximum client velocity. This motivates the search for strategies for moving a facility so as to closely approximate the Euclidean 1-center while guaranteeing low (relative) velocity.

We present lower bounds and efficient competitive algorithms for the exact and approximate maintenance of the Euclidean 1-center for a set of moving points in the plane. These results serve to accurately quantify the intrinsic velocity/ approximation quality tradeoff associated with the maintenance of the mobile Euclidean 1-center.

*Preliminary versions of some of the results in this paper first appeared in the 4th International ACM Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications (DIAL M for Mobility). This work has been supported by NSERC and MITACS. The work by Michael Segal was supported in part by the Pacific Institute for Mathematical Studies.

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1 Introduction

The goal of this paper is to formulate and study problems related to the location of mobile facilities (base stations) serving a set of mobile clients. The notion of mobile servers has been well studied in connection with stationary but transitory client sets [33]. Here we introduce problems related to the location of mobile facilities, where clients (or, more precisely, their associated sites) are modeled by points moving continuously in d -dimensional space, $d \geq 1$. Specific examples of such problems are the maintenance of the k -center and k -median for moving sites in some given metric space.

These problems have numerous potential applications, for example in mobile wireless communication networks when the broadcast range of base stations (which is directly related to energy utilization) needs to be minimized while still containing all the clients so as to provide service to the cellular phones. Map servers, or even servers whose sole purpose is to provide support for network control (such as those that maintain entity-to-address information) could be considered to be mobile facilities. Another potential application of these problems is the placement of mobile utilities (e.g. welding robots) in a manufacturing plant. In general, we anticipate that proximity problems related to systems of moving points will attract increasing interest from both theoretical and applied perspectives.

Facility location is a classical problem of operations research that has also been examined extensively in the computational geometry community. Most of the problems described in the facility location literature are concerned with finding a “desirable” facility location: the goal is to *minimize* a distance function between the facility (e.g., a service provider) and the sites (e.g., the client locations). Only recently has attention been paid to facility location problems for continuously moving points.

Surprisingly, the data structures and algorithms that have been developed for the *static* problems (i.e., clients are not moving) are not directly applicable to the setting of mobile clients when the motion of the facilities must satisfy natural constraints. In this paper we lay the foundations for the study of mobile facility location by treating mobile versions of the following classical facility location problem, for the special case where $k = 1$: given a set S of n *sites* (or demand points) in d -dimensional space ($d \geq 1$), the *k -center problem for S* asks for a set F of k *facilities* (or supply points) so that their associated *radius*, defined as the maximum distance between a site in S and its nearest facility in F , is minimized. Central to the research described here is the notion of an *approximate k -center*: a set F provides a λ -approximation of the k -center if its associated radius is at most a factor λ larger than that of an exact k -center.

The most familiar (and, for many applications, most appropriate) notion of distance is Euclidean distance (L_2 norm), although the notion of centers makes sense in (and we will describe several results for) other metrics (e.g. L_p norms) as well. Note that for some metrics (e.g. L_1) a k -center is not necessarily unique even for $k = 1$ and $d = 2$.

The k -center problem has been well studied in both the exact [4, 10, 11, 12, 15, 16, 19, 20, 29, 34] and $(1 + \varepsilon)$ -approximate [5, 13, 14, 15, 18, 26, 28, 31] versions. Dynamic facility location problems involving static but transient clients [33] or time varying domains (eg. networks whose edge distances change) also have been studied, see [24, 27].

Mobile facility location, on the other hand, addresses problems involving continuously moving clients. Let p be any continuously moving point specified by a continuous and piecewise differentiable function \hat{p} mapping the time interval $[0, T]$ to \mathbb{R}^d . We say that the *velocity* of p is bounded by v_{max} on $[0, T]$ if the magnitude of the derivative of \hat{p} , wherever it is defined on $[0, T]$, never exceeds v_{max} . In certain applications, there may be good reasons to choose different notions of distance to

measure the radius and velocity of center points, since the radius is constrained by the facility-client interaction medium and the velocity is constrained by the facility (and client) relocation medium. Nevertheless, we make the simplifying assumption that they are both captured by the same metric.

Let $S = \{s_1, s_2, \dots, s_n\}$ be a set of n continuously moving sites specified by continuous and piecewise differentiable functions $\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_n\}$, where $\hat{s}_i : [0, T] \rightarrow \mathbb{R}^d$ and the velocity of s_i is bounded by 1 for all i , $1 \leq i \leq n$. A *mobile λ -approximate k -center* for S is a set F of k continuously moving facilities $F = \{f_1, f_2, \dots, f_k\}$ specified by continuous and piecewise differentiable functions $\hat{f}_1, \dots, \hat{f}_k$; $\hat{f}_i : [0, T] \rightarrow \mathbb{R}^d$ such that at any given moment $t \in [0, T]$, the facilities at locations $\hat{f}_1(t), \dots, \hat{f}_k(t)$ form a λ -approximate k -center for the sites at locations $s_1(t), \dots, s_n(t)$. We refer to a mobile 1-approximate k -center as an *exact mobile k -center*.

The study of mobile facility location is still in its infancy. Agarwal and Har-Peled [3] build a kinetic data structure for maintaining an approximate Euclidean 1-center F in the plane using only $O(1/\varepsilon^{5/2})$ events (changes in the trajectory of the F). In general, their approximate center does not move continuously. Agarwal et al. [1] study an approximation of the 1-median in \mathbb{R}^1 within the kinetic framework. Here the approximation relates to rank among the sites, according to the natural total order, rather than the sum of distances. Gao et al. [23] design a randomized algorithm for maintaining a set of clusters among moving points in the plane. The centers of a given radius are selected among moving points so that their number is a constant-factor approximation of the minimum possible. As the points move, an event-based kinetic data structure updates the centers as necessary. This kinetic data structure is shown to be responsive, efficient, local, and compact. Har-Peled [25] found a scheme of determining centers in advance, i.e. if in the optimal solution the number of centers is k and r is the optimal radius for the points moving with the degree of motion l , then his scheme guarantees a 2^{l+1} -approximation of the radius with k^{l+1} centers chosen from the set of input points before the points start to move. Agarwal et al. [2] give efficient and compact kinetic data structures for maintaining the diameter, width, and smallest area or perimeter bounding rectangle of the points. Gao et al. [21] provides a good kinetic data structure for maintaining an 8-approximate k -center, for any $k \leq n$, where the centers are chosen from the set of input points.

In this paper we focus on exact and approximate versions of the 1-center problem where the facility is constrained to have bounded velocity. Unless otherwise specified, we assume that our problems are set in dimension 2. Even these quite restricted instances of mobile facility location raise interesting, and surprisingly challenging, algorithmic and geometric questions.

We demonstrate that the velocity of the exact Euclidean 1-center can be arbitrary large (Theorems 2 and 11). Motivated by this we analyse some simple approximation schemes (that trade lower accuracy of approximation for lower velocity of the facility) and develop general competitive strategies that guarantee, for any desired approximation bound, a motion strategy whose associated velocity is within a constant multiplicative factor of optimal (Theorems 10 and 11). Our approximation results using the simple bounding box strategy (Theorem 8) have recently been superseded by a significantly more involved strategy using what is known as the Gaussian center of a point set [17].

Although the analysis of our strategies is non-trivial they all have both simple specifications and efficient implementations within the kinetic framework introduced by Basch and Guibas [8, 22].

The remainder of this paper is organized as follows. In the next section we focus on the exact mobile 1-center problem in two different metrics. The essential result here, that motivates much of the rest of the paper, is that the Euclidean 1-center has arbitrarily high velocity in the worst case. Section 3 addresses approximate 1-centers and the tradeoff between velocity and approximation

quality. Section 4 concludes the paper with a summary of results and some remarks on related (open) problems.

2 Exact mobile 1-centers

Let S be any set of n mobile sites in \mathbb{R}^d , $d \geq 1$. We assume, without loss of generality, that although the velocity of a site may change during its motion (perhaps discontinuously), it never exceeds 1 in absolute value. Clearly, the exact 1-center in any metric must have velocity at least 1 in the worst case.

In the case where $d = 1$, it is easy to see that the average of the extrema of S provides a 1-center that has velocity at most 1 in any Minkowski (L_p) metric. Hence, in the remainder of this section we consider site sets S in dimensions $d > 1$. We begin by considering the mobile rectilinear (L_∞) 1-center, for which exact algorithms with efficient implementations are possible, and then turn to the mobile Euclidean (L_2) 1-center, whose computation is significantly more involved.

2.1 Rectilinear 1-center

Suppose that we want to maintain an exact *rectilinear* 1-center of the mobile site set S , i.e., a point c with the property that the maximum distance (in the L_∞ metric) between any site in S and c is minimized. We assume that the velocities of points (measured in the L_∞ metric) are bounded by 1. As previously noted, any rectilinear center must move with velocity at least 1 in the worst case (even in \mathbb{R}^1). It turns out that in this special case velocity 1 is sufficient as well.

Theorem 1 *The center of the smallest bounding hypercube of a site set S in \mathbb{R}^d , $d \geq 1$ provides a rectilinear 1-center of S whose velocity is bounded by 1.*

Proof. The smallest bounding hypercube $\Pi_{i=1}^d [a_i, b_i]$ of the site set S is defined by $2d$ sites (some of which may coincide). The center of this bounding hypercube is the point with coordinates $((a_1 + b_1)/2, \dots, (a_d + b_d)/2)$. Since the i -th coordinate of this hypercube center is determined by the extremal sites in dimension i , the i -th component of the velocity of the hypercube center is bounded by 1. Hence, the entire velocity of the hypercube center (in the L_∞ metric) is bounded by 1. ■

Remark. A similar result holds for the 1-center in \mathbb{R}^2 , under the L_1 metric, using the center of the smallest bounding diamond. We note that, like other exact and approximate mobile 1-center schemes discussed in this paper, the center of the smallest bounding hypercube, and thus the rectilinear 1-center, can be maintained using elementary data structures. The idea here is to maintain the extrema in each dimension under motion of points. Basch et al. [8] present several data structures to solve different variants of this maximum maintenance problem. For our purposes it suffices to use their kinetic swapping heap with $O(\log n)$ time responsiveness, $O(1)$ locality and $O(n \log^3 n)$ efficiency.

2.2 Euclidean 1-center

Although the static Euclidean 1-center problem (the 1-center problem under the L_2 metric), like the static rectilinear 1-center problem, can be solved in linear time [32], the mobile versions of these

problems are quite different. In contrast to the rectilinear 1-center problem we present an example proving that the Euclidean 1-center may move with arbitrarily high velocity.

Theorem 2 (Unbounded Velocity) *For any velocity $v \geq 0$ there is a set of three sites s_1, s_2, s_3 in $\mathbb{R}^d, d \geq 2$ such that a unit velocity motion of two of the sites induces an instantaneous velocity greater than v of the Euclidean 1-center.*

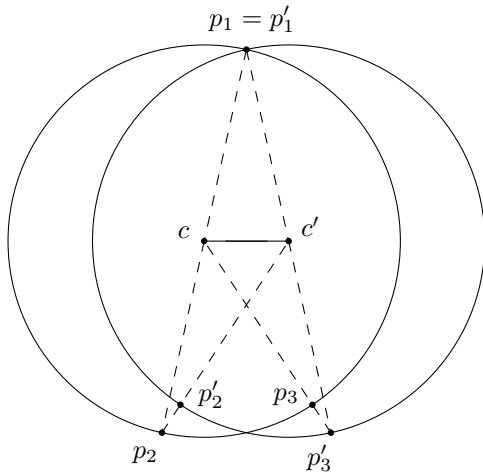


Figure 1: Euclidean 1-center may move with arbitrarily high velocity.

Proof. Figure 1 gives an example of three points p_1, p_2, p_3 lying on a unit radius circle in any plane of \mathbb{R}^d . These points move to points p'_1, p'_2, p'_3 , respectively. Points c and c' correspond to the Euclidean centers before and after the motions respectively.

Let $y = |c - c'|$ be the length of the path made by the 1-center and let $x = |p_2 - p'_2| = |p_3 - p'_3|$ be the length of the paths made by the points p_2 and p_3 . It suffices to show that $y/x > v$, for sufficiently small x . Indeed, if θ denotes the angle $p_2 c c'$ then $(1 + x)^2 = 1 + y^2 - 2y \cos \theta$, by the cosine law, and $\cos \theta = -\cos(\pi - \theta) = -y/2$. This implies that $(1 + x)^2 = 1 + 2y^2$ and hence $y > \sqrt{x}$. It follows that $y/x \geq v$ provided $x \leq 1/v^2$. ■

It is immediate from the theorem above that any bounded velocity approximations of the mobile Euclidean 1-center must, at some points in time, have an associated radius that exceeds the Euclidean radius. The next section explores this velocity/approximation-factor tradeoff in more detail.

3 Approximate Mobile Euclidean 1-centers

3.1 Unit velocity constraint

Suppose that we constrain an approximate mobile 1-center to move with velocity at most one. If, at the beginning of the motion of the sites we are allowed to put our facility f at any point in the space $\mathbb{R}^d, d \geq 1$, it is straightforward to achieve a 2-approximation of the 1-center by simply identifying the facility with any one mobile site. That this strategy provides a 2-approximation factor follows immediately from the fact that the Euclidean diameter of a site set is at most twice

its Euclidean radius. In fact the same 2-approximation factor can be achieved even in the case where the starting location of the facility is restricted to be any point inside the convex hull of S (by, for example, maintaining the facility at some fixed convex combination of three site locations.)

That the latter strategy guarantees a unit velocity bound is immediate from the following:

Observation 3 *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be fixed numbers such that $\alpha_i \geq 0$ for all i and $\sum_{i=1}^n \alpha_i = 1$. If all of the sites, s_1, \dots, s_n , move with velocity at most 1 then the point p defined as the convex combination $\sum_{i=1}^n \alpha_i s_i$ of the sites moves with velocity at most 1.*

Proof. Let $v_i \in \mathbb{R}^d$ be the velocity vector of s_i and let v be the velocity vector of p . Then $v = \sum_{i=1}^n \alpha_i v_i$ and $|v| \leq \sum_{i=1}^n \alpha_i \cdot |v_i| \leq \sum_{i=1}^n \alpha_i = 1$. ■

It turns out that remaining inside the convex hull of the sites, at all points in time, is crucial for any facility to achieve a bounded approximation factor. Suppose that at some point in time the facility lies outside the convex hull of the sites. Then there must exist some point r that is strictly closer to all of the sites than it is to the facility. (To see this suppose, without loss of generality, that the facility lies at the origin and all of the sites lie in the open halfspace $x > 0$. Provided $z > (x_i^2 + y_i^2)/2x_i$, the point $(z, 0)$ is strictly closer to the site s_i , located at (x_i, y_i) , than the facility. Thus, it suffices to choose $r = (z_{max}, 0)$, where $z_{max} = \max\{(x_i^2 + y_i^2)/2x_i\}$.) Hence, if all of the sites move with unit velocity to r the Euclidean radius of the site set reduces to zero while the radius associated with the facility remains strictly positive, which means its approximation factor becomes unbounded.

The arguments above are summarized in the following:

Lemma 4 (Unit velocity) *Let f be the initial position of a facility in \mathbb{R}^d .*

- (i) *If f is contained in the convex hull of S then there is an efficiently maintained unit velocity-bounded motion for f that guarantees a 2-approximation of the Euclidean 1-center.*
- (ii) *If f lies outside of the convex hull of S then no constant approximation factor can be guaranteed for any unit velocity-bounded motion for f .*

Following Lemma 4 we note that if the facility is restricted to move with velocity less than 1 then no approximation factor is achievable even for $d = 1$. Next we show that a slightly better approximation factor is achievable if the number of sites n is bounded and if the position of the facility is initialized and maintained as the *center of mass* of the sites (the convex combination $\sum_{i=1}^n s_i/n$).

Lemma 5 (Center of Mass) *The center of mass of a set of n sites S in $\mathbb{R}^d, d \geq 1$ provides a $(2 - \frac{2}{n})$ -approximation of the Euclidean 1-center of S .*

Proof. By Observation 1 the velocity of the center of mass of the sites does not exceed the maximum velocity of the clients.

Let c be the location of the center of mass. Let p be the farthest site from c in S . Let l be the line passing through c and p . We project all sites of S onto l (see Fig. 2). Let s' denote the projection of a site s of S . Clearly, there is a site s such that c lies on the segment $s'p$. Let q be the site of S that maximizes the length of the segment $q'p$. Since c is the center of mass of sites projected onto l ,

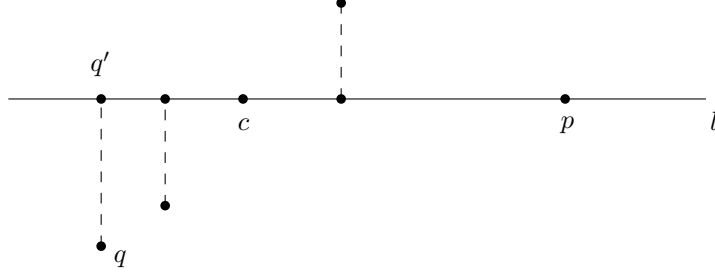


Figure 2: Center of mass approximation.

it follows that $|c - p| \leq (n - 1)|q' - c|$. Hence, $n|c - p| \leq (n - 1)|q' - p|$ or $|c - p| \leq (1 - 1/n)|q' - p|$. But the optimal 1-radius, r^* , is at least $|q' - p|/2$. Hence, $|c - p| \leq 2(1 - 1/n) \cdot r^*$. ■

Remark The preceding observation and lemmas can be easily seen to generalize to arbitrary Minkowski metrics L_p .

Surprisingly, the approximation factors stated in the previous lemmas are, in fact, asymptotically optimal.

Theorem 6 (Lower Bound) *There exist arbitrarily large sets S of mobile sites in \mathbb{R}^d , $d \geq 2$, with velocities bounded by 1, such that no mobile facility that moves with velocity at most 1 can maintain a λ -approximation of the Euclidean 1-center of S , for any $\lambda < 2$.*

Proof. It suffices to prove the lower bound for $d = 2$ since any two-dimensional scenario can be embedded into \mathbb{R}^d , $d \geq 3$. We prove this theorem by an adversary argument. The adversary picks a set S that contains arbitrarily large sets of sites at vertices A, B, C of an equilateral triangle of side length 2 (cf. Fig. 3). We can assume, by lemma 4, that the approximate Euclidean 1-center has its initial location inside the convex hull of S . The adversary proceeds in two phases. In the first phase the adversary moves the sites in S in a sequence of steps each of which moves all sites distance exactly 2 and culminates in the sites once again lying at the vertices of an equilateral triangle of side length 2.

In each step the adversary forces the approximate 1-center to move either outside of the current equilateral triangle (the convex hull of the sites) or closer to one of its corners. This goal can be achieved by the following strategy (cf. Fig. 3). The adversary checks which one of the triangles AoB, BoC, CoA contains the facility, where o is the center of ΔABC . Depending on location of the facility the adversary moves sites to an adjacent equilateral triangle. For example, assuming the facility is located in triangle AoB the adversary moves all the sites to the lower right triangle in Fig. 3. All sites at vertices A and B move towards vertex C and the sites at vertex C are split into two halves that move to the two remaining vertices of lower right triangle in Fig. 3. Similarly, if the facility is located in triangle BoC (AoC) adversary moves the sites to the lower left (resp. upper) triangle.

The adversary continues this strategy until the facility has moved either outside the current triangle or arbitrarily close to one of its vertices. To prove termination of this strategy it suffices to demonstrate that, while the facility remains inside the convex hull of the sites, the distance of the facility to the its closest triangle vertex converges to zero.

To show this we define the distance of the facility to the center of the current triangle using a hexagon centered at the center of the triangle. Assume that the facility is located in ΔABC on the

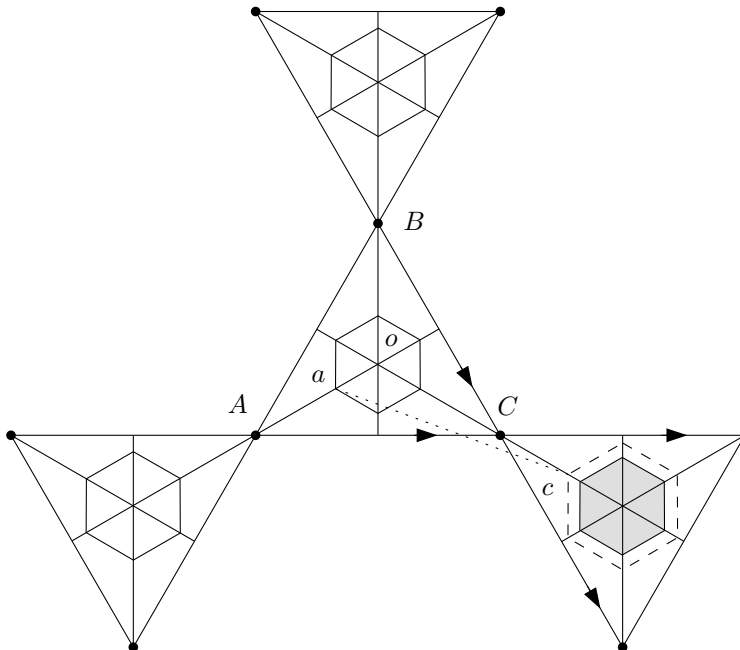


Figure 3: Adversary strategy for achieving lower bound for approximation factor.

left side of the hexagon centered at o . We show that at the end of the next step the facility will be outside the hexagon of the same size, shaded in Fig. 3, in the new triangle. Consider the smallest hexagon H (dashed in Fig. 3) in the new triangle that can be reached by the facility. Let a be the point of the hexagon centered at o closest to vertex A and let c be the point of H closest to vertex C , then $|ac| \leq 2$. Let $x_n = |Aa|$ and $x_{n+1} = |Cc|$. By Pythagorus's Theorem we have

$$((x_{n+1} - x_n)\sqrt{3}/2 + 2)^2 + \frac{(x_n + x_{n+1})^2}{4} \leq 4.$$

It follows that

$$x_{n+1} \leq \frac{x_n - 2\sqrt{3} + \sqrt{12 + 4\sqrt{3}x_n - 3x_n^2}}{2}$$

First we observe that $\sqrt{12 + 4\sqrt{3}x_n - 3x_n^2} < 2\sqrt{3} + x_n - x_n^2/(2\sqrt{3})$, if $x_n > 0$. Thus $x_{n+1} < x_n - x_n^2/(4\sqrt{3})$. It follows by induction that $x_n < 4\sqrt{3}/n$. Therefore, $\lim_{n \rightarrow \infty} x_n = 0$ and the facility is forced to be either outside the current triangle or arbitrarily close to one of the vertices of the current triangle.

Suppose that the facility is within distance ϵ of the vertex A . The adversary now moves the sites from the vertices B and C toward the point D at distance 2 from B and C (opposite A) and the sites from the vertex A distance 2 towards D . When this motion is complete the Euclidean radius of the sites is equal to $|A' - D|/2 = \sqrt{3} - 1$. Meanwhile the best strategy for the facility is to also move with unit velocity towards D . Since the facility ends up distance at least $2\sqrt{3} - 2 - \epsilon$ from D , the approximation factor is at least $2 - \epsilon/(\sqrt{3} - 1)$. ■

Remark. The lower bound above applies to *on line* facility relocation strategies (where the

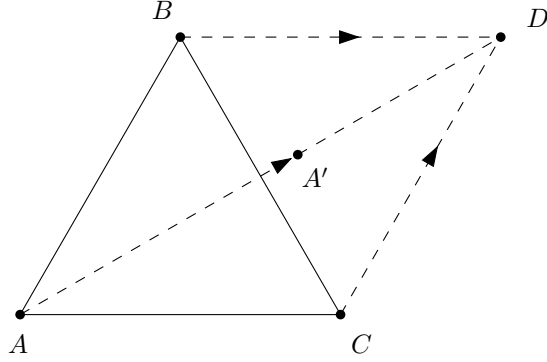


Figure 4: Final step of the adversary strategy.

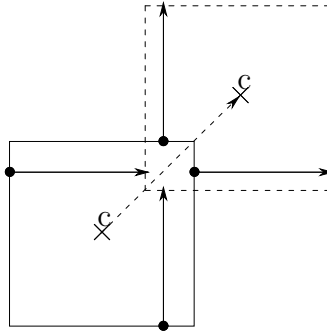


Figure 5: Rectilinear 1-center moves with velocity $\sqrt{2}$.

motion of clients is *not* known in advance). It remains an open question whether a c -approximation, for some $c < 2$, can be achieved using an off line strategy.

3.2 higher velocity approximations in the plane

We observed in Section 2.1 that the rectilinear 1-center has velocity at most 1 in the L_∞ metric and can be efficiently maintained. This makes it a natural candidate for an approximate Euclidean 1-center.

Observation 7 *The rectilinear 1-center of any set of sites in \mathbb{R}^2 moves with velocity at most $\sqrt{2}$. Furthermore, this bound is tight, i.e. there is a set of sites in \mathbb{R}^2 whose unique rectilinear 1-center moves with velocity $\sqrt{2}$.*

Proof. The bounding box $[a_1, b_1] \times [a_2, b_2]$ of the site set S is defined by 4 sites (some of them may coincide). The center of the bounding box has coordinates $((a_1 + b_1)/2, (a_2 + b_2)/2)$ and serves as a rectilinear 1-center. Since each coordinate of the center of bounding box moves with velocity at most 1, the center of bounding box can move with velocity at most $\sqrt{2}$. On the other hand, Fig. 3.2 shows an example when the rectilinear 1-center moves with velocity $\sqrt{2}$ ■

Remark. The observation above generalizes naturally to \mathbb{R}^d , giving a 1-center with velocity \sqrt{d} .

Theorem 8 (Bounding Box Strategy) *The center of the smallest bounding box of any set S of sites in \mathbb{R}^2 provides a $(1 + \sqrt{2})/2$ -approximation of the Euclidean 1-center of S . Furthermore, this bound is tight.*

Proof. Let r denote the radius associated with c , the Euclidean 1-center of S . Without loss of generality we assume that the smallest bounding box B of S has its center c_a at the origin. Its associated radius r_a satisfies $r_a = \max_{p \in S} \{|p|\}$. We will show that $r_a/r \leq (1 + \sqrt{2})/2$.

Suppose, again without loss of generality, that a site $p^* = (1, t)$ of S in the upper right quadrant of B realizes the maximum distance r_a from c_a . By interchanging the axes if necessary, we can assume that $t \geq 1$. Since $r_a = \sqrt{1 + t^2}$ and $r \geq t$ it is straightforward to confirm that

$$\frac{r_a}{r} \leq \frac{\sqrt{1 + t^2}}{t} < (1 + \sqrt{2})/2,$$

if $t \geq 2$. Hence we assume that $t \in [1, 2)$.

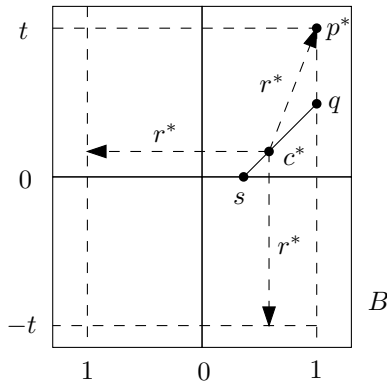


Figure 6: Bounding box strategy for approximating Euclidean 1-center.

Let s denote the point $(t - 1, 0)$ and q the point $(1, 2 - t)$. We note that $|p^* - s| > t$ and $|p^* - q| = 2t - 2 < 2$, since $t < 2$. Hence there exists a point c^* on the line segment between s and q that is equidistant (say r^*) from p^* and the lines $x = -1$ and $y = -t$ (see Fig. 6). It follows that $r \geq r^*$, since no circle with radius less than r^* can contain p^* as well as some point on the left side of B (which lies on or to the left of the line $x = -1$) and some point on the lower side of B (which lies on or below the line $y = -t$).

Since $|p^* - c^*| = r^*$ it follows that

$$(2 - r^*)^2 + (2t - r^*)^2 = (r^*)^2.$$

This can be transformed to the quadratic equation

$$(r^*)^2 - 4(t + 1)r^* + 4(t^2 + 1) = 0.$$

It follows that $r^* = 2(t + 1 - \sqrt{2t})$, since the root with “+” is greater than $2t$ which is impossible.

The distances r_a and r^* can be viewed as functions of t . In order to prove that $r_a/r \leq (1 + \sqrt{2})/2$ for $t \in [1, 2]$ we show that (i) $r_a/r^* = (1 + \sqrt{2})/2$, when $t = 1$, and (ii) $\frac{d(r_a/r^*)}{dt} \leq 0$ for any $t \in [1, 2]$. If $t = 1$ then $r_a = \sqrt{2}$ and $r^* = 2(2 - \sqrt{2})$ which implies (i). (Note that from this it is straightforward to construct an example that shows our bound is tight.)

To prove the condition (ii) we differentiate r^* and r_a with respect to t :

$$(r^*)' = 2(1 - 1/\sqrt{2t}) \quad \text{and} \quad r'_a = t/\sqrt{1+t^2}.$$

Since the derivative of r_a/r with respect to t has the same sign as $r'_a r - r_a r'$, it suffices to show that

$$\frac{2t}{\sqrt{1+t^2}} (t + 1 - \sqrt{2t}) \leq 2\sqrt{1+t^2} \left(1 - \frac{1}{\sqrt{2t}}\right).$$

Let $x = \sqrt{t/2}$. Then $t = 2x^2$ for $x \in [\sqrt{2}/2, 1]$ and the inequality above is equivalent to

$$2x^2(2x^2 + 1 - 2x) \leq (1 + 4x^4) \left(1 - \frac{1}{2x}\right).$$

Multiplying by $2x$ we obtain

$$(12x^4 - 4x^3) + (2x - 1) \geq 0$$

which clearly holds since $12x^4 - 4x^3 \geq 0$ and $2x - 1 \geq 0$. ■

We have observed that a 2-approximation is realizable with velocity 1 and a $(1 + \sqrt{2})/2$ -approximation is realizable with velocity $\sqrt{2}$. A natural question now is: what approximation factor can be achieved if we restrict the velocity of facility f to some constant between 1 and $\sqrt{2}$?

Suppose now, that the velocity of the facility is bounded by $v_{max} \in [1, \sqrt{2}]$. We can *mix* our two strategies — the center of mass of the points and the center of the bounding box — in the following way. Let (f_1, v_1) denote the location and velocity of the center of mass of the points. Similarly, (f_2, v_2) denote the location and velocity of the center of the bounding box. The mixing strategy maintains the *mixing center* (f, v) defined as $(\alpha f_1 + (1 - \alpha)f_2, \alpha v_1 + (1 - \alpha)v_2)$, where $\alpha = (\sqrt{2} - v_{max})/(\sqrt{2} - 1)$.

Lemma 9 (Mixing Strategy) *If the facility is allowed to move with velocity $v_{max} \in [1, \sqrt{2}]$, the corresponding mixing strategy achieves a λ -approximation of the Euclidean 1-center, for*

$$\lambda = \alpha \frac{1 + \sqrt{2}}{2} + (1 - \alpha) \left(2 - \frac{2}{n}\right), \quad \text{where} \quad \alpha = \frac{\sqrt{2} - v_{max}}{\sqrt{2} - 1}.$$

Proof. Let c_m be the center of mass and let c_b be the center of the bounding box. Let r be the exact radius of 1-center and let r_m, r_b be the radii determined by c_m and c_b , respectively. The mixing center is defined as $c_{mix} = \alpha c_m + (1 - \alpha)c_b$.

By Lemma 5 and Theorem 8 $r_m \leq (2 - 2/n)r$ and $r_b \leq (\sqrt{2} + 1)r/2$. Let p be any point in S . It suffices to prove that $|p - c_{mix}| \leq \lambda r$. The mixing center has a property that $|p - c_{mix}| \leq \alpha|p - c_m| + (1 - \alpha)|p - c_b| \leq \alpha(2 - 2/n)r + (1 - \alpha)(\sqrt{2} + 1)r/2 = \lambda r$. ■

Remark. The mixing strategy bounds from above the velocity/approximation quality tradeoff, for velocities in the range $[1, \sqrt{2}]$. In general it is not optimal. In particular, recent result [17] has shown that the so-called Gaussian center of a planar point set moves with velocity at most $4/\pi$ and achieves a 1.115-approximation of the Euclidean 1-center. Furthermore, the mixing strategy, like the bounding box strategy upon which it is built, has the property that it may in some situations move with much higher velocity than the Euclidean center that it is trying to approximate. While it is natural to try and modify the mixing strategy to take this into account there is another more general competitive strategy that applies across the full spectrum of feasible velocities (and associated approximations) and smoothly adapts to situations where the Euclidean center happens to move slowly relative to the motion of the sites.

Theorem 10 (Upper Bound) For any ε , $0 < \varepsilon < 1$, there is a strategy for moving a facility such that (i) the location of the facility provides an approximation of the Euclidean 1-center of a set S of points in \mathbb{R}^2 that is never worse than $1 + \varepsilon$, and (ii) the velocity of the facility never exceeds $\frac{(2+\varepsilon)(1+\varepsilon)}{\sqrt{2\varepsilon+\varepsilon^2}}$.

Proof. A velocity bound of $v = O(1/\sqrt{\varepsilon})$ can be demonstrated for the following simple discrete strategy. The facility moves in a sequence of phases where in phase i the facility is moved with velocity v to the location of the Euclidean 1-center at the end of phase $i - 1$. We maintain the invariant that, at the end of each phase, the facility f is located so that the radius r_a of the smallest circle, centered at f , enclosing all the client points is at most $(1 + \delta)$ (for some suitably chosen $\delta < \varepsilon$) times the Euclidean radius r .

The more precise bound asserted in the statement of the theorem is achieved by a similar *continuous* strategy: time is divided into infinitesimal intervals and in each interval the facility is moved with velocity v (to be specified below) towards the location of the Euclidean 1-center at the end of preceding interval. Since the motions of both the clients and the facility are continuous, the ratio r_a/r changes continuously. We show that in the limit, as the interval lengths approach 0, it is possible to maintain the invariant that $r_a/r \leq 1 + \varepsilon$. Since r_a/r changes continuously it suffices to show that whenever $r_a/r = 1 + \varepsilon$ its derivative (with respect to time) is negative.

Suppose that at some moment t_0 the Euclidean 1-center is located at the point c , the facility is located at point $a (\neq c)$, the exact Euclidean radius r equals 1, and the radius r_a associated with the facility satisfies $r_a = 1 + \varepsilon$. Suppose the circle C of radius r centered at c and the circle C_a of radius r_a centered at a intersect at points p and q . Let s denote the orthogonal projection of p onto the line through a and c and let x , y and z denote the lengths $|a - c|$, $|c - s|$ and $|p - s|$, respectively (see Fig. 7).

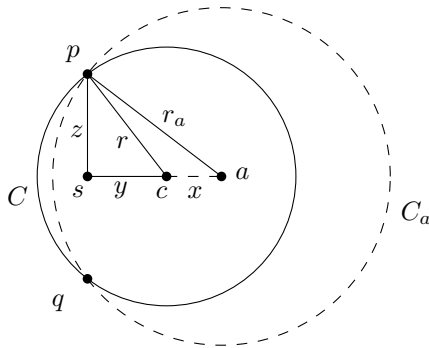


Figure 7: $(1 + \varepsilon)$ -approximation of Euclidean 1-center.

According to our strategy at time t_0 the facility moves from a towards c with its maximum velocity v . It remains to determine the value of v that will ensure that the derivative of r_a/r at time t_0 is negative. First note that any point in S that is distance r_a from a lies on the arc of C_a inside C . Such points could move away from a at a rate of at most 1. Meanwhile the facility is moving towards all of these points at a rate at least $v(x + y)/r_a$. It follows that the derivative of r_a at t_0 is less than or equal to $1 - v(x + y)/r_a$. Since the derivative of r at any time is at least -1 , it follows that the derivative of r_a/r at time t_0 is at most $(r(1 - v(x + y)/r_a) - r_a(-1))/r^2$ which is negative when $v \geq (2 + \varepsilon)r_a/(x + y)$ (since $(r + r_a)/r = 2 + \varepsilon$).

Since $z^2 + y^2 = r^2$ and $z^2 + (x + y)^2 = r_a^2$ (by Pythagorus), it follows that $r_a/(x + y) = (2r_ax)/(x^2 + r_a^2 - r^2)$. Thus, for fixed r and r_a , $r_a/(x + y)$ is maximized when $x^2 = r_a^2 - r^2$. Hence, $r_a/(x + y) \leq r_a/\sqrt{r_a^2 - r^2} = (1 + \varepsilon)/\sqrt{2\varepsilon + \varepsilon^2}$ and so the derivative of r_a/r is negative when $v \geq (2 + \varepsilon)(1 + \varepsilon)/\sqrt{2\varepsilon + \varepsilon^2}$. ■

Note that the strategies described in the preceding theorem are *on-line* strategies; the motion of the facility at any time depends only on its current location and the current location of the exact 1-center. In the following theorem we show that our bounds are asymptotically tight even for *off-line* strategies (i.e. strategies that have available complete knowledge of the past, present and future motions of all of the clients). Specifically, any fixed approximation factor λ is realized by our (on-line) strategy within a velocity bound that is to within a small constant multiple of the minimum velocity required of all (off-line) strategies that achieve approximation factor λ . This establishes that our strategies are *competitive* in the sense of [35, 33], achieving a small constant competitive ratio.

Theorem 11 (Lower Bound) *For every ε , $0 < \varepsilon < 1/64$, any $(1 + \varepsilon)$ -approximate mobile Euclidean 1-center has velocity at least $1/(8\sqrt{\varepsilon})$ in the worst case.*

Proof.

Consider any scheme for maintaining a facility at an approximate Euclidean 1-center with maximum velocity v . Let C_1 and C_2 be circles, both of radius 1, centered at the points c_1 and c_2 respectively, where $|c_1 - c_2| = s = 1/\sqrt{v^2 - 1}$. Initially clients are positioned at points p_1, \dots, p_4 on circle C_1 , where p_1p_2 forms a diameter of C_1 normal to the line joining c_1 and c_2 (see Fig. 8).

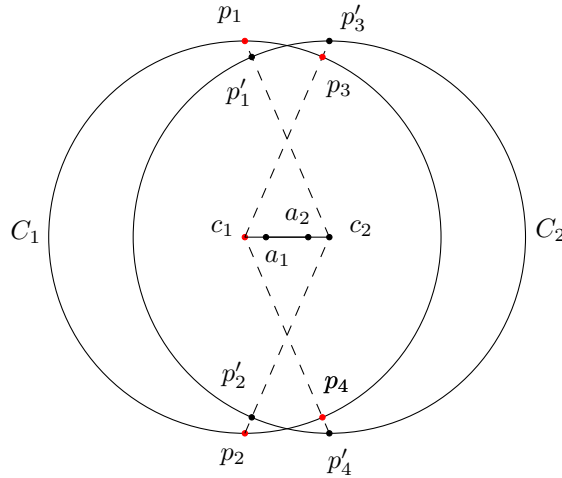


Figure 8: lower bound construction.

The clients at points p_1 and p_2 move with unit velocity towards point c_2 and the clients at points p_3 and p_4 move with unit velocity away from point c_1 (as illustrated) arriving at points p'_1, \dots, p'_4 respectively. We assume that points p_3 and p_4 were chosen in such a way that $p'_3p'_4$ forms a diameter of C_2 normal to the line joining c_1 and c_2 . Note that $|p_i - p'_i| = \sqrt{1 + s^2} - 1$ for $i = 1, \dots, 4$.

Let a_1 and a_2 denote the points on the line segment c_1c_2 at distance $(s - v(\sqrt{1 + s^2} - 1))/2 = (v - \sqrt{v^2 - 1})/2$ from c_1 and c_2 respectively. Note that $|a_1 - a_2| = v(\sqrt{1 + s^2} - 1)$ and $|a_1 - p_1| =$

$|a_1 - p_2| = |a_2 - p_3| = |a_2 - p_4| = \sqrt{1 + ((v - \sqrt{v^2 - 1})/2)^2}$. Furthermore, any pair of points b_1, b_2 satisfying $\max\{|b_1 - p_1|, |b_1 - p_2|, |b_2 - p_3|, |b_2 - p_4|\} \leq \sqrt{1 + ((v - \sqrt{v^2 - 1})/2)^2}$ must have $|b_1 - b_2| \geq v(\sqrt{1 + s^2})$. It follows that no matter how the facility moves (provided that its velocity never exceeds v) its approximation to the Euclidean 1-center either before or after the motion of the clients must be at least $\sqrt{1 + ((v - \sqrt{v^2 - 1})/2)^2}$. But

$$\begin{aligned} \sqrt{1 + ((v - \sqrt{v^2 - 1})/2)^2} &= \sqrt{1 + v^2(1 - \sqrt{1 - 1/v^2})^2/4} \\ &\geq \sqrt{1 + 1/(16v^2)} \\ &\geq 1 + 1/(64v^2). \end{aligned}$$

Hence if $v \leq 1/(8\sqrt{\varepsilon})$ an approximation of the Euclidean 1-center better than $1 + \varepsilon$ is impossible. ■

4 Conclusion

In this paper we introduced mobile versions the 1-center problem, a classical problem in facility location. We investigated the complexity of maintaining velocity-bounded mobile 1-centers in Euclidean space, identifying and asymptotically characterizing a natural velocity/approximation quality tradeoff.

It is natural to consider other standard facility location problems, such as the k -median problem, in the mobile setting as well. (The k -median problem, for a set S of sites, asks for a set F of facilities that minimize the sum, over all sites $s \in S$, of the distance from s to its closest facility $f \in F$.) Intuitively, the maintenance of the 1-median seems harder than that of the 1-center because, in this case, all sites participate in the definition of the median (while only three sites define the 1-center). Solutions to the static Euclidean 1-median problem, also known as the Fermat-Weber problem, have only been achieved by means of iterative algorithms (see, for example, [30]). Nevertheless, we have been able to achieve some preliminary results for the mobile 1-median problem [9] many of which closely parallel those developed for the mobile 1-center in this paper. Specifically, we can show that:

1. Like the Euclidean 1-center, the 2-d Euclidean 1-median can move with arbitrarily high velocity relative to that of the clients.
2. The center of mass provides a 2-approximation to the 2-d Euclidean 1-median using relative velocity at most 1. (Furthermore, there are examples where the approximation ratio of the center of mass is arbitrarily close to 2.)
3. The rectilinear (L_∞) 1-median provides a $\sqrt{2}$ -approximation of the Euclidean (L_2) 1-median using relative velocity at most $\sqrt{2}$.
3. An approximation ratio better than $2/\sqrt{3} \approx 1.154$ to the Euclidean 1-median is impossible for any facility constrained to move with relative velocity at most 1.

Future directions for the research in this area include: providing tighter bounds for mobile 1-center and 1-median problems, and the extension of our results to the mobile k -center and k -median problems for $k \geq 2$.

Acknowledgment

We thank the anonymous referees for numerous helpful comments, in particular for raising the question about off-line strategies that we discuss after Theorem 6.

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