k-Fault Resistance in Wireless Ad-Hoc Networks

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Abstract

Given a wireless network, we want to assign each node a transmission power, which will enable transmission between any two nodes (via other nodes). Moreover, due to possible faults, we want to have at least k vertex-disjoint paths from any node to any other, where k is some fixed integer, depending on the reliability of the nodes. The goal is to achieve this *directed* k-connectivity with a minimal overall power assignment. The problem is **NP-Hard** for any $k \geq 1$ already for planar networks. Here we first present an optimal power assignment for uniformly spaced nodes on a line for any $k \geq 1$. Based on it, we design an approximation algorithm for linear radio networks with factor $\min\{2, (\frac{\Delta}{\delta})^\alpha\}$, where Δ and δ are the maximal and minimal distances between adjacent nodes respectively and parameter $\alpha \geq 1$ being the distance-power gradient. We then extend it to the weighted version. Finally, we develop an approximation algorithm with factor $O(k^2)$, for planar case, which is, to the best of our knowledge, the first non-trivial result for this problem.

1 Introduction

A wireless ad-hoc network consists of several transceivers, communicating by radio. Each transceiver t is assigned a transmission power $p(t)$, which gives it some transmission range, denoted by r_t. This is customary to assume that the minimal transmission power required to transmit to a distance d is d^α, where the *distance-power gradient* α is usually taken to be in the interval $[2, 4]$ (see [22]). Thus, a transceiver s receives transmissions from t if $p(t) \geq d(t,s)^\alpha$, where $d(t,s)$ is the Euclidean distance between t and s. The transmission possibilities resulting from a power assignment induce a communication graph. Research efforts have focused on finding power assignments, for which the induced communication graph satisfies a certain connectivity property, while minimizing the total cost.

This paper is organized as follows. In the rest of this section we present the model, previous work and briefly describe our own contribution results. Sections 2 and 3 deal with linear radio networks and planar networks, respectively. Finally, we conclude in Section 4.

1.1 The model

We are given a system of n transceivers $t_1, t_2, \ldots, t_n$, positioned in $\mathbb{R}^d$, $d \geq 1$. Denote such a system by $S_n = (T, D)$, where T is the set of transceivers and D denotes the distances between them. When each transceiver is assigned a transmission power $p(t) = r_t^\alpha$, an ad-hoc network is created. A power assignment for S_n is a vector of transmission powers $\{p(t) \mid t \in T\}$ and is denoted by $A(S_n)$ (usually abbreviated to A). The resulting communication (directed) graph is denoted by $H_A = (T, E_A)$, where E_A is the set of directed edges resulting from the power assignment $A(S_n)$:

$$E_A = \{(t,s) \mid p(t) \geq d(t,s)^\alpha\}.$$

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Namely there is a directed edge from t to s if t has sufficient transmission power to reach s. Throughout this paper we address transceivers as nodes. The cost C_A of the assignment is the sum of all transmission powers:

$$C_A = \sum_{t \in T} p(t).$$

In [2], a weighted version of the model is considered. To each transceiver t we correspond an *energy unit cost* parameter γ_t, which measures the cost associated with assigning one unit of power to t. Note that different transceivers may have distinct energy unit costs due to differences between them, their environments and other factors. Let $W_n = \{\gamma_t \mid t \in T\}$ be the vector of unit costs. The cost of an assignment A with respect to a vector of weights W_n is:

$$C_{A,W_n} = \sum_{t \in T} \gamma_t p(t).$$

When the transceivers are positioned on a line, e.g. antennas along a highway, the resulting network is often called a *linear radio network*. In this kind of system we only need to consider the distances between adjacent transceivers rather than between any pair of transceivers. Put $d_i = d(t_i, t_{i+1})$, $1 \leq i \leq n-1$ and

$$d^L_{i,k} = \begin{cases} d(t_i, t_1), & 1 \leq i \leq k, \\ d(t_i, t_{i-k}), & k < i \leq n, \end{cases} \qquad d^R_{i,k} = \begin{cases} d(t_i, t_{i+k}), & 1 \leq i \leq n-k, \\ d(t_i, t_n), & n-k < i \leq n. \end{cases}$$

Denote such a linear system of transceivers by $L_n = (T, D)$ where $D = (d_i)_{i=1}^{n-1}$. A linear system with uniform distances $d_1 = d_2 = \ldots = d_{n-1}$ is denoted by U_n.
Recall that a graph $G = (V, E)$ is *k-connected* if for any two nodes $u, v \in V$ there are k vertex-disjoint paths connecting u to v. Equivalently, G is k-connected if it remains connected after omitting any set of up to $k-1$ vertices. Our main problem in this paper is:

Problem 1.1 ($K_C(S_n, k)$).

Input: A system S_n of transceivers.
Output: A power assignment $A(S_n)$, where H_A is k-connected, with a minimal possible cost C_A.

We shall also consider the following generalization:

Problem 1.2 ($WK_C(S_n, W_n, k)$).

Input: A system S_n of transceivers, a vector W_n of unit costs.
Output: A power assignment $A(S_n)$, where H_A is k-connected, with a minimal possible cost C_{A,W_n}.

1.2 Previous work

Kirousis et al. [19] introduced the `MinRange(SC)` problem, which is $K_C(S_n, k)$ for $k = 1$. They proved it to be **NP-Hard** for the three dimensional Euclidean space for any value of α. The same paper provided a 2-approximation algorithm for the planar case and an exact $O(n^4)$ time algorithm for the one dimensional case. In the planar case, the NP-hardness of the problem for every α has been proved in [13] and a simple 1.5-approximation algorithm for the case $\alpha = 1$ has been provided in [3]. Some researchers add an additional constraint parameter to the problem, the bounded diameter h of the induced communication graph, see results in [11, 14, 15]. Ambuhl et al. [2] presented some algorithms for the weighted power assignment, solving it optimally for the broadcast, multi-source broadcast and strong connectivity problems for the linear case (they achieved the same running time for the connectivity problem as in [19]). They also presented some approximation algorithms for the multi-dimensional case. An excellent survey covering many variations of the problem is given in [12].
A natural generalization of the strong connectivity requirement is k-connectivity. These networks also provide multi-path redundancy for load balancing or transmission fault tolerance. As power-optimal strong connectivity is **NP-Hard**, so is power-optimal k-connectivity. Two versions of the

problem arise: symmetric and asymmetric. In the symmetric version for any two nodes $t, s \in T$, $p(t) \geq d(t,s)^\alpha \Leftrightarrow p(s) \geq d(s,t)^\alpha$, that is a node t can reach node s if and only if s can reach node t, we can also refer to it as an undirected model. The asymmetric version allows directed links between two nodes. Krumke et al. [20] argued that the asymmetric version is harder than the symmetric version. Another possible connectivity property is k-edge connectivity, which implies that the removal of any k edges results in a disconnected graph. In [7], Calinescu and Wan presented various aspects of symmetric/asymmetric k-connectivity and k-edge connectivity. They first proved **NP-Hardness** of the symmetric two-edge and two-node connectivity and then provided a 4-approximation algorithm for both symmetric and asymmetric biconnectivity ($k = 2$) and a $2k$-approximation for both symmetric and asymmetric k-edge connectivity. Hajiaghayi et al. [17] give two algorithms for symmetric k-connectivity, with $O(k \log k)$ and $O(k)$-approximation factors and also some distributed approximation algorithms for $k = 2$ and $k = 3$ in geometric graphs. Jia et al. in [16] present various approximation factors (depending on k) for the symmetric k-connectivity, such as $3k$-approximation algorithm for any $k \geq 3$ and 6-approximation for $k = 3$. Additional results can be found in [1, 5, 6, 9, 14, 21]. It is worth mentioning that unless otherwise specified, all the algorithms are centralized.

Other relevant work in the area of energy efficient power assignment includes energy-efficient broadcasting and multicasting in wireless networks. The problem, given a source node s, is to find a minimum power assignment such that the induced communication graph contains a spanning tree rooted at s. This problem was proved to be **NP-Hard**. In [10, 18, 25, 26], authors presented some heuristic solutions and gave some theoretical analysis. Srinivas and Modiano in [24] provided a polynomial algorithm that optimally finds k node-disjoint paths for a given pair of nodes while minimizing the total node power needed on these k node-disjoint paths. They also provide a polynomial algorithm for solving the 2 edge-disjoint paths problem.

1.3 Our contribution

We provide an optimal solution for the $K_C(U_n, k)$ problem. Then we give a $\min\{2, (\frac{\Delta}{\delta})^\alpha\}$-approximation for the $K_C(L_n, k)$ problem, where $\Delta = \max_{1 \leq i \leq n-1} d_i$ and $\delta = \min_{1 \leq i \leq n-1} d_i$. Eventually we solve the weighted version of the linear radio networks – the $WK_C(L_n, W_n, k)$ problem and obtain an approximation factor of $\min\left\{\frac{\Gamma}{\gamma}(\frac{\Delta}{\delta})^\alpha, \left(1 + \frac{\Delta}{\delta}\right)^\alpha, 2\frac{\Gamma}{\gamma}\right\}$, where $\Gamma = \max_{1 \leq i \leq n} \gamma_i$ and $\gamma = \min_{1 \leq i \leq n} \gamma_i$. All our algorithms have an $O(\min\{n \log n, n(n-k)\})$ runtime. Finally we present a polynomial time $O(k^2)$ approximation algorithm for the two dimensional instance of $K_C(S_n, k)$.

2 Linear Radio Networks

Let $L_n = (T, D)$ be a linear radio network, and let $A(L_n) = \{p(t) \mid t \in T\}$. We assume that the nodes are sorted by their x coordinate, that is if $i < j$ then t_i is to the left of t_j. We denote by $N_R(t_i)$ the set of right neighbours of transceiver t_i : $N_R(t_i) = \{t_j : i < j, d(t_i, t_j)^\alpha \leq p(t_i)\}$). The left neighbors are defined similarly and denoted by $N_L(t_i)$. All neighbors are defined as $N(t_i) = N_R(t_i) \cup N_L(t_i)$.

Definition 2.1. For a given power assignment $A(L_n)$ a node t_i is *r-reachable from the left* if there are at least r nodes to the left of t_i with sufficient power assignment to reach t_i:

$$|\{t_j : j < i, \; t_i \in N_R(t_j)\}| \geq r$$

Similarly we can define *r-reachability from the right.*

Definition 2.2. A line is *r-reachable from the left* if every node t is r-reachable from the left. We define *r-reachability from the right* in the same manner.

Definition 2.3. A power assignment $A(L_n)$ is *k-reachable* if every node t_i is both $\min\{i - 1, k\}$-reachable from the left and $\min\{n - i, k\}$-reachable from the right.

Definition 2.4. For a given power assignment $A(L_n)$ node $t_i \in T$ is *k-connected to* node $t_j \in T$ if there are k vertex-disjoint paths from t_i to t_j. We say that nodes $t_i, t_j \in T$ are *k-connected* if node t_i is *k-connected* to node t_j and node t_j is *k-connected* to node t_i.

We say that a power assignment $A(L_n)$ forms a k-connected line if the communication graph H_A is k-connected. The same stands for k-reachability, that is a power assignment $A(L_n)$ forms a k-reachable line $A(L_n)$ if H_A is k-reachable.

2.1 Properties of k-connectivity and k-reachability

Property 2.5. Let $L_n = (T, D)$ be a linear radio network. Take any power assignment A that forms a k-connected line. It holds for all $t \in T$ that $|N(t)| \geq k$.

Proof. Let $L_n = (T, D)$ be a linear radio network. Take any power assignment $A(L_n)$ that forms a k-connected line. Take any pair of nodes $t_i, t_j \in T$. There are at least k paths from t_i to t_j. Let $F \subseteq N$ be the set of first nodes in each path (the first from each path). Since $F \subseteq N(t_i)$ and $|F| \geq k$ we conclude that $N(t_i) \geq k$. ■

Lemma 2.6. Let $L_n = (T, D)$ be a linear radio network. Take any power assignment $A(L_n)$ that forms a k-connected line. Every node t_i is $\min\{i-1, k\}$-reachable from left and $\min\{n-i, k\}$-reachable from right.

Proof. By symmetry, it is sufficient to prove left reachability. Suppose $i \leq k$. According to 2.5 every node t_j, $j < i$ should have a power assignment $p(t_j) \geq d(t_j, t_{k+1})^\alpha > d(t_j, t_i)^\alpha$. Therefore there are $i-1$ nodes that connect to i. Now assume that $i > k$. In order that node t_1 will be k-connected to t_i, there must exist k nodes having t_i as a right neighbour. As a result, every node t_i is $\min\{i-1, k\}$-reachable from left. ■

Corollary 2.7. If a power assignment $A(L_n)$ forms a k-connected line, then $A(L_n)$ forms a k-reachable line.

Lemma 2.8. Let $L_n = (T, D)$ be a linear radio network. The power assignment $A = \{p(t_i) = (d^R_{i,k})^\alpha \mid t_i \in T\}$ is an optimal assignment that forms a k-reachable line from the left.

Proof. It is easy to see that $A(L_n)$ forms a k-reachable line from the left. Let us prove the optimality of the assignment. We prove it by contradiction. Suppose there exist a power assignment $A'(L_n) = \{p'(t) = {r'_t}^\alpha \mid t \in T\}$ such that $C_{A'} < C_A$. Let t_j be the rightmost node with a transmission range $r'_{t_j} = d^R_{j,f} < d^R_{j,k}$, where $0 \leq f < k$. Since A' is k-reachable from the left, there exists a node t_i, $1 \leq i < j$ with a range assignment of $r'_{t_i} = d^R_{i,j+k-i}$, otherwise t_{j+k} is not k-reachable from the left. Note that because A' is optimal the range assignment of t_i cannot exceed reaching t_{j+k}. Let $A''(L_n) = \{p''(t) \mid t \in T\}$, where $p''(t_i) = (d^R_{i,j+f-i})^\alpha$, $p''(t_j) = (d^R_{j,k})^\alpha$ and $p''(t) = p'(t)$, for $t \neq t_i, t_j$ (see Figure 1). It is clear that k-reachability of the line remains unchanged so we need to compare $C_{A'}$ and $C_{A''}$. We compare the power assignments of t_i and t_j as the rest remain unchanged. We use the fact that for any convex function f it holds

$$f(a+b) + f(b+c) \leq f(a+b+c) + f(b) \quad \forall a, b, c \geq 0$$

Let $x = d^R_{i,j-i}$, $y = d^R_{j,f-j}$, $z = d^R_{f+j,k-f}$. Thus,

$$p'(t_i) + p'(t_j) - (p''(t_i) + p''(t_j)) = (x+y+z)^\alpha + b^\alpha - (x+y)^\alpha - (y+z)^\alpha \geq 0$$

Therefore $C_{A''} \leq C_{A'}$. We can continue to improve the cost of the assignment by taking the rightmost node t_j with assigned power that is less than $d^R_{j,k}$ and increase it to $d^R_{j,k}$, while decreasing the power of some other node. In this fashion we obtain an assignment $\tilde{A} = A$ since in the end every node t_i has a power assignment $d^R_{i,k}$ just like in A. In addition, because we are constantly improving we have $C_{\tilde{A}} \leq C_{A'}$. But $C_{\tilde{A}} = C_A$. We have reached a contradiction, and therefore $A(L_n)$ is optimal. ■

Due to symmetry, the optimal assignment that forms a k-reachable line from the right is $A = \{p(t_i) = (d^L_{i,k})^\alpha \mid t_i \in T\}$.

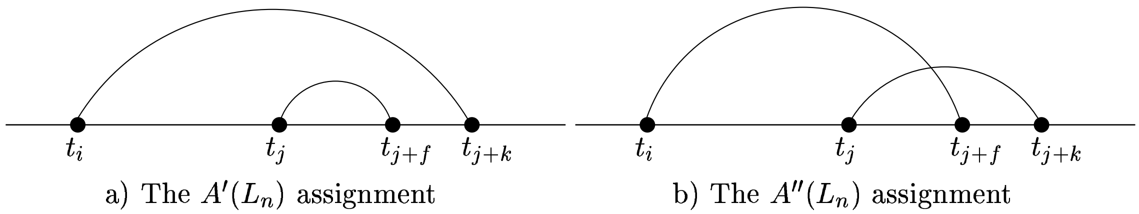

Figure 1: $C''_A < C'_A$

2.2 Unit distances

First we provide an optimal solution for the $K_C(U_n, k)$ problem.

Theorem 2.9. Let $U_n = (T, D)$ be a linear radio network with distances $d_i = \delta$, for $1 \leq i \leq n-1$. Then the power assignment $A^*(U_n) = \{p(t) = (k\delta)^\alpha \mid t \in T\}$ is unique and optimal solution for the $K_C(U_n, k)$ problem.

We prove the following lemma first.

Lemma 2.10. The communication graph H_A resulting from the power assignment $A^*(U_n)$ in Theorem 2.9 is k-connected.

Proof. We need to prove that there are k disjoint paths from t_i to t_j for arbitrary $1 \leq i, j \leq n$. Note that every node t_i has $|N_R(t_i)| \geq \min\{n-i, k\}$ and $|N_L(t_i)| \geq \min\{i-1, k\}$. Without loss of generality assume $i < j$. We consider three possibilities (see Figure 2). The first possibility is $j - i \geq k$: then we simply jump right to the neighbours of t_i (there are at least k) and then continue to jump k neighbours to the right until we go over t_j. The second possibility is $j - i < k$ and $i \leq n - k$: then our first jump is to one of the neighbours of t_i to the right and then immediately to t_j (unless we landed on t_j straightaway). The third and final possibility is $j - i < k$ and $i > n - k$. In this case we jump right to $n - i$ neighbours and then to t_j, which gives us $n - i$ paths. We also jump to $k - (n - i)$ neighbours to the left and immediately to t_j. ■

Proof of Theorem 2.9. According to Corrolary 2.7 every power assignment A that forms a k-connected line also forms a k-reachable line. It easy to see that A^* forms a k-reachable line from both left and right and is optimal by Lemma 2.8. According to Lemma 2.10 the assignment A^* forms a k-connected line. We conclude that A^* is an optimal solution for the $K_C(U_n, k)$ problem. The assignment is unique because in A^* every node $t \in T$ is assigned $p_t = (k\delta)^2$. ■

2.3 Arbitrary distances

Next, we provide an approximation algorithm for the more general problem $K(L_n, k)$. Given a linear radio network $L_n = (D, T)$, let $\delta = \min_{1 \leq i \leq n-1} d_i$ and $\Delta = \max_{1 \leq i \leq n-1} d_i$. Let A^* be the optimal solution for the $K(L_n, k)$.

Lemma 2.11. $C_{A^*} \geq n(k\delta)^\alpha$.

Proof. Let $U'_n = (D', T)$ be a linear radio network with distances $d'_i = \delta$, for $1 \leq i \leq n-1$. Also let A' be the optimal solution for the $K(U'_n, k)$ problem. It is easy to see that $C_{A^*} \geq C_{A'}$. According to Theorem 2.9 $C_{A'} = n(k\delta)^\alpha$ and therefore $C_{A^*} \geq n(k\delta)^\alpha$. ■

Lemma 2.12. Let $L_n = (T, D)$ be a linear radio network. The power assignment $A(U_n) = \{p(t) = (k\Delta)^\alpha \mid t \in T\}$ is a solution for the $K_C(L_n, k)$ problem and it holds $C_A \leq \left(\frac{\Delta}{\delta}\right)^\alpha C_{A^*}$.

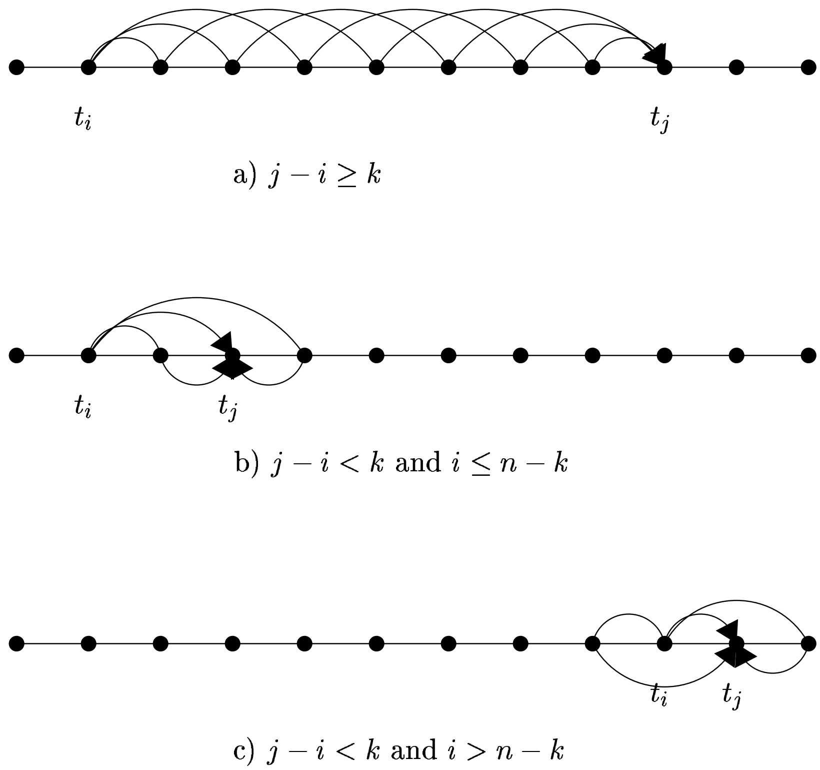

Figure 2: The three possibilities with $k = 3$

Proof. It is easy to see that power assignment A forms a k-connected line. The proof is similar to the one presented in Lemma 2.10, which uses the fact that every node t_i has at least $\min\{i - 1, k\}$ ($\min\{n - i, k\}$) neighbours to its left (right), respectively. According to Lemma 2.11 $C_{A^*} \geq n(k\delta)^\alpha$ and therefore

$$\frac{C_A}{C_{A^*}} = \frac{n(k\Delta)^\alpha}{C_{A^*}} \leq \frac{nk^\alpha \Delta^\alpha}{nk^\alpha \delta^\alpha} = \left(\frac{\Delta}{\delta}\right)^\alpha$$

■

Next we try to improve on the ratio of the approximation. We propose an assignment that is just enough for every node to reach k neighbours from either side. We argue that such an assignment is no worse than 2 times the optimal cost.

Theorem 2.13. Let $L_n = (T, D)$ be a linear radio network. The power assignment $A(L_n) = \left\{p(t_i) = (\max\{d^L_{i,k}, d^R_{i,k}\})^\alpha \mid t_i \in T\right\}$ is a solution for the $K_C(L_n, k)$ problem and it holds $C_A \leq 2C_{A^*}$.

Proof. The power assignment A forms a k-connected line, based on the proof in Lemma 2.10. Let A_L be the the optimal assignment that forms a line k-reachable from the left, and let A_R be the corresponding assignment providing right k-reachability. According to Lemma 2.8

$$C_{A_R} = \sum_{i=1}^{n} (d^L_{i,k})^\alpha \text{ and } C_{A_L} = \sum_{i=1}^{n} (d^R_{i,k})^\alpha$$

By Corollary 2.7 we conclude that $C_{A^*} \geq C_{A_L}$ and $C_{A^*} \geq C_{A_R}$. Finally we note that $C_A \leq C_{A_L} + C_{A_R}$ and as a result we obtain $C_A \leq 2C_{A^*}$. ■

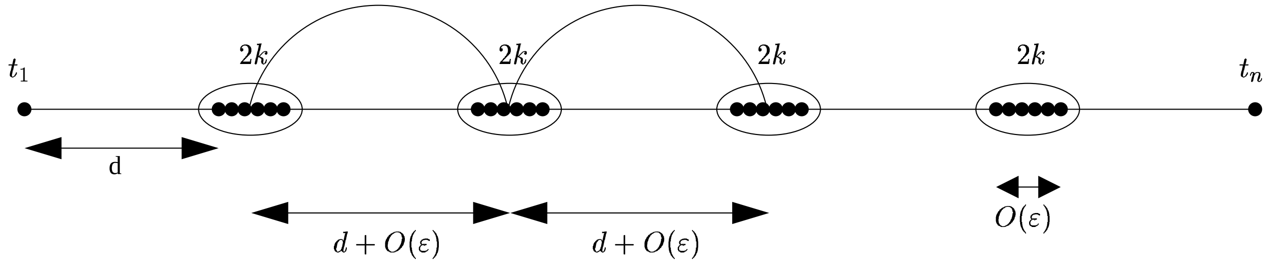

Figure 3: Ratio $\frac{C_A}{C_{A^*}} = 2 + O(\varepsilon^\alpha)$

Figure 3 shows that the approximation factor 2 of our algorithm is tight. The nodes are spread on the line in groups of $2k$ except for the first and last. In every group the nodes are equaly spaced at a distance of ε. Assume $n >> k$, $d >> \varepsilon$ and ε being infinitely small.
We obtain a ratio infinitely close to 2 in Figure 3 because the optimal solution has to assign the k middle nodes of every group a range assignment of $d^\alpha + O((d+\varepsilon)^{\alpha-1})$, and the rest of the nodes in the group should be assigned a transmission power of $O(\varepsilon^\alpha)$. In addition $p(t_1) = p(t_n) = d^\alpha + O((d+\varepsilon)^{\alpha-1})$. The total cost would be

$$C_{A^*} = \left(\frac{n-2}{2} + 2\right)(d^\alpha + O(d+\varepsilon)^{\alpha-1}) + \frac{n-2}{2}O(\varepsilon^\alpha)$$

By using our algorithm each node is assigned a transmission power of $d^\alpha + O((d+\varepsilon)^{\alpha-1})$, because it needs to reach the k-th neighbour. And so the total cost would be

$$C_A = n(d^\alpha + O(d+\varepsilon^\alpha))$$

It is easy to see that the approximation ratio is tight.

Corollary 2.14. Let $L_n = (T, D)$ be a linear radio network. The power assignment $A(L_n) = \left\{p(t_i) = (\max\{d^L_{i,k}, d^R_{i,k}\})^\alpha \mid t_i \in T\right\}$ is a solution for the $K_C(L_n, k)$ problem and it holds

$$C_A \leq \min\left\{2, \left(\frac{\Delta}{\delta}\right)^\alpha\right\} C_{A^*}$$

Proof. Follows immediately from Lemma 2.12 and Theorem 2.13. ∎

2.4 Weighted instance

We finish the linear radio networks section by giving an approximate solution to the $WK_C(L_n, W_n, k)$ problem. Given a linear radio network $L_n = (D, T)$ and the energy unit cost vector W_n, let δ and Δ be as before and let $\gamma = \min_{1\leq i\leq n} \gamma_i$ and $\Gamma = \max_{1\leq i\leq n} \gamma_i$. First we prove two lemmas based on the results achieved for unit and arbitrary distances. Let A^* be the optimal solution for the $WK(L_n, k)$.

Lemma 2.15. $C_{A^*} \geq n\gamma(k\delta)^\alpha$.

Proof. Let $L'_n = (D', T)$ be a linear radio network with distances $d'_i = \delta$ for $1 \leq i \leq n-1$ and an energy unit costs vector W_n with weights $\gamma_i = \gamma$, for $1 \leq i \leq n$. Let $U''_n = (D'', T)$ be a linear radio network with distances $d''_i = \delta\sqrt[\alpha]{\gamma}$ for $1 \leq i \leq n-1$. Let A' and A'' be the optimal solutions for the $K_C(L'_n, k)$ and $K_C(U''_n, k)$ problems respectively. Easy to see that $C_{A'} = C_{A''}$ and that $C_{A^*} \geq C_{A'} = C_{A''}$. According to Theorem 2.9 $C_{A''} = n(k\delta\sqrt[\alpha]{\gamma})^\alpha = n\gamma(k\delta)^\alpha$ and therefore $C_{A^*} \geq n\gamma(k\delta)^\alpha$. ∎

Lemma 2.16. Let $L_n = (T, D)$ be a linear radio network with energy unit costs vector W_n. The power assignment $A(L_n) = \{p(t) = (k\Delta)^\alpha \mid t \in T\}$ is a solution for the $WK_C(L_n, W_n, k)$ problem and it holds $C_A \leq \frac{\Gamma}{\gamma}\left(\frac{\Delta}{\delta}\right)^\alpha C_{A^*}$.

Proof. The proof is similar to the one given in Lemma 2.12 and it uses Lemma 2.15 for the lower bound. ∎

Lemma 2.17. Let $L_n = (T, D)$ be a linear radio network with energy unit costs vector W_n. The power assignment $A(L_n) = \left\{p(t_i) = (\max\{d^L_{i,k}, d^R_{i,k}\})^\alpha \mid t_i \in T\right\}$ is a solution for the $WK_C(L_n, W_n, k)$ problem and it holds $C_A \leq 2\frac{\Gamma}{\gamma}C_{A^*}$.

Proof. As in Lemma 2.10, the assignment $A(L_n)$ is a solution. Let $L^1_n = (T, D)$ be a linear radio network with energy unit costs vector W^1_n so that $\gamma^1_i = 1$. Similar to the proof of Theorem 2.13, let A^1_L be the the optimal assignment that forms a line k-reachable from the left given a linear radio network L^1 and let A^1_R be the corresponding assignment providing right k-reachability. Easy to see that $\gamma C_{A^1_L} \leq C_{A^*}$ and $\gamma C_{A^1_R} \leq C_{A^*}$ due to Corollary 2.7. It is also clear that $C_A \leq \Gamma\left(C_{A^1_L} + C_{A^1_R}\right)$. As a result we have $C_A \leq 2\frac{\Gamma}{\gamma}C_{A^*}$. ∎

Now we prove another bound. The 2-approximation achieved in the previous section will not fit here, since the optimal assignment that forms a k-reachable line unnecessarily also forms a k-connected line as in Theorem 2.13. As before we try to assign a node just enough power to reach k neighbours from either side. For any node t_i let $d_{i,k}$ be the radius so that if $p(i) = (d_{i,k})^\alpha$ then $|N(t_i)| \geq k$. We start from the following simple observation.

Observation 2.18. For any $1 \leq l \leq n-1$ and $1 \leq i, j \leq n-l$, given two intervals of $l+1$ nodes long each, (t_i, t_{i+l}) and (t_j, t_{j+l}), it holds

$$d(t_i, t_{i+l}) \leq \frac{\Delta}{\delta}d(t_j, t_{j+l}) \text{ and } d(t_j, t_{j+l}) \leq \frac{\Delta}{\delta}d(t_i, t_{i+l})$$

Lemma 2.19. For any node $t_i \in T$ it holds $\max\{d^L_{i,k}, d^R_{i,k}\} \leq (1 + \frac{\Delta}{\delta})d_{i,k}$.

Proof. Take some node t_i. Suppose that as a result of a power assignment $p(t_i) = (d_{i,k})^\alpha$, t_i reaches f nodes to its left (i.e. $|N_L(t_i)| = f$) and $k - f$ nodes to its right(i.e. $|N_R(t_i)| = k - f$).
Without loss of generality we prove $d^L_{i,k} \leq (1+\frac{\Delta}{\delta})d_{i,k}$. Let $l = i-f$ be the index of the leftmost node reached from t_i. In order to reach the k-th node to its left t_i needs to reach additional $m = k - f$ nodes from that side (see Figure 4). We can conclude that $d^L_{i,k} \leq d_{i,k} + d^L_{l,m}$. By Observation 2.18 we have $d^L_{l,m} \leq \frac{\Delta}{\delta}d_{i,k}$. As a result we have $d^L_{i,k} \leq (1 + \frac{\Delta}{\delta})d_{i,k}$. ∎

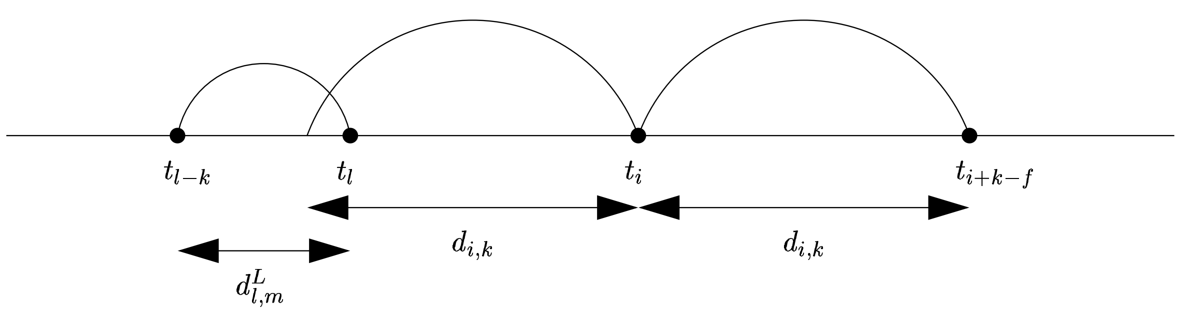

Figure 4: $\max\{d^L_{i,k}, d^R_{i,k}\} \leq (1 + \frac{\Delta}{\delta})d_{i,k}$

We are ready to prove the main Theorem.

Theorem 2.20. Let $L_n = (T, D)$ be a linear radio network with energy unit costs vector W_n. The power assignment $A(L_n) = \left\{p(t_i) = (\max\{d^L_{i,k}, d^R_{i,k}\})^\alpha \mid t_i \in T\right\}$ is a solution for the $WK_C(L_n, W_n, k)$ problem and it holds $C_A \leq \left(1 + \frac{\Delta}{\delta}\right)^\alpha C_{A^*}$.

Proof. According to definition of $d_{i,k}$ and Property 2.5 we have $C_{A^*} \geq \sum_{i=1}^{n} \gamma_i (d_{i,k})^\alpha$. By Lemma 2.19 we have $\max\{d^L_{i,k}, d^R_{i,k}\} \leq (1 + \frac{\Delta}{\delta}) d_{i,k}$ and therefore

$$
\begin{aligned}
C_A &= \sum_{i=1}^{n} \gamma_i p(t_i) = \sum_{i=1}^{n} \gamma_i (\max\{d^L_{i,k}, d^R_{i,k}\})^\alpha \leq \sum_{i=1}^{n} \gamma_i \left(1 + \frac{\Delta}{\delta}\right)^\alpha (d_{i,k})^\alpha \\
&= \left(1 + \frac{\Delta}{\delta}\right)^\alpha \sum_{i=1}^{n} \gamma_i (d_{i,k})^\alpha \leq \left(1 + \frac{\Delta}{\delta}\right)^\alpha C_{A^*}
\end{aligned}
$$

■

Corollary 2.21. Let $L_n = (T, D)$ be a linear radio network with energy unit costs vector W_n. The power assignment $A(L_n) = \left\{p(t_i) = (\max\{d^L_{i,k}, d^R_{i,k}\})^\alpha \mid t_i \in T\right\}$ is a solution for the $WK_C(L_n, W_n, k)$ problem and it holds

$$C_A \leq \min\left\{\frac{\Gamma}{\gamma}\left(\frac{\Delta}{\delta}\right)^\alpha, \left(1 + \frac{\Delta}{\delta}\right)^\alpha, 2\frac{\Gamma}{\gamma}\right\} C_{A^*}$$

Proof. Follows immediately from Lemma 2.16, Lemma 2.17 and Theorem 2.20. ■

All our algorithms need to compute for each node the k-th neighbour to its left and right. It can be done very fast using an approach of posets described by Segal and Kedem [23], which has a running time of $O(\min\{n \log n, n(n-k)\})$.

3 Planar case

We move on to a problem in which the transceivers are located in the plane. As we mentioned in the introduction, the problem is hard for the planar case even if $k = 1$. Here we provide a $O(k^2)$-approximation algorithm which is based on finding a Hamiltonian cycle first and then assigning each transceiver enough power to reach $k/2$ transceivers in both directions in the cycle. A Hamiltonian cycle, is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once. A graph possessing a Hamiltonian circuit is said to be a Hamiltonian graph or simply Hamiltonian. Below we prove the correctness of the above scheme. For simplicity, we assume $\alpha = 2$. We start our analysis from definitions.

3.1 Definitions

Given a system of n transceivers S_n we define $G_c(S_n) = (T, E, \vec{c})$ to be a complete graph where $T = \{t \mid t \in T\}$ is a set of nodes, E is the set of directed edges between all pairs of nodes and a cost vector $\vec{c} = \left\{c(t,s) = d(t,s)^2 \mid t, s \in T\right\}$. The complete graph $G_c(S_n)$ is said to follow the r-triangle inequality if for any nodes $t, s, q \in T$ it holds $c(t,q) \leq r \cdot (c(t,s) + c(s,q))$. Easy to see that in our case $r = 2$. This is because the nodes are placed in the Euclidean plane, that is for every $t, s, q \in T$ it holds that $d(t,q) \leq d(t,s) + d(s,q)$. We use Cauchy-Shwartz to obtain for every $t_i, t_j, t_l \in T : d(t_i, t_l)^2 \leq 2\left(d(t_i, t_j)^2 + d(t_j, t_l)^2\right)$. Moreover, it can be proved that $r = 2^{\alpha-1}$ for any $\alpha \geq 2$.
Let us denote by A^*_k the optimal solution of the $K_c(S_n, k)$ problem. We say that an assignment A forms a k-connected graph if H_A is k-connected. We denote by $p_A(t)$ the power assignment of node t in A.

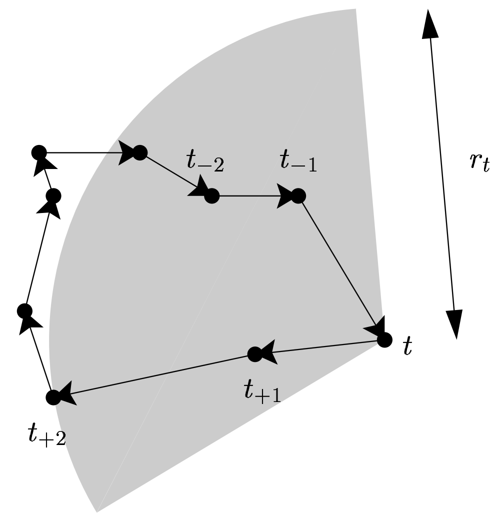

Figure 5: Power assignment of t with $k = 2$

Given a graph $G = (V, E, \vec{c})$ we denote by $h(G)$ some Hamiltonian cycle in G. We use the following notation for a Hamiltonian cycle in G starting at some node t

$$h(G) = \{t = t_{+0}, t_{+1}, \ldots, t_{+(n-1)}, t_{+n} = t\}$$
$$\text{or}$$
$$h(G) = \{t = t_{-n}, t_{-(n-1)}, t_{-(n-2)}, \ldots, t_{-1}, t_{-0} = t\}$$

Where nodes $(t_{+i})_{i=0}^{n}$ and $(t_{-i})_{i=0}^{n}$ are disjoint and $t_{+i}, t_{+(i+1)}$ are two consecutive nodes in the Hamiltonian cycle $h(G)$. We denote by $C_{h(G)} = \sum_{i=0}^{n-1} c(t_{+i}, t_{+(i+1)})$ the cost of a Hamiltonian cycle. We denote the optimal Hamiltonian (with the minimized cost) by $h^*(G)$. For any Hamiltonian cycle $h(G)$ let $d_k^+(t) = \sum_{i=0}^{k-1} d(t_{+i}, t_{+(i+1)})$ and $d_k^-(t) = \sum_{i=0}^{k-1} d(t_{-i}, t_{-(i+1)})$.

3.2 The algorithm

First we want to compute a 2-connected communication graph. Since finding A_2^* is **NP-Hard** we find a minimal cost asymmetric biconnectivity as described in [7]. Then we build a Hamiltonian cycle as described in [4]. Afterwards we assign power to nodes according to Hamiltonian cycle construction.

Lemma 3.1. Let S_n be a system of transceivers in the plane. Let $A_2(S_n)$ be a power assignment so that H_{A_2} is 2-connected. Then there exists a Hamiltonian cycle h' so that $C_{h'} \leq 8C_{A_2}$

Proof. Let $UD(A_2)$ be an *improved* assignment of assignment A_2, that is if t reaches s in A_2 then the power of s in $UD(A_2)$ is increased (if necessary) to reach t in $UD(A_2)$. It is clear that $C_{UD(A_2)} \leq 2C_{A_2}$. We follow the proof in Theorem 1 in [4] and build a Hamiltonian cycle h' based on $UD(A_2)$ with cost $C_{h'} \leq 2rC_{UD(H_{A_2})}$, where $r = 2$. The inequality follows from the proof of Theorem 1 in [4] because for any node $t \in T$ it holds $d(t, t_{+1})^2 \leq 2r \cdot p_{UD(A_2)}(t)$. That is each node is assigned no more than $2r$ times its power in $UD(A_2)$ while constructing the Hamiltonian cycle. As a result we obtain $C_{h'} \leq 4C_{UD(A_2)} \leq 8C_{A_2}$. ∎

Our algorithm uses the same approach as the one used for constructing a k-connected line. The original direction of a Hamiltonian cycle will be called the *clockwise* direction and the opposite *counter-clockwise* direction. We want each node to reach $k/2$ nodes in clockwise and $k/2$ nodes in a counter-clockwise direction (for simplicity we assume k is even). Figure 5 gives an example for $k = 2$.

Algorithm(PLANE-K-CONNECT)

1. Find a Hamiltonian cycle h' as described in Lemma 3.1, where A_2 is the assignment given in [7] for asymmetric biconnectivity.

2. Assign $A_k = \left\{p(t) = \max\left\{(d^+_{k/2}(t))^2, (d^-_{k/2}(t))^2\right\} \quad | \quad t \in T\right\}$

Theorem 3.2. The assignment A_k in the algorithm PLANE-K-CONNECT solves the $K_C(S_n, k)$ problem and it holds $C_{A_k} \leq 16k^2 C_{A^*_k}$.

Proof. First we proof that H_{A_k} is k-connected. It is clear that for every node t the set $\{t_{+1}, \dots, t_{+k/2}\} \subseteq N_R(t)$ and $\{t_{-k/2}, t_{-(k/2-1)}, \dots, t_{-1}\} \subseteq N_L(t)$. Now it is easy to see that H_{A_k} is k-connected. This is due to the same arguments as given in 2.10. This time for any pair of nodes $t, s \in T$, t can reach s by using $k/2$ paths going in clockwise direction and using $k/2$ pathes going in counter-clockwise direction.
Next we prove the approximation ratio. Due to Cauchy-Shwartz inequality we have that

$$(d^+_{k/2}(t))^2 \leq \frac{k}{2}\sum_{i=0}^{\frac{k}{2}-1} d(t_{+i}, t_{+(i+1)})^2 \text{ and also } (d^-_{k/2}(t))^2 \leq \frac{k}{2}\sum_{i=0}^{\frac{k}{2}-1} d(t_{-i}, t_{-(i+1)})^2$$

And as a result we have:

$$C_{A_k} = \sum_{t\in T} \max\left\{(d^+_{k/2}(t))^2, (d^-_{k/2}(t))^2\right\} \leq \sum_{t\in T}(d^+_{k/2}(t))^2 + \sum_{t\in T}(d^-_{k/2}(t))^2 \leq \sum_{t\in T}\frac{k}{2}\sum_{i=0}^{\frac{k}{2}-1} d(t_{+i}, t_{+(i+1)})^2 +$$
$$\sum_{t\in T}\frac{k}{2}\sum_{i=0}^{\frac{k}{2}-1} d(t_{-i}, t_{-(i+1)})^2 = \left[\text{Notice that each edge in the cycle is "bought"} \frac{k}{2} \text{ times in each direction}\right]$$
$$\frac{k}{2}\left(\frac{k}{2}\sum_{t\in T} d(t_{+0}, t_{+1})^2 + \frac{k}{2}\sum_{t\in T} d(t_{-0}, t_{-1})^2\right) = \frac{k^2}{4}(2C_{h'}) \leq \frac{k^2}{4}(16C_{A_2}) = 4k^2 C_{A_2} \leq 16k^2 C_{A^*_2}$$

The last inequality is due to asymmetric biconnectivity results in [7]. Since $A^*_k \geq A^*_2$ we obtain $C_{A_k} \leq 16k^2 C_{A^*_k}$ ∎

Remark: The result above can be improved by a constant factor using a theorem by Chvatal and Erdos in [8] that proves an existence of Hamiltonian cycle for every k-connected graph G with $n \geq 3$ and $k \geq \beta$, where β is the size of maximum independent set of G, has a Hamiltonian cycle. Notice that for every k-connected graph we have $\beta \leq \frac{n}{k}$ (follows immediately from Lemma 2.5). If $k \in \Omega(\sqrt{n})$ then it holds $C_{A^*_k} \geq C_{h^*(G_c)}$. Following a similar method to the one described above we can obtain an assignment A_k that forms a k-connecetd line so that $C_{A_k} \leq 8k^2 C_{h^*(G_c)}$. As a result we obtain an approximation factor of $8k^2$.

4 Conclusions and Future Work

In this paper we addressed the problem of power assignment in wireless ad-hoc networks. We have presented an optimal solution for a uniform linear radio network and provided fast constant factor approximation algorithms for the more general case of linear networks. We have also given a non-trivial $O(k^2)$ approximation algorithm for the planar case.

Our work opens several directions for future research. First, it is of interest to find a better approximation factor for the planar case. It would also be of interest to explore the problem of asymmetric k-connectivity for multicast and broadcast problems. Here, given a source node s we would like to have a power assignment so that in the induced communication graph there are k vertex-disjoint pathes from s to every other node.

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References

[1] E. Althaus, G. Calinescu, I. Mandoiu, S. Prasad, N. Tchervenski and A. Zelikovsly, "Power Efficient Range Assignment in Ad-Hoc Wireless Networks", *IEEE Wireless Communications and Networking Conference (WCNC03)*, 2003

[2] C. Ambuhl, A. Clementi, M. Ianni, G. Rossi, A. Monti, R. Silvestri, "The Range Assignment Problem in Non-Homogeneous Static Ad-Hoc Networks , *2nd International Conference on AD-HOC Networks and Wireless (AdHoc-Now 03)*, 2004

[3] C.Ambuhl. A. Clementi, P. Penna, G. Rossi and R. Silvestri, "Energy Consumption in Radio Networks: Selfish Agents and Rewarding Mechanisms", *Proceedings of the 10th International Colloquium on Structural Information & Communication Complexity (SIROCCO)*, pp. 1–16, 2003

[4] M. A. Bender and C. Chekuri, "Performance Guarantees for the TSP with a Parameterized Triangle Inequality", *Information Processing Letters*, Vol. 73(1-2), pp. 17-21, 2000

[5] D. Blough, M. Leoncini, G. Resta, and P. Santi, "On the Symmetric Range Assignment Problem in Wireless Ad Hoc Networks", *Proceedings of the 2nd IFIP International Conference on Theoretical Computer Science (TCS)*, pp. 71–82, 2002

[6] G. Calinescu, I. Mandoiu and A. Zelikovsly, "Symmetric Connectivity with Minimum Power Consumption in Radio Networks", *Proc. 2nd IFIP International Conference on Theoretical Computer Science*, pp. 119–130, 2002

[7] G. Calinescu and P. J. Wan, "Range assignment for High Connectivitity in Wireless Ad Hoc Networks", *2nd International Conference on AD-HOC Networks and Wireless (AdHoc-Now 03)*, pp. 235–246, 2003

[8] V. Chvatal and P. Erdos, "A Note on Hamiltonian Circuits", *Discrete Math*, Vol. 2, pp. 111–113, 1972

[9] J. Cheriyan, S. Vempala and A. Vetta, "Approximation algorithms for minimum-cost k-vertex connected subgraphs", *Proceedings of the thiry-fourth annual ACM symposium on Theory of computing*, pp. 306-312, 2002

[10] A. E. F. Clementi, P. Crescenzi, P. Penna, G. Rossi and P. Vocca, "On the Complexity of Computing Minimum Energy Consumption Broadcast Subgraphs", *18th Annual Symposium on Theoretical Aspects of Computer Science*, pp. 121–131, 2001

[11] A. E. F. Clementi, A. Ferreira, P. Penna, S. Perennes, and R. Silvestri, "The minimum range assignment problem on linear radio networks", *European Symposium on Algorithms*, pp. 143–154, 2000

[12] A. E. F. Clementi, G. Huiban, P. Penna, G. Rossi and Y. C. Verhoeven, "Some Recent Theoretical Advances and Open Questions on Energy Consumption in Ad-Hoc Wireless Networks", *3rd Workshop on Approximation and Randomization Algorithms in Communication Networks*, 2003

[13] A. E. F. Clementi, P. Penna and R. Silvestri, "Hardness Results for the Power Range Assignment Problem in Packet Radio Networks", *Proceedings of the 2nd International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, pp. 197–208, 1999

[14] A. E. F. Clementi, P. Penna and R. Silvestri, "On the Power Assignment Problem in Radio Networks", *Electronic Colloquium on Computational Complexity (ECCC)*, 2000

[15] A. E. F. Clementi, P. Penna and R. Silvestri, "The Power Range Assignment Problem in Radio Networks on the Plane", *Proceedings of the 17th Annual Symposium on Theoretical Aspects of Computer Science*, pp. 651-660, 2000

[16] X. Jia, D. Kim, P. Wan and C. Yi, "Power Assignment for k-Connectivity in Wireless Ad Hoc Networks", *Proceedings of the IEEE INFOCOM*, 2005

[17] M. T. Hajiaghayi, N. Immorlica and V. S. Mirrokni, "Power Optimization in Fault-Tolerant Topology Control Algorithms for Wireless Multi-Hop Networks", *Proceedings of the 9th Annual International Conference on Mobile Computing and Networking*, pp. 300–312, 2003

[18] G. Huiban and Y. C. Verhoeven, "A Self-Stabilized Distributed Algorithm for the Range Assignment in Ad-Hoc Wireless Networks", *Soumis Parallel Processing Letters*, ??

[19] L. M. Kirousis, E. Kranakis, D. Krizanc and A. Pelc, "Power Consumption in Packet Radio Networks" *Theoretical Computer Science*, Vol. 243(1-2), pp. 289-305, 2000

[20] S. O. Krumke, R. Liu, E. L. Lloyd, M. V. Marathe, R. Ramanathan and S. S. Ravi, "Topology Control Problems under Symmetric and Asymmetric Power Thresholds" *ADHOC-NOW 2003*, pp. 187–198, 2003

[21] E. L. Lloyd, R. Liu, M. V. Marathe, R. Ramanathan and S. S. Ravi, "Algorithmic Aspects of Topology Control Problems for Ad Hoc Networks", *MONET 10(1-2)*, pp. 19–34, 2005

[22] K. Pahlavan and A. Levesque, "Wireless Information Networks", *Wiley-Interscience*, 1995

[23] M. Segal and K. Kedem, "Geometric Applications Of Posets", *WADS*, pp. 402–417, 1997

[24] A. Srinivas and E. Modiano, "Minimum Energy Disjoint Path Routing in Wireless Ad-hoc, *Proceedings of the 9th annual international conference on Mobile computing and networking*, pp. 122-133, 2003

[25] P. J. Wan, G. Calinescu, X. Y. Li and O. Frieder, "Minimum-Energy Broadcast Routing in Static Ad Hoc Wireless Networks", *ACM Wireless Networks*, 2002

[26] J. Wieselthier, G. Nguyen and A. Ephremides, "On the Construction of Energy-Efficient Broadcast and Multicast Trees in Wireless Networks", *Proceedings of the IEEE INFOCOM*, pp. 586-594, 2000