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## European Journal of Combinatorics

journal homepage: [www.elsevier.com/locate/ejc](http://www.elsevier.com/locate/ejc)

# On the non-existence of lattice tilings by quasi-crosses<sup>☆</sup>

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## ARTICLE INFO

## Article history:

Received 14 December 2012

Accepted 4 May 2013

## ABSTRACT

We study necessary conditions for the existence of lattice tilings of  $\mathbb{R}^n$  by quasi-crosses. We prove general non-existence results using a variety of number-theoretic tools. We then apply these results to the two smallest unclassified shapes, the  $(3, 1, n)$ -quasi-cross and the  $(3, 2, n)$ -quasi-cross. We show that for dimensions  $n \leq 250$ , apart from the known constructions, there are no lattice tilings of  $\mathbb{R}^n$  by  $(3, 1, n)$ -quasi-crosses except for 13 remaining unresolved cases, and no lattice tilings of  $\mathbb{R}^n$  by  $(3, 2, n)$ -quasi-crosses except for 19 remaining unresolved cases.

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## 1. Introduction

Problems involving tilings of  $\mathbb{R}^n$  by clusters of cubes have a long history, as is evident from the early work of Minkowski [14]. In this context, let

$$Q = \{(x_1, \dots, x_n) \mid 0 \leq x_i < 1, x_i \in \mathbb{R}\}$$

denote the *unit cube*, which, we shall also say, is *placed at the origin*. A unit cube placed at  $\bar{e} \in \mathbb{R}^n$  is a *translate* of the cube

$$\bar{e} + Q = \{\bar{e} + \bar{x} \mid \bar{x} \in Q\},$$

and a *cluster of cubes* is a union of disjoint translates of cubes

$$c = \mathcal{E} + Q = \{\bar{e} + Q \mid \bar{e} \in \mathcal{E}\},$$

for some  $\mathcal{E} \subseteq \mathbb{R}^n$ .

<sup>☆</sup> This work was supported in part by ISF grant 134/10.

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A set of disjoint translates of  $\mathcal{C}$  is called a *packing* of  $\mathbb{R}^n$  by  $\mathcal{C}$ . If, in addition, the union of the translates is the entire space  $\mathbb{R}^n$ , we say it is a *tiling*. If the set of translates forming the packing (tiling) forms an additive subgroup of  $\mathbb{Z}^n$ , we shall say it is a *lattice<sup>2</sup> packing* (lattice tiling).

Several types of clusters have been considered in the past. The two most studied clusters are the  $(k, n)$ -cross and the  $(k, n)$ -semi-cross. The  $(k, n)$ -cross is defined by the following set of translates

$$\mathcal{E}_{\text{cross}} = \{i\bar{e}_j \in \mathbb{R}^n \mid i \in [-k, k], j \in [n]\}$$

where  $[a, b] = \{a, a + 1, \dots, b\} \subseteq \mathbb{Z}$ ,  $[a]$  is short for  $[1, a]$ ,  $[a, b]^* = [a, b] \setminus \{0\}$ , and  $\bar{e}_j$  is the  $j$ -th standard unit vector. That is, a  $(k, n)$ -cross contains a center cube, and arms of length  $k$  cubes in the positive and negative directions along each axis. In contrast, the  $(k, n)$ -semi-cross has arms only in the positive directions and is defined by

$$\mathcal{E}_{\text{semi-cross}} = \{i\bar{e}_j \in \mathbb{R}^n \mid i \in [0, k], j \in [n]\}.$$

Packings of  $\mathbb{R}^n$  by crosses and semi-crosses, mainly using lattices, were studied by Hamaker and Stein [5,6], Hickerson and Stein [8], and Stein [18]. For an excellent survey the reader is referred to Stein and Szabó [19]. We also note that a  $(1, n)$ -cross is also a Lee sphere of radius 1. Apart from radius 1 or dimension 2, the non-existence of tilings of  $\mathbb{R}^n$  by Lee spheres is a long-standing conjecture by Golomb and Welch [4] (see [3,9,10,15], as well as the more recent [11] for a survey on the current status of the conjecture).

Motivated by an application to error-correcting codes for non-volatile memories, Schwartz suggested in [16] a generalization of both the cross and semi-cross to a shape called the  $(k_+, k_-, n)$ -quasi-cross defined by the set of translations

$$\mathcal{E}_{\text{quasi-cross}} = \{i\bar{e}_j \in \mathbb{R}^n \mid i \in [-k_-, k_+], j \in [n]\}.$$

Namely, in a  $(k_+, k_-, n)$ -quasi-cross the center cube has arms of length  $k_+$  in the positive directions, and arms of length  $k_-$  in the negative directions (see Fig. 1). Thus, a  $(k, 0, n)$ -quasi-cross is simply a  $(k, n)$ -semi-cross, while a  $(k, k, n)$ -quasi-cross is a  $(k, n)$ -cross. To avoid the two well-studied cases (for example, see [5,6,8,12,13,18]) we shall assume throughout that  $1 \leq k_- < k_+$ .

A few constructions were given in [16] for lattice tilings of  $\mathbb{R}^n$  by quasi-crosses, and in particular, a full classification was provided of the dimensions in which there exist lattice tilings by  $(2, 1, n)$ -quasi-crosses. Recently, Yari, Kløve, and Bose [22], gave other constructions for lattice packings and tilings by quasi-crosses, and in particular, new constructions for tilings by  $(3, 1, n)$ -quasi-crosses.

The motivation given in [16] is that of producing perfect 1-error-correcting codes for the unbalanced limited magnitude channel, a natural extension to the earlier work of [2]. The dual case of  $(n - 1)$ -error-correcting codes gives rise to a tiling problem of a cluster of cubes called a “chair”, which is described by Buzaglo and Etzion in [1].

The goal of this work is to derive new general necessary conditions for the existence of tilings of  $\mathbb{R}^n$  by quasi-crosses. To demonstrate these tools we shall apply them to the two smallest unclassified cases of the  $(3, 1, n)$ -quasi-cross and the  $(3, 2, n)$ -quasi-cross.

The paper is organized as follows. We begin in Section 2 by providing the notation and definitions used throughout the paper. We shall also cite relevant results from previous works. We continue in Section 3 with a list of the main results. We conclude in Section 4 with the application of the general results to the specific case of tilings by  $(3, 1, n)$ -quasi-crosses and tilings by  $(3, 2, n)$ -quasi-crosses.

## 2. Preliminaries

In this section we provide notation and definitions used throughout this work, and cite specific relevant results which will be used in the following sections.

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<sup>2</sup> This is, in fact, an *integer* lattice, but we shall omit this throughout the paper.

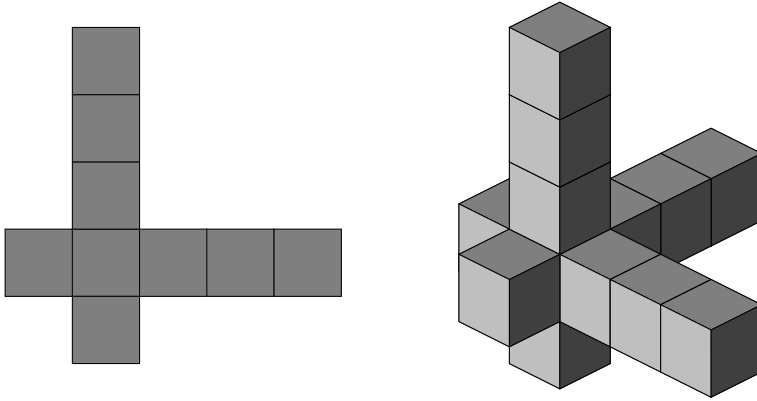


Fig. 1. A (3, 1, 2)-quasi-cross and a (3, 1, 3)-quasi-cross.

2.1. Abelian-group splitting and lattice tiling

While we may use geometric arguments to prove necessary conditions for a shape to tile  $\mathbb{R}^n$ , stronger results may be obtained using the algebraic structure of a lattice tiling. An equivalence between lattice tilings and Abelian-group splitting was described in [8,17,18], which we describe here for completeness.

Let  $G$  be a finite Abelian group, where we shall denote the group operation as  $+$ . Given some  $s \in G$  and a non-negative integer  $m \in \mathbb{Z}$ , we denote by  $ms$  the sum  $s + s + \dots + s$ , where  $s$  appears in the sum  $m$  times. The definition is extended in the natural way to negative integers  $m$ .

A splitting of  $G$  is a pair of sets,  $M \subseteq \mathbb{Z} \setminus \{0\}$ , called the multiplier set, and  $S = \{s_1, s_2, \dots, s_n\} \subseteq G$ , called the splitter set, such that the elements of the form  $ms$ ,  $m \in M$ ,  $s \in S$ , are all distinct, non-zero, and cover all the non-zero elements of  $G$ . We shall denote such a splitting as  $G = (M, S)$ . It follows that  $|M| \cdot |S| = |G| - 1$ .

Next, we define a homomorphism  $\phi : \mathbb{Z}^n \rightarrow G$  by

$$\phi(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i s_i.$$

If the multiplier set is  $M = [-k_-, k_+ ]^*$ , then it may be easily verified that  $\ker \phi$  is a lattice tiling of  $\mathbb{R}^n$  by  $(k_+, k_-, n)$ -quasi-crosses. The fact that  $\ker \phi$  is a lattice is obvious. To show that the lattice is a packing by  $(k_+, k_-, n)$ -quasi-crosses, assume to the contrary two such distinct quasi-crosses, one placed at  $\bar{x} = (x_1, \dots, x_n)$  and one placed at  $\bar{y} = (y_1, \dots, y_n)$ , have a non-empty intersection, i.e.,

$$\bar{x} + m_1 \bar{e}_i = \bar{y} + m_2 \bar{e}_j,$$

where  $m_1, m_2 \in M$ , then

$$m_1 s_i = \phi(\bar{x} + m_1 \bar{e}_i) = \phi(\bar{y} + m_2 \bar{e}_j) = m_2 s_j$$

which is possible only if  $m_1 = m_2$  and  $i = j$ , resulting in the two quasi-crosses being the same one, a contradiction.

Finally, to show that the packing is in fact a tiling let  $\bar{x} \in \mathbb{R}^n$  be some point in the space. Obviously,  $\bar{x} \in [\bar{x}] + Q$ . If  $\phi([\bar{x}]) = 0$  then  $[\bar{x}] \in \ker \phi$  and  $\bar{x}$  is in the  $(k_+, k_-, n)$ -quasi-cross cube cluster placed at  $[\bar{x}]$ . Otherwise, by the properties of the splitting there exist  $m \in M$  and  $s_i \in S$  such that  $\phi([\bar{x}]) = ms_i$ . It follows that  $[\bar{x}] - m\bar{e}_i \in \ker \phi$  and  $\bar{x}$  is in the  $(k_+, k_-, n)$ -quasi-cross cube cluster placed at  $[\bar{x}] - m\bar{e}_i$ .

Group splitting as a method for constructing error-correcting codes was also discussed, for example, in the case of shift-correcting codes [20] and integer codes [21].

2.2. Previous results

Several results from previous works are relevant to this one. Some apply directly to quasi-crosses, while others will be used as a basis for our new results, appearing in the next sections. The first theorem we cite is the only one whose proof uses geometric arguments to derive a necessary condition on lattice tilings of  $\mathbb{R}^n$  by quasi-crosses.

**Theorem 1** ([16, Theorem 9]). For any  $n \geq 2$ , if

$$\frac{2k_+(k_- + 1) - k_-^2}{k_+ + k_-} > n,$$

then there is no lattice tiling of  $\mathbb{R}^n$  by  $(k_+, k_-, n)$ -quasi-crosses.

When looking for a lattice tiling using the group-splitting equivalence, the question is which finite Abelian group to split. It was demonstrated in [16] that splitting different Abelian groups of the same size may result in different lattice tilings. However, since we are only interested in finding necessary conditions for the existence of lattice tilings, the following theorem from [16] (which is a generalization of a theorem from [19]) shows that we may focus only on cyclic groups.

**Theorem 2** ([16, Theorem 15]). Let  $G$  be a finite Abelian group, and let  $M = [-k_-, k_+]^*$  be the multiplier set corresponding to the  $(k_+, k_-, n)$ -quasi-cross. If there is a splitting  $G = (M, S)$ , then there is a splitting of the cyclic group of the same size  $\mathbb{Z}_{|G|} = (M, S')$ .

Using Theorem 2 we can say that the  $(k_+, k_-, n)$ -quasi-cross lattice tiles  $\mathbb{R}^n$  if and only if  $\mathbb{Z}_q = (M, S)$ , where  $q = (k_+ + k_-)n + 1$  and  $M = [-k_-, k_+]^*$ . Furthermore, the expressions  $ms$ , for  $m \in M$  and  $s \in S$ , simply denote integer multiplication in the ring  $\mathbb{Z}_q$ . To avoid confusion, we shall denote the multiplicative semi-group of the ring  $\mathbb{Z}_q$  as  $R_q$ .

Another useful result is the following.

**Theorem 3** ([16, Theorem 16]). Let  $k \geq 2$  be some positive integer, and let  $M = [-(k - 1), k]^*$ . If  $G = (M, S)$  is a splitting of an Abelian group  $G$ ,  $|G| > 1$ , then  $\gcd(k, |G|) \neq 1$ .

A notion we shall find useful is that of a *character*, as defined by Stein [17]: a character is a homomorphism  $\chi : G \rightarrow R$  from a semi-group  $G$  into a (multiplicative) semi-group  $H$ . The following theorem, with a one-line proof that we bring for completeness, is due to Stein.<sup>3</sup>

**Theorem 4** ([17, Theorem 4.1]). Let us consider a splitting  $\mathbb{Z}_q = (M, S)$  and let  $\chi : R_q \rightarrow R$  be a character from  $R_q$  into a ring  $R$ . Then

$$\left( \sum_{m \in M} \chi(m) \right) \cdot \left( \sum_{s \in S} \chi(s) \right) = \sum_{a \in R_q} \chi(a).$$

**Proof.**

$$\sum_{a \in R_q} \chi(a) = \sum_{\substack{m \in M \\ s \in S}} \chi(ms) = \sum_{\substack{m \in M \\ s \in S}} \chi(m)\chi(s) = \left( \sum_{m \in M} \chi(m) \right) \cdot \left( \sum_{s \in S} \chi(s) \right). \quad \square$$

<sup>3</sup> The version due to Stein is somewhat more general, but we shall not require the full generality of the original claim.

Several characters will be of interest in the following section, the first is the Legendre symbol: for an odd prime  $p$  we define the character  $\left(\frac{\cdot}{p}\right) : R_p \rightarrow \mathbb{C}$  as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & a \equiv x^2 \pmod{p} \text{ for some } x \in R_p, \\ -1 & \text{otherwise.} \end{cases}$$

If  $\left(\frac{a}{p}\right) = 1$  we call  $a$  a *quadratic residue modulo  $p$*  (QR), and otherwise we call  $a$  a *quadratic non-residue modulo  $p$*  (QNR). Using the Legendre symbol and [Theorem 4](#) Stein proved the following theorem.

**Theorem 5** ([17, Corollary 4.3]). *If  $\mathbb{Z}_p = (M, S)$  is a splitting,  $p$  an odd prime, then in at least one of  $M$  or  $S$  the number of quadratic residues equals the number of quadratic non-residues.*

We recall some well-known facts about the Legendre symbol, which we shall use later. For a proof, see for example [7].

**Lemma 6.** *Let  $p$  be an odd prime, and let  $\ell$  denote some integer. Then*

$$\begin{aligned} \left(\frac{-1}{p}\right) &= 1 \quad \text{iff} \quad p = 4\ell + 1, \\ \left(\frac{2}{p}\right) &= 1 \quad \text{iff} \quad p = 8\ell \pm 1, \\ \left(\frac{3}{p}\right) &= 1 \quad \text{iff} \quad p = 12\ell \pm 1, \\ \left(\frac{5}{p}\right) &= 1 \quad \text{iff} \quad p = 10\ell \pm 1. \end{aligned}$$

### 3. Main results

In this section we list our new results, where we group them according to the method employed to derive the necessary condition for lattice tiling  $\mathbb{R}^n$  by  $(k_+, k_-, n)$ -quasi-crosses.

#### 3.1. The generalized Legendre symbol

We can extend the methods used in [Theorem 5](#) by considering higher-order Legendre symbols to obtain necessary conditions for  $(k_+, k_-, n)$ -quasi-cross to lattice tile  $\mathbb{R}^n$  when  $k_+ + k_-$  is a prime. To that end we first need a simple lemma.

**Lemma 7.** *Let  $p$  be a prime and set  $\omega = e^{2\pi i/p}$ ,  $i = \sqrt{-1}$ . If  $a_0, \dots, a_{p-1} \in \mathbb{Q}$  are rational numbers such that  $\sum_{j=0}^{p-1} a_j \omega^j = 0$ , then  $a_0 = a_1 = \dots = a_{p-1}$ .*

**Proof.** Define the polynomial  $a(x) = \sum_{j=0}^{p-1} a_j x^j \in \mathbb{Q}[x]$ . It is therefore given that  $a(\omega) = 0$ , and hence all the conjugates of  $\omega$  relative to  $\mathbb{Q}$  are also roots of  $a(x)$ . It is well-known (see for example [7]) that these are  $\omega^j$  where  $\gcd(j, p) = 1$ . Since  $p$  is a prime, we have that all of  $\omega^j$ ,  $1 \leq j \leq p-1$ , are also roots of  $a(x)$ , i.e.,

$$(x - \omega^1)(x - \omega^2) \cdots (x - \omega^{p-1}) \mid a(x).$$

However,

$$(x - \omega^1)(x - \omega^2) \cdots (x - \omega^{p-1}) = \frac{x^p - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{p-1}.$$

We now have

$$1 + x + x^2 + \cdots + x^{p-1} \mid a(x)$$

while the degree of  $a(x)$  is at most  $p - 1$ , resulting in

$$a(x) = c(1 + x + x^2 + \dots + x^{p-1}),$$

for some constant  $c \in \mathbb{Q}$ .  $\square$

**Theorem 8.** Let  $1 \leq k_- < k_+$  be positive integers such that  $k_+ + k_-$  is an odd prime. If the  $(k_+, k_-, n)$ -quasi-cross lattice tiles  $\mathbb{R}^n$ , and  $(k_+ + k_-)n + 1$  is a prime, then  $k_+ + k_- \mid n$ .

**Proof.** Denote  $q = (k_+ + k_-)n + 1$ , and assume  $\mathbb{Z}_q = (M, S)$  is a splitting with  $M = [-k_-, k_+)^*$ . Let  $g$  be a primitive element in  $\mathbb{Z}_q$ , which is a field as  $q$  is prime.

We also denote  $p = k_+ + k_-$ , an odd prime, and let  $\omega = e^{2\pi i/p}$  be a complex  $p$ -th root of unity. We define the character  $\chi : R_q \rightarrow \mathbb{C}$  as  $\chi(g^j) = \omega^j$ . We note that  $p \mid q - 1$ . By Theorem 4 we have

$$\left(\sum_{m \in M} \chi(m)\right) \cdot \left(\sum_{s \in S} \chi(s)\right) = \sum_{j=1}^{q-1} \chi(j) = \sum_{j=0}^{q-2} \chi(g^j) = \sum_{j=0}^{q-2} \omega^j = \frac{\omega^{q-1} - 1}{\omega - 1} = 0$$

since  $\omega^{q-1} = 1$  and  $\omega \neq 1$ . It then follows that

$$\sum_{m \in M} \chi(m) = 0 \quad \text{or} \quad \sum_{s \in S} \chi(s) = 0.$$

We first check the characters of the elements in  $M$ . We have  $1 \in M$  and necessarily  $\chi(1) = 1 = \omega^0$ . We also have  $-1 \in M$ , and since  $(-1)^2 = 1$ , we get  $\chi(-1)^2 = \chi(1) = 1$ , but  $p$  is an odd prime and so  $\chi(-1) = 1 = \omega^0$  also. If  $\sum_{m \in M} \chi(m) = 0$  then by Lemma 7 each power of  $\omega$  appears an equal number of times, and since we have  $p$  powers and  $p$  summands, each should appear exactly once. However,  $\omega^0$  appears at least twice, and so  $\sum_{m \in M} \chi(m) \neq 0$ .

It now follows that we must have  $\sum_{s \in S} \chi(s) = 0$ , which again by Lemma 7 implies that  $k_+ + k_- = p \mid n$ , as claimed.  $\square$

### 3.2. The power character

A different flavor of necessary conditions is obtained by examining the power character which we now define: for any fixed positive integer  $r$ , the function  $\chi_r : R_q \rightarrow R_q$  defined by  $\chi_r(a) = a^r$ , is a character we call the *power character*.

**Theorem 9.** Let  $\mathbb{Z}_q = (M, S)$  be a splitting,  $n = |S| < q - 1$ . If  $q$  is a prime, then

$$\sum_{m \in M} m^i \equiv 0 \pmod{q}$$

for some  $1 \leq i \leq n$ .

**Proof.** For every  $1 \leq i \leq n$  we consider the power character  $\chi_i : R_q \rightarrow R_q$  defined by  $\chi_i(a) = a^i$ . By Theorem 4 we therefore have

$$\left(\sum_{m \in M} m^i\right) \cdot \left(\sum_{s \in S} s^i\right) \equiv \sum_{j=1}^{q-1} j^i \pmod{q} \tag{1}$$

for all  $1 \leq i \leq n$ .

Let  $g$  be a primitive element in  $\mathbb{Z}_q$ , which is a field as  $q$  is prime. We can then write

$$\sum_{j=1}^{q-1} j^i \equiv \sum_{j=0}^{q-2} g^{ij} \equiv \frac{g^{i(q-1)} - 1}{g^i - 1} \equiv 0 \pmod{q}$$

since  $g^i \not\equiv 0 \pmod{q}$  for all  $1 \leq i \leq n < q - 1$ .

Since  $\mathbb{Z}_q$  is a field, it now follows from (1), that for all  $1 \leq i \leq n$  we have

$$\sum_{s \in S} s^i \equiv 0 \pmod{q} \quad \text{or} \quad \sum_{m \in M} m^i \equiv 0 \pmod{q}.$$

Assume to the contrary that for all  $1 \leq i \leq n$  we have

$$\sum_{m \in M} m^i \not\equiv 0 \pmod{q}.$$

If we define the matrix

$$V = \begin{pmatrix} s_1^1 & s_1^2 & \cdots & s_1^n \\ s_2^1 & s_2^2 & \cdots & s_2^n \\ \vdots & \vdots & \ddots & \vdots \\ s_n^1 & s_n^2 & \cdots & s_n^n \end{pmatrix}$$

then it follows that

$$(1, 1, \dots, 1)V \equiv (0, 0, \dots, 0) \pmod{q}$$

and so

$$\det(V) \equiv 0 \pmod{q}.$$

However,  $V$  is clearly a Vandermonde matrix, and the elements of  $S$  are distinct and non-zero, which implies

$$\det(V) = \prod_{j=1}^n s_j \cdot \prod_{j < j'} (s_j - s_{j'}) \not\equiv 0 \pmod{q},$$

a contradiction.  $\square$

Slightly less general results may be achieved by considering specific power characters.

**Theorem 10.** *There is no lattice tiling of  $\mathbb{R}^n$  by  $(4k - 1, 1, n)$ -quasi-crosses for all positive integers  $k$  such that  $kn \equiv 5, 8 \pmod{9}$ .*

**Proof.** Let us assume to the contrary that there exists a splitting  $\mathbb{Z}_{4kn+1} = (M, S)$  with  $M = [-1, 4k - 1]^*$  and  $|S| = n$ . Consider the power character  $\chi_2 : \mathbb{R}_{4kn+1} \rightarrow \mathbb{R}_{4kn+1}$  defined by  $\chi_2(a) = a^2$ . By Theorem 4 it follows that

$$\left( \sum_{m \in M} m^2 \right) \cdot \left( \sum_{s \in S} s^2 \right) \equiv \sum_{i=1}^{4kn} i^2 \pmod{4kn + 1}.$$

By a simple induction one can easily prove that

$$3 \left| (-1)^2 + \sum_{i=1}^{4k-1} i^2 \right.$$

for all  $k \geq 1$ . Thus, we can write

$$3t \sum_{s \in S} s^2 \equiv \frac{4kn(4kn + 1)(8kn + 1)}{6} \pmod{4kn + 1} \tag{2}$$

for some integer  $t$ , where we used the well-known identity

$$\sum_{i=1}^a i^2 = \frac{a(a + 1)(2a + 1)}{6}.$$

We now note that  $kn \equiv 5, 8 \pmod{9}$  implies  $4kn + 1 \equiv 3, 6 \pmod{9}$ , and so 3 is a zero divisor in  $R_{4kn+1}$ . The LHS of (2) is a multiple of 3. On the other hand, in the RHS of (2),  $\frac{4kn}{2}$ ,  $\frac{4kn+1}{3}$ , and  $8kn + 1$ , are all integers leaving non-zero residue modulo 3. This is a contradiction.  $\square$

**Theorem 11.** *There is no lattice tiling of  $\mathbb{R}^n$  by  $(4k + 2, 1, n)$ -quasi-crosses for all positive integers  $k$ , and  $n \equiv 3, 7 \pmod{8}$ .*

**Proof.** The proof is similar to that of Theorem 10. Assume to the contrary that there exists a splitting  $\mathbb{Z}_{(4k+3)n+1} = (M, S)$  with  $M = [-1, 4k + 2]^*$  and  $|S| = n$ . Consider the power character  $\chi_3 : R_{(4k+3)n+1} \rightarrow R_{(4k+3)n+1}$  defined by  $\chi_3(a) = a^3$ . By Theorem 4 it follows that

$$\left(\sum_{m \in M} m^3\right) \cdot \left(\sum_{s \in S} s^3\right) \equiv \sum_{i=1}^{(4k+3)n} i^3 \pmod{(4k+3)n+1}.$$

By a simple induction one can easily prove that

$$8 \mid (-1)^3 + \sum_{i=1}^{4k+2} i^3$$

for all  $k \geq 1$ . Thus, we can write

$$8t \sum_{s \in S} s^3 \equiv \frac{((4k+3)n)^2((4k+3)n+1)^2}{4} \pmod{(4k+3)n+1} \tag{3}$$

for some integer  $t$ , where we used the identity

$$\sum_{i=1}^a i^3 = \frac{a^2(a+1)^2}{4}.$$

At this point we note that  $n \equiv 3, 7 \pmod{8}$  implies  $(4k+3)n+1 \equiv 2, 6 \pmod{8}$ , and so 2 is a zero divisor in  $R_{(4k+3)n+1}$ . The LHS of (3) is a multiple of 2. On the other hand, the RHS of (3) is odd since both  $((4k+3)n)^2$ , and  $\frac{((4k+3)n+1)^2}{4}$ , are odd integers. This is a contradiction.  $\square$

More elaborate results of the same flavor may be reached by using other power characters.

### 3.3. Unique representation

By carefully examining the way specific elements of the split group are represented we may sometimes reach a contradiction to the unique representation of the group elements required by the splitting. The following few results illustrate this method. We introduce the useful notion of the *primorial*, defined as

$$n\# = \prod_{\substack{p \text{ prime} \\ p \leq n}} p.$$

**Lemma 12.** *If an integer  $d$  divides  $(k_+ + k_-)n + 1$ ,  $\gcd(d, k_+\#) = 1$ , and  $n < d < (k_+ + k_-)n + 1$ , then the  $(k_+, k_-, n)$ -quasi-cross does not lattice tile  $\mathbb{R}^n$ .*

**Proof.** Denote  $q = (k_+ + k_-)n + 1$ . Assume to the contrary that there is a splitting  $\mathbb{Z}_q = (M, S)$  with  $M = [-k_-, k_+]^*$ . We note that  $d$  is a zero divisor in  $\mathbb{Z}_q$  but not zero itself. According to the splitting, there is a unique representation  $d \equiv ms \pmod{q}$  with  $m \in M$  and  $s \in S$ . Since  $\gcd(d, k_+\#) = 1$  it follows that  $\gcd(d, m) = 1$  and therefore  $d \mid s$ . Denote, then,  $s = ds'$ .

Since  $d > n$  we have

$$\frac{q}{d} = \frac{(k_+ + k_-)n + 1}{d} \leq k_+ + k_-.$$



Thus, there exist  $m_1, m_2 \in M, m_2 \leq k_-$ , such that

$$m_1 + m_2 = \frac{q}{d}.$$

Then,

$$m_1s + m_2s = \frac{q}{d}s = qs' \equiv 0 \pmod{q},$$

and so

$$m_1s \equiv -m_2s \pmod{q}.$$

Since  $m_1, -m_2 \in M$  we have a contradiction to the splitting.  $\square$

The previous lemma gives rise to the following theorem.

**Theorem 13.** For any  $1 < r < k_+ + k_-$ , the  $(k_+, k_-, n)$ -quasi-cross does not lattice tile  $\mathbb{R}^n$  when

$$(k_+ + k_-)n + 1 \equiv ru \pmod{r \cdot k_+\#}$$

for some integer  $u$  such that  $\gcd(u, k_+\#) = 1$ .

**Proof.** We first note that reducing the requirement on  $n$  modulo  $r$  gives

$$(k_+ + k_-)n + 1 \equiv 0 \pmod{r}.$$

Thus,  $\frac{(k_+ + k_-)n + 1}{r}$  is an integer and

$$\frac{(k_+ + k_-)n + 1}{r} \equiv u \pmod{k_+\#}.$$

We can now use Lemma 12 with  $d = \frac{(k_+ + k_-)n + 1}{r} > n$ , and the claim follows.  $\square$

### 3.4. Recursion

Recursion is also a powerful tool for formulating necessary conditions for tilings. We present a simple recursion which may be used in several ways to rule out lattice tilings.

**Theorem 14.** If there is a splitting  $\mathbb{Z}_q = (M, S)$ , with  $M = [-k_-, k_+ ]^*$ , and some positive integer  $d \mid q, \gcd(d, k_+\#) = 1$ , then

$$(k_+ + k_-)d \mid q - d,$$

and there is a splitting  $\mathbb{Z}_{q/d} = (M, S')$ .

**Proof.** Let us consider the subgroup of  $\mathbb{Z}_q$  defined by

$$H = d\mathbb{Z} \cap \mathbb{Z}_q = \left\{ 0, d, 2d, \dots, \left(\frac{q}{d} - 1\right)d \right\}.$$

Each element  $id \in H, 1 \leq i \leq q/d - 1$ , has a unique representation as

$$id \equiv ms \pmod{q} \tag{4}$$

with  $m \in M$  and  $s \in S$ . Since  $d$  is a zero divisor in  $\mathbb{Z}_q$ , and  $\gcd(d, k_+\#) = 1$ , it follows that  $\gcd(d, m) = 1$  and  $d \mid s$ . Denote  $s = dt$  and reduce (4) modulo  $q/d$  to get

$$i \equiv mt \pmod{\frac{q}{d}}.$$

Define  $S' = \{s' \mid ds' \in S\}$ . Since every element of  $H$  has a unique factorization as in (4), it follows that  $\mathbb{Z}_{q/d} = (M, S')$  is indeed a splitting. Furthermore, the size of  $S'$ ,

$$|S'| = \frac{|\mathbb{Z}_{q/d}| - 1}{|M|} = \frac{q - d}{(k_+ + k_-)d}$$

must be an integer.  $\square$

The following two corollaries follow immediately from **Theorem 14**: the first is in fact a recursive construction, while the second may be used to prove the non-existence of lattice tilings.

**Corollary 15.** *If the  $(k_+, k_-, n)$ -quasi-cross lattice tiles  $\mathbb{R}^n$ , and for some positive integer  $d \mid (k_+ + k_-)n + 1$  we have  $\gcd(d, k_+\#) = 1$ , then the  $(k_+, k_-, n')$ -quasi-cross lattice tiles  $\mathbb{R}^{n'}$ ,  $n' = \frac{(k_+ + k_-)n + 1 - d}{(k_+ + k_-)d}$ .*

**Corollary 16.** *If there exists a positive integer  $d \mid (k_+ + k_-)n + 1$ ,  $\gcd(d, k_+\#) = 1$ , but  $\frac{(k_+ + k_-)n + 1 - d}{(k_+ + k_-)d}$  is not an integer, then the  $(k_+, k_-, n)$ -quasi-cross does not lattice tile  $\mathbb{R}^n$ .*

We can turn **Corollary 16** into a more convenient form of the non-existence result in the following theorem.

**Theorem 17.** *Let  $p > k_+$  be a prime,  $p \not\equiv 1 \pmod{k_+ + k_-}$ , and  $p \neq k_+ + k_-$ . Then the  $(k_+, k_-, n)$ -quasi-cross does not lattice tile  $\mathbb{R}^n$  for  $n \equiv -(k_+ + k_-)^{-1} \pmod{p}$ , where  $(k_+ + k_-)^{-1}$  is the multiplicative inverse of  $k_+ + k_-$  in  $\mathbb{Z}_p$ .*

**Proof.** We start by noting that  $1 \leq k_- < k_+ < p$  and  $p \neq k_+ + k_-$  which means  $p \nmid k_+ + k_-$  and so  $k_+ + k_-$  has a multiplicative inverse in  $\mathbb{Z}_p$ . If

$$n \equiv -(k_+ + k_-)^{-1} \pmod{p}$$

then

$$(k_+ + k_-)n + 1 \equiv 0 \pmod{p}.$$

Thus,  $p \mid (k_+ + k_-)n + 1$ . However,

$$p \not\equiv 1 \pmod{k_+ + k_-}$$

implies

$$(k_+ + k_-)n + 1 - p \not\equiv 0 \pmod{(k_+ + k_-)p}.$$

Since  $p > k_+$  we must have  $\gcd(p, k_+\#) = 1$ . We now use **Corollary 16** with  $d = p$ .  $\square$

Even though **Corollary 15** was phrased as a recursive construction, it can also be used to prove the non-existence of a lattice tiling, as shown in the following theorem.

**Theorem 18.** *Let  $p$  be a prime,  $p \equiv 1 \pmod{k_+ + k_-}$ . If the  $(k_+, k_-, n)$ -quasi-cross does not lattice tile  $\mathbb{R}^n$ , then the  $(k_+, k_-, n')$ -quasi-cross does not lattice tile  $\mathbb{R}^{n'}$ ,*

$$n' = \frac{((k_+ + k_-)n + 1)p^i - 1}{k_+ + k_-},$$

for all positive integers  $i$ .

**Proof.** Assume to the contrary that there is a lattice tiling of  $\mathbb{R}^{n'}$  by  $(k_+, k_-, n')$ -quasi-crosses, where  $n' = pn + \frac{p-1}{k_+ + k_-}$ . We note that  $p \mid (k_+ + k_-)n' + 1$ , and that  $p > k_+$  and so  $\gcd(p, k_+\#) = 1$ . We now use **Corollary 15** and get that there must be a lattice tiling of  $\mathbb{R}^{n''}$  by  $(k_+, k_-, n'')$ -quasi-crosses, where

$$n'' = \frac{(k_+ + k_-)n' + 1 - p}{(k_+ + k_-)p} = n,$$

a contradiction. Thus, there is no lattice tiling of  $\mathbb{R}^{n'}$  by  $(k_+, k_-, n')$ -quasi-crosses. Repeating this argument  $i$  times, for any positive integer  $i$ , completes the proof.  $\square$

#### 4. Application to small quasi-crosses

In this concluding section we apply the general methods described in the previous sections to the problem of finding necessary conditions for the existence of lattice tilings of  $\mathbb{R}^n$  by  $(3, 1, n)$ -quasi-crosses and by  $(3, 2, n)$ -quasi-crosses. These are the smallest unclassified cases, since lattice tilings by the smaller  $(2, 1, n)$ -quasi-cross have been completely classified in [16].

We begin with the following theorem which is a straightforward application of Lemma 6.

**Theorem 19.** *The  $(3, 1, n)$ -quasi-cross does not lattice tile  $\mathbb{R}^n$  when  $4n + 1$  is a prime,  $n \equiv 3 \pmod{6}$ .*

**Proof.** Assume to the contrary that  $\mathbb{Z}_{4n+1} = (M, S)$  is a splitting with  $M = [-1, 3]^*$ . Obviously, 1 is a QR. Using Lemma 6 we note that  $-1$  and 3 are also QRs, while 2 is a QNR, when  $n \equiv 3 \pmod{6}$ . Thus  $S$  should have an equal number of QRs and QNRs, but  $|S| = n$  is odd, a contradiction.  $\square$

A generalization for higher power residues (generalizing the Legendre symbol) can be made, as is seen in the next theorem, which uses quartic residues.<sup>4</sup>

**Theorem 20.** *Let  $4n + 1$  be a prime, with  $n$  being an odd integer. If*

$$6^n \not\equiv 1 \pmod{4n + 1}$$

*then the  $(3, 1, n)$ -quasi-cross does not lattice tile  $\mathbb{R}^n$ .*

**Proof.** Let  $g$  be a primitive element in  $\mathbb{Z}_{4n+1}$ , which is a field as  $4n + 1$  is prime. We define the character  $\chi : \mathbb{Z}_{4n+1} \rightarrow \mathbb{C}$  as

$$\chi(g^j) = e^{\frac{2\pi ij}{4}},$$

where  $i = \sqrt{-1}$ .

Assume to the contrary that there exists a splitting  $\mathbb{Z}_{4n+1} = (M, S)$  under the conditions of the theorem. By Theorem 4 we have

$$\left( \sum_{m \in M} \chi(m) \right) \cdot \left( \sum_{s \in S} \chi(s) \right) = \sum_{j=1}^{4n} \chi(j). \tag{5}$$

We also have

$$\sum_{j=1}^{4n} \chi(j) = \sum_{j=0}^{4n-1} \chi(g^j) = \sum_{j=0}^{4n-1} \chi(g)^j = \frac{\chi(g)^{4n} - 1}{\chi(g) - 1} = 0. \tag{6}$$

It follows from (5) and (6) that

$$\sum_{m \in M} \chi(m) = 0 \quad \text{or} \quad \sum_{s \in S} \chi(s) = 0.$$

We first note that 1 and  $-1$  are quadratic residues in  $\mathbb{Z}_{4n+1}$ . If 2 or 3 are quadratic residues, then by Lemma 6 the set  $S$  must contain an equal number of quadratic residues and quadratic non-residues, but  $|S| = n$  is odd. We therefore need to consider only the case where both 2 and 3 are quadratic non-residues.

We now turn to check the characters of the elements of  $M$ . It is easily seen that  $\chi(1) = 1$ . Since  $n$  is odd, we deduce  $\chi(-1) = -1$ , i.e.,  $-1$  is a quadratic residue in  $\mathbb{Z}_{4n+1}$  but is a quartic non-residue. Since both 2 and 3 are quadratic non-residues, we have  $\chi(2), \chi(3) \in \{i, -i\}$ .

We note that the quartic residues form a multiplicative subgroup

$$\{g^{4j} \mid 0 \leq j \leq n - 1\} \subseteq \mathbb{Z}_{4n+1}.$$

<sup>4</sup> Quartic residues are sometimes also called *biquadratic* residues.

It is also easily seen that

$$(g^j)^n = (g^{4\lfloor j/4 \rfloor + (j \bmod 4)})^n = (g^{4n})^{\lfloor j/4 \rfloor} g^{n(j \bmod 4)} = g^{n(j \bmod 4)}.$$

Since  $n$  is odd, we get that an element  $a \in \mathbb{Z}_{4n+1}$ ,  $a \neq 0$ , is a quartic residue, i.e.,  $\chi(a) = 1$ , if and only if  $a^n \equiv 1 \pmod{4n+1}$ .

We are given that  $6^n \not\equiv 1 \pmod{4n+1}$ , and thus  $1 \neq \chi(6) = \chi(2)\chi(3)$ . It follows that  $\chi(2) = \chi(3)$ . We now have

$$\sum_{m \in M} \chi(m) = \chi(-1) + \chi(1) + \chi(2) + \chi(3) = \pm 2i \neq 0.$$

Therefore,  $\sum_{s \in S} \chi(s) = 0$ , which is only possible if  $|S| = n$  is even, a contradiction.  $\square$

In contrast to the non-existence results, we do know of some constructions for lattice tilings by quasi-crosses. For the first shape, the  $(3, 1, n)$ -quasi-cross, we recall that there exists a construction of lattice tilings from [16] for dimensions  $n = (5^i - 1)/4$ ,  $i \geq 1$ . In addition, certain primes were shown in [22] to induce lattice tilings, as well as a recursive construction, though a closed analytic form for the dimension appears to be hard to obtain. Using a computer to verify the requirements for the construction from [22], for  $n \leq 250$  we also have lattice tilings of  $\mathbb{R}^n$  by  $(3, 1, n)$ -quasi-crosses for dimensions

$$n = 37, 43, 97, 102, 115, 139, 163, 169, 186, 199, 216.$$

The only non-existence result with a nice analytic form is that of Theorem 10, ruling out lattice tilings by  $(3, 1, n)$ -quasi-crosses when  $n \equiv 5, 8 \pmod{9}$ . However, especially for the  $(3, 1, n)$ -quasi-cross, numerous other non-existence results lacking a nice analytic form ensue from the previous section. Aggregating the entire set of necessary conditions, for  $n \leq 250$ , apart from the dimensions mentioned above that allow a lattice tiling, no other lattice tiling of  $\mathbb{R}^n$  by  $(3, 1, n)$ -quasi-crosses exists except perhaps in the remaining unclassified cases of

$$n = 20, 22, 24, 60, 101, 111, 114, 121, 144, 182, 220, 234, 235.$$

For the second shape, the  $(3, 2, n)$ -quasi-cross, no lattice tiling is known except for the trivial tiling of  $\mathbb{R}^1$ . The combined non-existence results with a nice analytic form are much stronger in this case.

**Corollary 21.** *If the  $(3, 2, n)$ -quasi-cross lattice tiles  $\mathbb{R}^n$  then  $n \equiv 1, 7, 13, 16 \pmod{18}$ .*

**Proof.** The proof is obtained by combining the following results.

- $5n + 1 \equiv 0 \pmod{3}$  by Theorem 3.
- $n \not\equiv 4, 10 \pmod{18}$  by Theorem 13 with  $r = 3$ .  $\square$

Aggregating this result with the other recursive necessary conditions, for  $2 \leq n \leq 250$ , no lattice tiling of  $\mathbb{R}^n$  by  $(3, 2, n)$ -quasi-crosses exists except perhaps in the remaining unclassified cases of

$$n = 7, 13, 19, 37, 43, 49, 73, 79, 85, 97, 115, 121, 145, 157, 181, 211, 217, 223, 229.$$

## Acknowledgment

The author would like to thank Muriel Médard for hosting him at MIT during his sabbatical.

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