Network Coding Solutions for the Combination Network and its Subgraphs

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***Abstract*—The combination network is one of the simplest and insightful networks in coding theory. The vector network coding solutions for this network and some of its sub-networks are examined. For a fixed alphabet size of a vector network coding solution, an upper bound on the number of nodes in the network is obtained. This bound is an MDS bound for subspaces over a finite field. A family of sub-networks of combination networks is defined. It is proved that for this family of networks, which are minimal multicast networks, there is a gap in the minimum alphabet size between vector network coding solutions and scalar network coding solutions. This gap is obtained for any number of messages and is based on coloring of the q-Kneser graph and a new hypergraph generalization for it.**

I. Introduction

Network coding has been attracting increased attention for almost two decades since the seminal papers [1], [16]. Multicast networks have received most of this attention. A recent survey on the foundation of multicast network coding can be found in [13]. The multicast network-coding problem can be formulated as follows: given a network with one source which has h messages, for each edge find a function of the packets received at the starting node of the edge, such that each receiver can recover all the messages from its received packets. Such an assignment of a function to each edge is called a *solution* for the network. Therefore, the received packets on an edge can be expressed as functions of the source messages. If these functions are linear, we obtain a *linear network coding solution*, otherwise we have a *nonlinear solution*. In linear network coding, each linear function on an edge consists of coding coefficients for each incoming packet. If the coding coefficients and the packets are scalars, it is called a *scalar network coding solution*. If the messages and the packets are vectors and the coding coefficients are matrices then it is called a *vector network coding solution*. A network which has a solution is called a *solvable* network. It is well-known that a multicast network with one source, h messages, and N receivers, is solvable if and only if the min-cut between the source and each receiver is at least h [13].

The functions on the edges of the network form the network code. The coding coefficients form the network coding vectors on the edges. The vector of coding coefficients is called the *local coding vector* when the function on the edge is considered as a linear combination of the packets received at the starting node of the edge. When we consider the function on the edge as a linear combination of the h messages, the vector of coding coefficients (for the h messages) is called the *global coding vector*. To recover the h messages, a receiver R should obtain h global coding vectors whose linear span has dimension h. In other words, the $h \times h$ matrix formed by these h global coding vectors should be invertible. This $h \times h$ matrix is called a *transfer matrix* of R. The previous description constitutes the framework for scalar linear network coding. The framework for vector network coding was presented in [9]. Each message and each packet is a vector of length t and the coding coefficients are $t \times t$ matrices. The global coding vectors, on the edges, consist of h matrices of size $t \times t$, which together form $t \times (ht)$ matrices. W.l.o.g., we assume that each $t \times (ht)$ matrix of a global coding vector is a generator matrix of a t-subspace of $\mathbb{F}_q^{ht}$. To recover the h messages, a receiver R should have on its ℓ incoming edges, $\ell \geqslant h$, h such global coding vectors which form together an $(\ell t) \times (ht)$ transfer matrix of rank ht.

The *field size* of the solution is an important parameter that directly influences the complexity of the calculations at the network nodes. It is known that any field size $q \geqslant N$ suffices for a solution. However, it is conjectured that the smallest field size allowing a solution is much smaller [12], [13]. An efficient algorithm to find such a field size and the related network code was given in [15]. It is conjectured that the minimum alphabet size is much smaller, but this was proved only for two messages [12]. For this purpose we distinguish between the smallest alphabet size required for each one of the three types of network coding solutions. Given a network $\mathcal{N}$, we define $q_s(\mathcal{N})$ to be the smallest field size q for which $\mathcal{N}$ has a scalar linear solution. Similarly, $q_n(\mathcal{N})$ is the smallest alphabet size q (q not necessarily a prime power) for which $\mathcal{N}$ has a scalar nonlinear solution, and $q_v(\mathcal{N})$ is the smallest value q^t, q a prime power, such that $\mathcal{N}$ has a vector solution over $\mathbb{F}_q^t$. By definition, $q_s(\mathcal{N}) \geqslant q_v(\mathcal{N}) \geqslant q_n(\mathcal{N})$, and we define the *vector gap* by

$$\text{gap}_v(\mathcal{N}) \triangleq q_s(\mathcal{N}) - q_v(\mathcal{N}).$$

Two other gaps ($q_s(\mathcal{N}) - q_n(\mathcal{N})$ and $q_v(\mathcal{N}) - q_n(\mathcal{N})$) are defined similarly, but this paper will be mostly devoted to the vector gap.

One of the most celebrated families of networks is the family of combination networks [22], which were used for various topics in network coding. The $\mathcal{N}_{h,r,s}$ combination network, where $s \geqslant h$, is shown in Fig. 1. The network has three layers: the first layer consists of a single source with h messages. The source transmits r packets to the r nodes of the middle layer. Any s nodes in the middle layer are connected to a receiver, and each one of the $\binom{r}{s}$ receivers demands all the h messages. It was proved in [22] that a solution for such a network exists if and only if a related error-correcting code exists. This network was also generalized to compare scalar and vector network coding [11]. Its sub-networks were used to prove that finding the minimum required field size of a (linear or nonlinear) scalar network code for a certain multicast network is NP-complete [21].

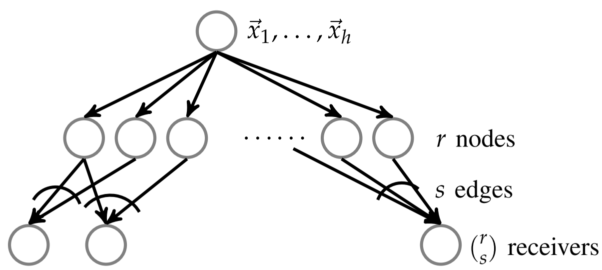

Figure 1. The $\mathcal{N}_{h,r,s}$ combination network: it has an edge from the source to each of the r nodes in the middle layer. Each of the $\binom{r}{s}$ receivers is connected to a unique set of s middle-layer nodes, and demands all of the h source messages.

The goal of this work is to consider two problems which are related to vector coding solutions for combination networks and their sub-networks. In Section II, we describe network coding solutions (vector, scalar, linear and nonlinear) for the $\mathcal{N}_{h,r,s}$ combination network. In particular, we consider the $\mathcal{N}_{h,r,h}$ combination network and the maximum number of nodes in the middle layer for such a network. This number is related to the largest length of certain MDS codes. While there exists a proof on the upper bound of such length for linear and nonlinear codes, we are not aware of any proof based on the properties of the subspaces. These codes are also MDS array codes which were considered in the past for storage [5] and are very popular today as distributed-storage codes, e.g., see [8], [24] and references therein. In Section III, the vector gap is considered. Such vector gaps, which are very large, were considered in [11] for any number of messages $h > 2$. The networks which were used for the proof are generalizations of the combination networks in which for each receiver there are some redundant edges on the paths between the source and the receiver. The extra edges were used to distribute the (ht)-space formed by the h vector messages of length t on more than h edges. This enables some edges to transmit only a fraction of a one-dimensional space. However, a similar idea cannot be used for scalar linear network coding. The question whether such gaps can be obtained if there are no such redundant edges remained open. In Section III, we give a positive answer to this question and prove that there exists a vector gap in such networks called *minimal multicast networks*, for any number of messages. The gap is increasing with the number of messages. This also proves the existence of a gap for two messages which was left open in [11]. The networks which will be used for this purpose are sub-networks of the combination networks. The proof will be based on the chromatic number of the q-Kneser graph and a generalized version of it, the q-Kneser hypergraph, which was not defined before. The coloring problem raises an intriguing combinatorial problem which has independent intellectual merit. Several more related problems will be presented in Section IV and will be considered in the full version of this paper. The same is true for some proofs of claims in the paper.

II. Vector Solution and Bound for MDS Codes

In this section, we first describe the three types of solutions for the $\mathcal{N}_{h,r,s}$ combination network. The key result is the following theorem proved in [22]. Let $(r, q^h, r-s+1)_q$ denote a code over $\mathbb{F}_q$ of length r with q^h codewords and minimum Hamming distance $r-s+1$. If this code is linear, it is denoted by $[r, h, r-s+1]_q$.

Theorem 1. *([22]) The $\mathcal{N}_{h,r,s}$ combination network is solvable over $\mathbb{F}_q$ if and only if there exists an $(r, q^h, r-s+1)_q$ code.*

In view of Theorem 1, what are the functions on the edges of the $\mathcal{N}_{h,r,s}$ combination network in the three types of solutions?

1) For the scalar nonlinear solution, an $(r, q^h, r-s+1)_q$ code, each coordinate in a codeword is a function of h information symbols which are represented by the h messages. The function for the ith symbol of a codeword is the function on the link from the source to the ith node in the middle layer.
2) For the scalar linear solution, an $[r, h, r-s+1]_q$ code is required. It has an $r \times h$ generator matrix and the h entries of its ith column are the coding coefficients of the linear function on the link from the source to the ith node in the middle layer.

In both cases, the nodes of the middle layer transmit their information to the related receivers. Each receiver obtains s symbols from the middle-layer nodes, each one has the same global coding vector on its incoming and outgoing edges. Since the minimum Hamming distance of the code is $r-s+1$, it follows that for each two different sets of h messages, each receiver obtains a different s-tuple of symbols from the middle layer nodes. Hence, it can recover the h messages.

For the vector network coding solution, the h matrices of size $t \times t$ on the edges from the source to the middle-layer nodes form together a $t \times (ht)$ matrix which has dimension t, i.e., it represents a t-subspace of $\mathbb{F}_q^{ht}$. Now, to have a solution for the $\mathcal{N}_{h,r,s}$ combination network, each s subspaces, related to the edges between the source and the middle-layer nodes, span the (ht)-space defined by the messages of the source.

A fundamental combinatorial structure that underpins some of the generalized combination networks is a structure we call a $(t; h, \alpha)_q$-independent configuration. We use $\begin{bmatrix} V \\ t \end{bmatrix}$ to denote all the t-dimensional subspaces of a vector space V, and $\begin{bmatrix} a \\ b \end{bmatrix}$

to denote the Gaussian coefficient (where the field size q is understood from context).

Definition 2. *Let q be a prime power, t, h, α be positive integers, $\alpha \leqslant h$, and denote $V = \mathbb{F}_q^{ht}$. A $(t; h, \alpha)_q$-independent configuration (IC) is a set $\mathcal{C} = \{V_1, V_2, \ldots, V_m\} \subseteq \begin{bmatrix} V \\ t \end{bmatrix}$, such that for all $1 \leqslant i_1 < i_2 < \cdots < i_\alpha \leqslant m$,*

$$\dim(V_{i_1} + V_{i_2} + \cdots + V_{i_\alpha}) = \alpha t.$$

We say $|\mathcal{C}| = m$ is the size of the IC.

Lemma 3. *Let $\mathcal{C}$ be a $(t; h, \alpha)_q$-IC. If $\alpha \geqslant 2$ then*

$$|\mathcal{C}| \leqslant \frac{q^{(h-\alpha+2)t} - 1}{q^t - 1} + \alpha - 2.$$

Proof: If $\alpha = 2$ the claim is immediate by considering the size of a t-spread [10].

Assume now $\alpha > 2$, and denote $V \triangleq \mathbb{F}_q^{ht}$. Let us write $\mathcal{C} = \{V_1, V_2, \ldots, V_m\}$, and define

$$W_1 \triangleq V_1 + V_2 + \cdots + V_{\alpha-2},$$

where $\dim(W_1) = (\alpha - 2)t$. By the definition of an IC, $\mathbb{F}_q^{ht} = W_1 + W_2$, where $W_2 \in \begin{bmatrix} V \\ (h-\alpha+2)t \end{bmatrix}$. It follows that any vector $v \in V_j$, $\alpha - 1 \leqslant j \leqslant m$, may be written uniquely as $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$. We now define

$$V_j' \triangleq \{v_2 \ : \ v_1 + v_2 \in V_j, \, v_1 \in W_1, \, v_2 \in W_2\},$$

for all $\alpha - 1 \leqslant j \leqslant m$. It is easily seen that $\dim(V_j') = t$.

Furthermore, for any $\alpha - 1 \leqslant j_1 < j_2 \leqslant m$,

$$\dim(W_1 + V_{j_1}' + V_{j_2}') = \alpha t \Rightarrow \dim(V_{j_1}' + V_{j_2}') = 2t.$$

Thus, the set $\{V_i'\}_{\alpha-1 \leqslant i \leqslant m}$ contains $|\mathcal{C}| - \alpha + 2$ pairwise disjoint t-subspaces of W_2. Thus,

$$|\mathcal{C}| - \alpha + 2 \leqslant \frac{\begin{bmatrix} (h-\alpha+2)t \\ 1 \end{bmatrix}}{\begin{bmatrix} t \\ 1 \end{bmatrix}}.$$

■

We now make the connection between ICs and a certain family of combination networks.

Lemma 4. *The $\mathcal{N}_{h,r,h}$ combination network has a vector solution over $\mathbb{F}_q$ with messages of length t if and only if there exists a $(t; h, h)_q$-IC of size r.*

Proof: In the first direction assume that a vector solution over $\mathbb{F}_q$ with messages of length t exists. We note that by construction, any node i in the middle layer has a subspace $V_i \subseteq V \triangleq \mathbb{F}_q^{ht}$, with $\dim(V_i) \leqslant t$. If the terminal R_j gets from the middle layer the subspaces $V_{j_1}, \ldots, V_{j_h}$, then

$$\dim(V_{j_1} + \cdots + V_{j_h}) = ht,$$

which implies that $\dim(V_i) = t$. Thus, $\{V_i\}_{1 \leqslant i \leqslant r}$ is a $(t; h, h)_q$-IC.

In the other direction, assume $\mathcal{C} = \{V_1, \ldots, V_r\}$ is a $(t; h, h)_q$-IC. We can easily construct a vector network coding solution to the $\mathcal{N}_{h,r,h}$ combination network. Simply send V_i to the ith middle layer node. Since $\mathcal{C}$ is a $(t; h, h)_q$-IC it follows that receiver has a full rank $(ht) \times (ht)$ transfer matrix from which it can recover the h messages. ■

Lemmas 3 and 4 form a generalization for an upper bound on the length of MDS code (use $\alpha = h$ in Lemma 3). The related results for (scalar) linear codes are given in [18]. Corollary 7 [18, p. 321] asserts that for an $[n, k, n-k+1]_q$ MDS code, we have that $n \leqslant q + k - 1$. This result is strengthened in Theorem 11 [18, p. 326] by using a more complicated proof based on projective geometry. The theorem asserts that if $k \geqslant 3$ and q is odd then $n \leqslant q + k - 2$. A more complicated proof for the same result is given for nonlinear codes in [20, pp. 12-13].

Lemmas 3 and 4 can be generalized for a family of networks which generalize the combination network [11]. Some interesting consequences implied by this generalization will be discussed in the full version of this paper.

We can use Lemma 4 to upper bound the vector gap in the $\mathcal{N}_{h,r,h}$ combination networks. For this we will use Bertrand's postulate (e.g., see [2]) that the interval $[n, 2n]$ contains a prime power for any integer n; and that the interval $[x, x + x^{21/40}]$ contains a prime for all large enough x [3]. This implies the following result.

Theorem 5. *For all positive integers h and r, let $\mathcal{N}$ denote the $\mathcal{N}_{h,r,h}$ combination network. Then $\text{gap}_v(\mathcal{N}) \leqslant r + h - 3$, and for all large enough r, $\text{gap}_v(\mathcal{N}) \leqslant (r-1)^{21/40} + h - 2$.*

III. Minimal Multicast Networks

In this section we will prove that for each number of messages $h \geqslant 2$, there exists a minimal multicast network for which vector network coding outperforms scalar network coding. A *minimal* multicast network can deliver h messages from the source to the receivers, but if any edge is removed, it can deliver at most $h - 1$ messages to at least one of the receivers. From a practical point of view, considering such minimal networks is interesting as it minimizes the used network resources. From a theoretical point of view, minimal networks can be regarded as a fair setting for a comparison between the three types of network coding solutions.

Definition 6. *A multicast network $\mathcal{N}$ is said to be* minimal *if every edge crosses a cut of size h.*

Thus, in a minimal network, the removal of any edge makes at least one cut have size strictly less than h, and therefore the new network is incapable of a solution.

To achieve the goal of this section, a sub-network of the $\mathcal{N}_{h,r,h}$ combination network, denoted by $\mathcal{N}_{h,r,h}^*$, will be used. The network $\mathcal{N}_{h,r,h}^*$ has one source in the first layer, and $\begin{bmatrix} ht \\ t \end{bmatrix}$ nodes in the middle layer, each node represents a different t-subspace of $\mathbb{F}_q^{ht}$. From each h nodes in the middle layer which represent the t-subspaces $V_{i_1}, \ldots, V_{i_h}$ for which

$$V_{i_1} + \cdots + V_{i_h} = \mathbb{F}_q^{ht}, \qquad (1)$$

there are links to a unique receiver.

For the remainder of this work, let $G(X)$ be any $t \times (ht)$ generator matrix for a t-subspace X of $\mathbb{F}_q^{ht}$. Also, the *splitting* of a $t \times (ht)$ matrix G is the h matrices of size $t \times t$ obtained by taking the first t columns of G, then the next t columns, and so on.

It is obvious from the definition of $\mathcal{N}^*_{h,r,h}$ that for vector network coding, the minimum alphabet size for which it is solvable is q^t. The coding coefficients on the edge from the source to the node represented by the t-subspace X are formed by splitting of $G(X)$ into h matrices of size $t \times t$. The global coding vector from a node u of the middle layer to a receiver R_j is the same one as the global coding vector (which coincides with the local coding vector) from the source to u. It implies by (1) that the $(ht) \times (ht)$ transfer matrix of each receiver is of full rank. It is not difficult to prove that a smaller alphabet size is impossible.

For the scalar solution we form a new hypergraph $G = (V, E)$, where V is the set of middle-layer vertices of $\mathcal{N}^*_{h,r,h}$. Each set of h vertices from the middle layer from which there are links to a joint receiver R_j of the third layer (i.e (1) is satisfied), are connected in G by a hyperedge. When $h = 2$ this hypergraph is the well-known q-Kneser graph $qK_{2t:t}$. Hence, we will denote the general hypergraph by $qK^h_{ht:t}$ and call it the q-Kneser hypergraph. This is not to be confused with the q-Kneser graph $qK_{ht:t}$ whose vertices are t-subspaces of $\mathbb{F}_q^{ht}$ and two vertices are connected by an edge if their subspaces are disjoint.

A *coloring* of a graph $G = (V, E)$ is an assignment of a set of colors to the set of vertices V such that for each edge $\{u, v\} \in E$, the vertices u and v are assigned different colors. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors in which we can color G. Before we discuss the q-Kneser hypergraph we will concentrate on the q-Kneser graph [6], [7] which is related to $\mathcal{N}^*_{2,r,2}$, i.e. a sub-network of a combination network with two messages.

The network $\mathcal{N}^*_{2,r,2}$ has two messages and in a scalar network coding solution on the link between the source and each node in the middle layer there is a global coding vector from $\mathbb{F}_q^2$. The two vectors on two distinct such edges, which transmit information to two middle layer nodes (that represent two disjoint t-subspaces of $\mathbb{F}_q^{2t}$), must be linearly independent. The set of such pairs of nodes is exactly the pairs of vertices which define edges in $qK_{2t:t}$. Hence, each color of a vertex in $qK_{2t:t}$ will be associated with a vector of $\mathbb{F}_q^2$, such that two different colors will be associated with two linearly independent vectors. Since the cardinality of largest set of vectors in $\mathbb{F}_q^2$ which are pairwise linearly independent is $q+1$, it follows that if the chromatic number of $qK_{2t:t}$ is c then the alphabet size for the linear scalar solution is the smallest prime power greater than or equal to $c - 1$.

This coloring method and the reduction from the network to the q-Kneser graph are similar to some ideas in [12], [21]. Our method is unique by using the q-Kneser graph and also its new generalization.

Theorem 7. *For a prime power q and an integer $t \geqslant 2$, $t \in \{2, 3\}$ or $t < q \log q - q$, there exists a minimal network $\mathcal{N}$ with two messages for which* $\mathrm{gap}_v(\mathcal{N}) \geqslant q^{t-1} - 1$.

Proof: By [6], [7], the chromatic number of $qK_{2t:t}$ is

$$\chi(qK_{2t:t}) = q^t + q^{t-1} \qquad \text{for } t = 2,3 \text{ or } t < q \log q - q.$$

Thus, for these cases

$$q_s(\mathcal{N}) \geqslant q^t + q^{t-1} - 1,$$

and therefore we have a gap of at least

$$\mathrm{gap}_v(\mathcal{N}) = q_s(\mathcal{N}) - q_v(\mathcal{N}) \geqslant q^{t-1} - 1.$$

■

The scope of Theorem 7 is somewhat limited due to the restrictions on the value of t. We can remove these restrictions, but severely reduce the guaranteed vector gap to merely 1.

Theorem 8. *For a prime power q and any integer $t \geqslant 2$, there exists a minimal network $\mathcal{N}$ with two messages for which* $\mathrm{gap}_v(\mathcal{N}) \geqslant 1$.

Proof: We will prove that $\chi(qK_{2t:t}) > q^t + 1$. Recall that the vertex set of $qK_{2t:t}$ is $\begin{bmatrix} V \\ t \end{bmatrix}$, where $V = \mathbb{F}_q^{2t}$. Assume a coloring of $qK_{2t:t}$ with c colors. Let $U_i \subset \begin{bmatrix} V \\ t \end{bmatrix}$, $1 \leqslant i \leqslant c$, be the set of vertices colored with color i. Then each U_i is a 1-intersecting family in the language of [14], and an anticode of diameter $t - 1$ in the language of [23]. Also, the set $\{U_i\}_{1 \leqslant i \leqslant c}$ forms a tiling (partition) of $\begin{bmatrix} V \\ t \end{bmatrix}$.

By [14], for all $1 \leqslant i \leqslant c$, $|U_i| \leqslant \begin{bmatrix} 2t-1 \\ t-1 \end{bmatrix}$, and U_i is either

$$\left\{ U \in \begin{bmatrix} V \\ t \end{bmatrix} \ : \ V_1 \subseteq U \right\} \text{ or } \left\{ U \in \begin{bmatrix} V \\ t \end{bmatrix} \ : \ U \subseteq V_{2t-1} \right\},$$

where V_1, V_{2t-1} are subspaces of V of dimensions 1 and $2t - 1$, respectively. However, by [23], there is no tiling of V by U_i of these shapes. Thus,

$$\chi(qK_{2t:t}) > \frac{\begin{bmatrix} 2t \\ t \end{bmatrix}}{\begin{bmatrix} 2t-1 \\ t-1 \end{bmatrix}} = q^t + 1,$$

since such a tiling of V will have size $q^t + 1$. It follows that

$$\mathrm{gap}_v(\mathcal{N}) = q_s(\mathcal{N}) - q_v(\mathcal{N}) \geqslant (q^t + 2 - 1) - q^t = 1.$$

■

For h messages, the vector network code for the $\mathcal{N}^*_{h,r,h}$ network is exactly as in the $\mathcal{N}^*_{2,r,2}$ network. The coding coefficients on the edge from the source to the middle-layer node represented by the t-subspace X is formed by splitting $G(X)$ to h matrices of size $t \times t$. For the scalar linear network code we consider the q-Kneser hypergraph $qK^h_{ht:t}$. Our generalization is different from other generalizations, e.g. [19] and references therein. The chromatic number of $qK_{ht:t}$ is related to different colors in the hyperedges of $qK^h_{ht:t}$.

Theorem 9. *For any prime power q and integers $t \geqslant 2$, $h \geqslant 3$,*

$$\chi(qK_{ht:t}) \geqslant \frac{q^{ht} - 1}{q^t - 1} = \sum_{i=0}^{h-1} q^{it}.$$

Proof: Let U be a set of $(q^{ht} - 1)/(q^t - 1)$ pairwise-disjoint subspaces of $\mathbb{F}_q^{ht}$. Such a set is called a *spread* and it exists for all h and t [10]. The vertices of $qK_{ht:t}$ related to the t-subspaces that are in U should be colored in a different colors, which implies the claim of the theorem. ■

Corollary 10. *For any prime power q and integers $t \geqslant 2, h \geqslant 3$, there exists a minimal network $\mathcal{N}$ with h messages for which* $\mathrm{gap}_v(\mathcal{N}) \geqslant \sum_{i=2}^{h-1} q^{it}$.

In [11] it was proved that for even $h \geqslant 4$ there exists a multicast network (not minimal) $\mathcal{N}$ for which $\mathrm{gap}_v(\mathcal{N}) = q^{(h-2)t^2/h+o(t)}$, and for odd $h \geqslant 5$ there exists a multicast network (not minimal) $\mathcal{N}$ for which $\mathrm{gap}_v(\mathcal{N}) = q^{(h-3)t^2/(h-1)+o(t)}$. Corollary 10 implies that if t is fixed a vector gap larger than $q^{f(t)}$ for any function $f(t)$ can be obtained. It is well-known [9], [11] that a scalar linear network coding solution can be translated to a vector coding solution with vectors of length t over $\mathbb{F}_q$. Corollary 10 and the vector gaps proved in [11] imply that a translation from a vector coding solution with vectors of length t over $\mathbb{F}_q$ to a scalar linear solution will require an alphabet of size $q^{f(h,t)}$, with an interesting trade-off between h and t in $f(h,t)$.

Finally, it is possible to improve the bound in Theorem 9 and as a consequence also the bound in Corollary 10. A lower bound on the chromatic number of $qK_{ht:t}$ is obtained by using a normal spread (also called a geometric spread) [4], [17] and the chromatic number of $qK_{2t:t}$ as given in [6], [7]. For example we have:

Theorem 11. *For a prime power q and an integer $t \geqslant 2$, $t \in \{2,3\}$ or $t < q \log q - q$, there exists a minimal network $\mathcal{N}$ with h messages for which* $\mathrm{gap}_v(\mathcal{N}) \geqslant q^{(h-1)t} + \sum_{i=1}^{h-1} q^{it-1} - 1$.

In the full version of the paper we will also prove that the vector-gap problem, for minimal multicast networks with two messages can be reduced to sub-networks of the combination network.

IV. Conclusion

The family of combination networks and their sub-networks were used to prove two results. The first one is an upper bound on the number of nodes in the middle layer for a vector network coding solution. The second one is that for any number of messages vector network coding outperforms scalar network coding for minimal multicast networks with respect to the field size. The first result is an MDS bound for vector spaces and the proof is based on vector spaces and is simpler than the one for nonlinear MDS codes. The second result induces an interesting question on the chromatic number of q-Kneser hypergraphs.

There are a few more problems which are induced directly from our discussion.

1) Can the vector gap in minimal multicast networks with more than two messages be reduced to subgraphs of the combination networks?
2) What is the maximum vector gap for minimal multicast networks, with h messages and vectors of length t? Is it the one obtained by using the chromatic number $\chi(qK_{ht:t})$?
3) Can vector gaps for multicast networks with two messages be larger than the one obtained for minimal multicast networks?
4) What is the largest possible vector gap as a function of h and t for a multicast network with h messages?

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