

Bounds on the Size of Balls over Permutations with the Infinity Metric

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Abstract—We study the size (or volume) of balls in the metric space of permutations, S_n , under the infinity metric. We focus on the regime of balls with radius $r = \rho \cdot (n-1)$, $\rho \in [0, 1]$, *i.e.*, a radius that is a constant fraction of the maximum possible distance. We provide new bounds on the size of such balls. These bounds reduce the asymptotic gap between the upper and lower bound to at most 0.06 bits per symbol.

I. INTRODUCTION

Given a metric space (M, d) , perhaps one of the most basic constructs is that of a ball

$$B_r(x) \triangleq \{x' \in M \mid d(x, x') \leq r\},$$

where $x \in M$ is the ball's center and r is the ball's radius. Since many coding-theoretic problems may be viewed as the study of packing or covering of a metric space by balls, properties of balls and their parameters have been studied extensively in a wide range of metrics [10], [11], [31].

An important feature of a ball is its size (or volume), *i.e.*, the number of points in the ball. It is an important component in many bounds on code parameters, most notably, the ball-packing bound and the Gilbert–Varshamov bound [31]. Thus, the exact size, the asymptotic size, or bounds on the size of balls in various metrics are of interest.

Lately, metric spaces over permutations have received increased attention. This is motivated, in particular, by the recent application of rank modulation to non-volatile memories [22]: in such applications, the charge levels of memory cells are compared against each other, and a permutation is induced by the relative ranking of the cells' charge levels. For designing error-correcting codes or covering codes over the space of permutations, one needs to choose a suitable metric and so several metrics have been studied for the space of permutations, including Hamming's metric [1], [3], [4], [7], [8], [13], [14], [24], [33], Kendall's τ -metric [2], [5], [6], [9], [23], [32], [44], [46], and Ulam's metric [18].

This paper focuses on the infinity metric, whose definition will follow in the next section. Spaces of permutations with this metric have been used for error-correction [28], [37], [40], [46], code relabeling [41], anticodes [35], covering codes [17], [44], and snake-in-the-box codes [45]. It is therefore surprising

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that the asymptotic size of a ball in this metric space is (to the best of our knowledge) unknown, and a considerable gap exists between the known lower and upper bounds.

The goal of this paper is to reduce the gap between the lower and upper bounds on the asymptotic size of balls in the space of permutations with the infinity metric. To that end, we exploit a well-known connection between the size of the aforementioned balls, and permanents of binary Toeplitz matrices. We carefully employ recent advances in bounds on permanents of doubly-stochastic matrices to obtain the desired results, along with using some older bounds on permanents.

The paper is organized as follows. In Section II we present notations and definitions, in Section III we analyze the asymptotic gap between the known bounds on the size of balls, in Section IV we present the new bounds, and we conclude in Section V with a discussion of the results and some open questions.

II. NOTATION

For the rest of this paper, n will denote a positive integer. With this, we define $[n] \triangleq \{1, 2, \dots, n\}$ and let S_n be the set of all permutations over $[n]$. The identity permutation in S_n is denoted by Id_n . Additionally, the composition of any two permutations $f, g \in S_n$ is denoted by fg and represents the mapping $i \mapsto f(g(i))$.

For any $f, g \in S_n$, the infinity metric (or infinity distance) between them, denoted $d_\infty(f, g)$, is defined as

$$d_\infty(f, g) \triangleq \max_{i \in [n]} |f(i) - g(i)|.$$

Since $d_\infty(\cdot, \cdot)$ is the only metric we will be using, we shall simply denote it by $d(\cdot, \cdot)$. Observe that for any $f, g \in S_n$, we have $0 \leq d(f, g) \leq n - 1$.

We define the *ball* of radius r centered at $f \in S_n$ as the set

$$B_{r,n}(f) \triangleq \{g \in S_n \mid d(f, g) \leq r\}.$$

The infinity metric over S_n is right invariant [12], *i.e.*, for all $f, g, h \in S_n$ we have $d(fh, gh) = d(f, g)$. Thus, the size of a ball depends only on r and n , and not on the choice of the center. We will therefore denote by $|B_{r,n}|$ the size of a ball of radius r in S_n .

For an $n \times n$ matrix, $M = (m_{i,j})$, the *permanent* of M is defined as

$$\text{per}(M) \triangleq \sum_{f \in S_n} \prod_{i \in [n]} m_{i,f(i)}.$$

A particular matrix of interest is the following binary Toeplitz matrix $A_{r,n} = (a_{i,j})$ of size $n \times n$ defined by

$$a_{i,j} \triangleq \begin{cases} 1 & |i-j| \leq r \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in [n]. \quad (1)$$

It is well known [25], [27], [34], [40] that

$$\begin{aligned} \text{per}(A_{r,n}) &= \sum_{f \in S_n} \prod_{i \in [n]} a_{i,f(i)} \\ &= |\{f \in S_n \mid \forall i \in [n] : |i - f(i)| \leq r\}| \\ &= |\{f \in S_n \mid d(\text{Id}_n, f) \leq r\}| \\ &= |B_{r,n}(\text{Id}_n)| = |B_{r,n}|. \end{aligned}$$

Note that for any *fixed* radius r , tight asymptotic bounds on $|B_{r,n}|$ are known [26], [29], [34], [39]. However, in this paper we are interested in the case of radius $r = \rho \cdot (n-1)$, where $\rho \in [0, 1]$ is a real constant. In expressions like $r = \rho \cdot (n-1)$ we always implicitly assume that ρ is such that r is an integer. We call ρ the *normalized radius*.

III. ANALYSIS OF KNOWN BOUNDS

Given any upper and lower bounds on the ball size,

$$\varphi(r, n) \leq |B_{r,n}| \leq \Phi(r, n),$$

we are interested in the following measure of asymptotic gap between the two:

Definition 1. For bounding functions φ and Φ as above, we define

$$\text{Gap}_\varphi^\Phi(\rho) \triangleq \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 \frac{\Phi(\rho \cdot (n-1), n)}{\varphi(\rho \cdot (n-1), n)},$$

for any real constant $\rho \in [0, 1]$.

The following proposition summarizes, to the best of our knowledge, the tightest known bounds for balls in (S_n, d_∞) .

Proposition 2. It holds that

$$\varphi_1(r, n) \leq |B_{r,n}| \leq \Phi_1(r, n),$$

where

$$\begin{aligned} \varphi_1(r, n) &\triangleq \begin{cases} \frac{(2\rho n)^n}{2^{2\rho n} e^n} \cdot 2^{o(n)} & 0 \leq r \leq \frac{n-1}{2} \\ \frac{n^n}{e^n 2^{2n(1-\rho)}} \cdot 2^{o(n)} & \frac{n-1}{2} \leq r \leq n-1 \end{cases}, \\ \Phi_1(r, n) &\triangleq \begin{cases} \frac{(2^{2\rho+1}\rho n)^n}{e^{n(2\rho+1)}} \cdot 2^{o(n)} & 0 \leq r \leq \frac{n-1}{2} \\ \frac{n^n}{e^{n(3-2\rho)} \rho^{2\rho n}} \cdot 2^{o(n)} & \frac{n-1}{2} \leq r \leq n-1 \end{cases}. \end{aligned}$$

Proof: These bounds are a rearrangement into asymptotic form of the following results:

- For the range $0 \leq r \leq \frac{n-1}{2}$, the upper and lower bounds were given in [25], [27].

- For the full range of possible radii, the upper bound was given in [40].
- For the full range of possible radii, the lower bound was given in [17].

Note that [27] conjectured a stronger lower bound for the range $0 \leq r \leq \frac{n-1}{2}$.

Our first goal is to analyze the gap between the upper bound Φ_1 and the lower bound φ_1 . In order to do so, we recall Stirling's approximation of $n!$ (see, e.g., [19])

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n 2^{O(1/n)}.$$

Corollary 3. It holds that

$$\text{Gap}_{\varphi_1}^{\Phi_1}(\rho) = \begin{cases} (4-2\log_2 e) \cdot \rho & 0 \leq \rho \leq \frac{1}{2} \\ 2(1-\rho)(1-\log_2 e) - 2\rho \log_2 \rho & \frac{1}{2} \leq \rho \leq 1 \end{cases}.$$

Proof: See [36].

The result of Corollary 3 is visualized by curve (a) in Fig. 1.

IV. NEW BOUNDS

In this section we present new bounds on the size of balls in (S_n, d_∞) .

- We first use a new lower bound on the permanent of non-negative matrices based on the Bethe permanent. This lower bound will imply improved bounds in the entire range of normalized radii, $0 \leq \rho \leq 1$.
- We then focus on a well-known lower bound on the permanent of non-negative matrices, where the matrix must be such that there are diagonal matrices D and D' with positive diagonal elements such that left-multiplication by D and right-multiplication by D' yields a doubly-stochastic matrix. This approach will enable us to further reduce the asymptotic gap between the lower and upper bounds, but so far only for the range $\frac{1}{2} \leq \rho \leq 1$.

A. Bethe-Permanent-Based Lower Bound

We recall that a *doubly-stochastic matrix* is a square $n \times n$ matrix with non-negative real entries for which the sum of each row and each column is 1. Let $\text{per}_B(M)$ be the Bethe permanent of an $n \times n$ -matrix with non-negative entries [42]. The following theorem is due to Gurvits [20]. (See also the discussion in [42].)

Theorem 4. Let $M \triangleq (m_{i,j})$ be an $n \times n$ matrix with non-negative entries and $\text{per}(M) > 0$. Let $Q \triangleq (q_{i,j})$ be any $n \times n$ doubly-stochastic matrix. Then

$$\log_2 \text{per}(M) \geq \log_2 \text{per}_B(M) \geq \sum_{i,j \in [n]} \left(-q_{i,j} \log_2 \frac{q_{i,j}}{m_{i,j}} + (1 - q_{i,j}) \log_2 (1 - q_{i,j}) \right).$$

For Theorem 4 to be meaningful, we note that $q_{i,j} = 0$ whenever $m_{i,j} = 0$, i.e., the support of Q should be a subset of the support of M .

In our case, we are interested in $M = A_{r,n}$ as defined in (1). Our goal is to find a doubly-stochastic matrix Q to use with Theorem 4.

For convenience, let us define the function $\tilde{H} : [0, 1] \rightarrow \mathbb{R}$ as

$$\tilde{H}(x) \triangleq -x \log_2 x + (1-x) \log_2(1-x).$$

We note that \tilde{H} is *not* the binary entropy function since the second summand appears with a flipped sign.

Theorem 5. Fix some $r, 0 \leq r \leq n-1$. It holds that

$$|B_{r,n}| \geq \varphi_2(r, n),$$

where for $0 \leq r \leq \frac{n-1}{2}$ we define

$$\begin{aligned} \log_2 \varphi_2(r, n) &\triangleq r(r+1)\tilde{H}\left(\frac{2}{2r+1}\right) \\ &\quad + (n(2r+1) - 2r(r+1))\tilde{H}\left(\frac{1}{2r+1}\right), \end{aligned}$$

whereas for $\frac{n-1}{2} \leq r \leq n-1$ we define

$$\begin{aligned} \log_2 \varphi_2(r, n) &\triangleq (n-r-1)(n-r)\tilde{H}\left(\frac{2}{n}\right) \\ &\quad + (n^2 - 2(n-r-1)(n-r))\tilde{H}\left(\frac{1}{n}\right). \end{aligned}$$

Proof: See [36]. ■

The lower bound $\varphi_2(r, n)$ presents an asymptotic improvement over the lower bound $\varphi_1(r, n)$, as shown in the following corollary.

Corollary 6. It holds that

$$\text{Gap}_{\varphi_2}^{\Phi_1}(\rho) = \begin{cases} (3-2\log_2 e) \cdot \rho & 0 \leq \rho \leq \frac{1}{2} \\ 2(1-\rho)(1-\rho - \log_2 e) & \\ -2\rho \log_2 \rho & \frac{1}{2} \leq \rho \leq 1 \end{cases}.$$

Proof: With the help of Theorem 5, it is a simple exercise to show that

$$\varphi_2(\rho \cdot (n-1), n) = \begin{cases} \frac{(2\rho n)^n}{2^{2\rho n}} \cdot 2^{o(n)} & 0 \leq \rho \leq \frac{1}{2} \\ \frac{n^n}{e^{n2^{2n(1-\rho)^2}}} \cdot 2^{o(n)} & \frac{1}{2} \leq \rho \leq 1 \end{cases}.$$

Together with the asymptotic form of $\Phi_1(r, n)$, which was given in the proof of Corollary 3, the claim follows immediately. ■

The result of Corollary 6 is visualized by curve (b) in Fig. 1.

B. Sinkhorn-Balancing-Based Lower Bound

The following theorem is well known.

Theorem 7. Let $M \triangleq (m_{i,j})$ be an $n \times n$ matrix with non-negative entries and $\text{per}(M) > 0$. Additionally, we require that there are two diagonal matrices D and D' with positive diagonal elements such that $D \cdot M \cdot D'$ is a doubly-stochastic matrix. Let $Q \triangleq (q_{i,j})$ be an $n \times n$ doubly-stochastic matrix. Then

$$\log_2 \text{per}(M) \geq \log_2 \frac{n!}{n^n} + \sum_{i,j \in [n]} \left(-q_{i,j} \log_2 \frac{q_{i,j}}{m_{i,j}} \right). \quad (2)$$

Proof: Let D and D' be given by

$$D \triangleq \text{diag}(d_1, \dots, d_n),$$

$$D' \triangleq \text{diag}(d'_1, \dots, d'_n),$$

where $d_i, i \in [n]$, and $d'_j, j \in [n]$, are positive real numbers. Note that the element in the i -th row and the j -th column of $D \cdot M \cdot D'$ is given by $d_i \cdot m_{i,j} \cdot d'_j$. Then

$$\begin{aligned} \log_2 \text{per}(M) &= \log_2 \text{per}(D \cdot M \cdot D') - \sum_{i \in [n]} \log_2(d_i) - \sum_{j \in [n]} \log_2(d'_j) \\ &\geq \log_2 \frac{n!}{n^n} - \sum_{i \in [n]} \log_2(d_i) - \sum_{j \in [n]} \log_2(d'_j) \\ &\geq \log_2 \frac{n!}{n^n} - \sum_{i \in [n]} \log_2(d_i) - \sum_{j \in [n]} \log_2(d'_j) \\ &\quad - \sum_{i,j \in [n]} q_{i,j} \log_2 \frac{q_{i,j}}{d_i \cdot m_{i,j} \cdot d'_j} \\ &= \log_2 \frac{n!}{n^n} - \sum_{i,j \in [n]} q_{i,j} \log_2 \frac{q_{i,j}}{m_{i,j}}, \end{aligned}$$

where the first inequality follows from van der Waerden's conjecture (proven by Falikman [16] and by Egorychev [15]) and where the second inequality follows from the non-negativity of relative entropy. ■

We note that D and D' are auxiliary matrices in Theorem 7. Only their existence matters, while their entries do not play a role in (2). Additionally, for Theorem 7 to be meaningful, we note that $q_{i,j} = 0$ whenever $m_{i,j} = 0$, i.e., the support of Q should be a subset of the support of M . The right-hand side of (2) can be maximized with the help of Sinkhorn's balancing algorithm [38], see, e.g., the discussions in [21], [30], [43].

As in Section IV-A, we are interested in $M = A_{r,n}$ as defined in (1). Our goal is to find a doubly-stochastic matrix Q to use with Theorem 7. Such a matrix is found in the upcoming Lemma 8 and it is used in Theorem 9 to establish a lower bound on $\log_2 \text{per}(A_{r,n})$, and with that a lower bound on $\log_2 |B_{r,n}|$.

Lemma 8. Fix some $r, \frac{n-1}{2} \leq r \leq n-1$. The matrix $Q^* \triangleq (q_{i,j}^*)$ which maximizes the right-hand side of (2) for $M = A_{r,n} \triangleq (a_{i,j})$, is given by

$$q_{i,j}^* = a_{i,j} \cdot C \cdot \exp_2(\lambda_i) \cdot \exp_2(\lambda'_j), \quad i, j \in [n], \quad (3)$$

where

- if $r = n-1$, then $C \triangleq 1/n$, $\lambda_i \triangleq 0$ for $i \in [n]$, and $\lambda'_j \triangleq 0$ for $j \in [n]$;
- if $\frac{n-1}{2} \leq r \leq n-2$, then

$$\lambda_i \triangleq \begin{cases} ((n-r) - i) \cdot \log_2(\alpha_{r,n}) & 1 \leq i \leq n-r \\ 0 & n-r \leq i \leq r+1, \\ (i - (r+1)) \cdot \log_2(\alpha_{r,n}) & r+1 \leq i \leq n \end{cases}$$

$$\lambda'_j \triangleq \lambda_j, \quad j \in [n],$$

$$C \triangleq (\alpha_{r,n} - 1) \cdot \alpha_{r,n}^{-(n-r)} \triangleq \frac{\alpha_{r,n} - 1}{(2r - n + 2) - (2r - n) \cdot \alpha_{r,n}},$$

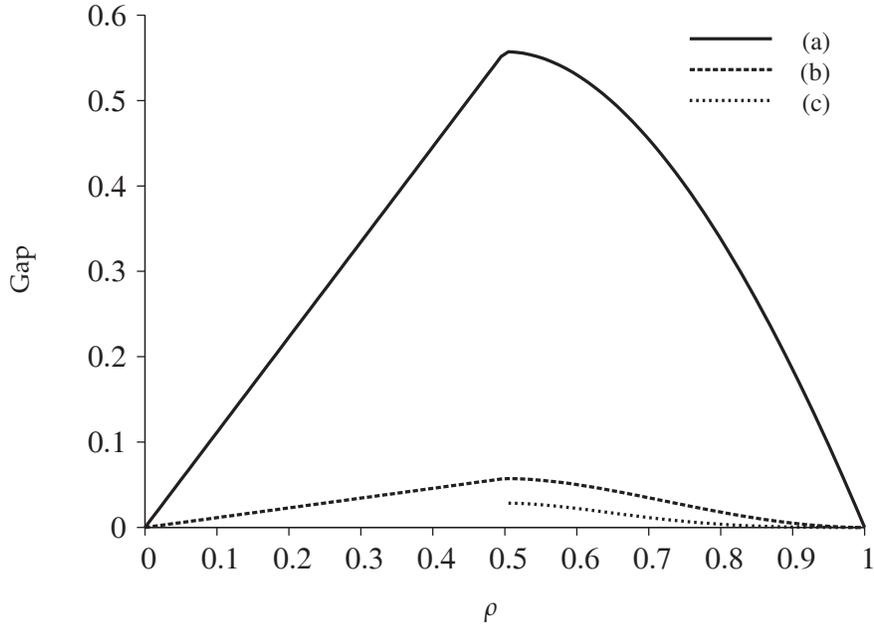


Figure 1. (a) $\text{Gap}_{\varphi_1}^{\Phi_1}(\rho)$ of Corollary 3; (b) $\text{Gap}_{\varphi_2}^{\Phi_1}(\rho)$ of Corollary 6; (c) $\text{Gap}_{\varphi_3}^{\Phi_1}(\rho)$ of Corollary 12.

and $\alpha_{r,n}$ is the unique positive solution of the equation

$$\alpha_{r,n}^{n-r} + (2r-n) \cdot \alpha_{r,n} - (2r-n+2) = 0.$$

In either case, note that

- $\lambda_{n+1-i} = \lambda_i, i \in [n]$,
- $\lambda'_{n+1-j} = \lambda'_j, j \in [n]$.

Proof: See [36].

Theorem 9. Fix some $r, \frac{n-1}{2} \leq r \leq n-1$. We have the following lower bound on $\log_2 |B_{r,n}|$:

- For $r = n-1$:

$$\log_2 |B_{r,n}| \geq \log_2(n!).$$

(Because $|B_{r,n}| = n!$, this lower bound is tight.)

- For $\frac{n-1}{2} \leq r \leq n-2$:

$$\log_2 |B_{r,n}| \geq \log_2(n!) - n \log_2(n) - n \cdot \log_2(\alpha_{r,n}-1) + (n-r) \cdot (2r-n+2) \cdot \log_2(\alpha_{r,n}),$$

where $\alpha_{r,n}$ was specified in Lemma 8.

Proof: See [36].

Note that the lower bound in Theorem 9 contains the constant $\alpha_{r,n}$. In order to get rid of this constant, the upcoming Lemma 10 suitably approximates this constant and Theorem 11 will then show the updated expression for the lower bound.

Lemma 10. Let $r = \rho \cdot (n-1)$, with $\frac{1}{2} < \rho < 1$ a constant. Then $\alpha_{r,n}$ from Lemma 8 is

$$\alpha_{r,n} = 1 + \left(\hat{t} + \Theta \left(\frac{1}{n} \right) \right) \left(2^{\frac{1}{(n-1)(1-\rho)+1}} - 1 \right),$$

where

$$\hat{t} \triangleq \frac{1}{\ln(2)} \left(\frac{2(1-\rho)}{2\rho-1} - W \left(\frac{(1-\rho) \exp \left(\frac{2(1-\rho)}{2\rho-1} \right)}{2\rho-1} \right) \right), \quad (4)$$

and where $W(\cdot)$ denotes Lambert's function, i.e., $W(\cdot)$ is defined by $z = W(z) \exp(W(z))$.

Proof: See [36].

Theorem 11.

Let $r = \rho \cdot (n-1)$, with $\frac{1}{2} < \rho < 1$ a constant. It holds that

$$|B_{r,n}| \geq \varphi_3(r,n) \cdot 2^{o(n)},$$

where

$$\varphi_3(r,n) \triangleq \frac{n^n \cdot 2^{\hat{t}(2\rho-1)n} \cdot (1-\rho)^n}{(e^{\hat{t}} \ln(2))^n},$$

and where \hat{t} is given by (4).

Proof: See [36].

Finally, the following corollary computes the asymptotic gap between the upper bound Φ_1 and the lower bound φ_3 .

Corollary 12. For $\frac{1}{2} < \rho < 1$,

$$\text{Gap}_{\varphi_3}^{\Phi_1}(\rho) = \log_2(\hat{t} \ln(2)) - \hat{t}(2\rho-1) - \log_2(1-\rho) - 2(1-\rho) \log_2(e) - 2\rho \log_2(\rho).$$

Proof: The proof is straightforward and is therefore omitted.

The result of Corollary 12 is visualized by curve (c) in Fig. 1.

V. CONCLUSION

As is evidenced by Fig. 1, we have been able to significantly reduce the asymptotic gap between the best known upper and lower bounds on $|B_{r,n}|$. One wonders if this asymptotic gap can be closed even further. In particular:

- In the case of the Bethe-permanent-based lower bound (Section IV-A), can one find analytically tractable matrices Q that yield better lower bounds?
- In the case of the Sinkhorn-balancing-based lower bound (Section IV-B), can one find an analytically tractable matrix Q also for the case $0 \leq r \leq \frac{n-1}{2}$? (For the case $\frac{n-1}{2} \leq r \leq n-1$, no improvement is possible as Lemma 8 found the matrix Q that maximizes the right-hand side of (2).)

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