Two-Dimensional Burst-Correcting Codes

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Abstract — **We consider two-dimensional error-correcting codes capable of correcting unrestricted bursts of size b. We construct optimal 2-burst-correcting codes in three connectivity models: the rectangular grid with 4 or 8 neighbors, and the hexagonal graph. We also give optimal, or nearly optimal, 2-burst-correcting codes in all dimensions. We then construct 3-burst-correcting codes with 3 redundancy bits above the sphere-packing bound, followed by b-straight-burst-correcting codes with $b-2$ redundancy bits above the sphere-packing bound. We conclude by improving the Reiger bound for two-dimensional unrestricted-burst-correcting codes.**

I. Basic Definitions

A two-dimensional linear code, C, is a linear subspace of the $n_1 \times n_2$ binary matrices. If the subspace is of dimension $n_1n_2 - r$, we say that the code is $[n_1 \times n_2, n_1n_2 - r]$. The code may be also defined by its parity-check matrix. Let $H = (h_{ijk})$ be a $n_1 \times n_2 \times r$ three-dimensional binary matrix, and let $c = (c_{ij})$ denote a binary $n_1 \times n_2$ matrix. The linear subspace defined by $\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} c_{ij}h_{ijk} = 0$, is an $[n_1 \times n_2, n_1n_2 - r]$ code. We say that r is the redundancy of the code.

We define an unrestricted burst of size b as a connected set of b coordinates such that all errors occur in coordinates belonging to the set. Different connectivity models produce different shapes of error bursts. All the neighbor sets of the different models are summarized in Figure 1.

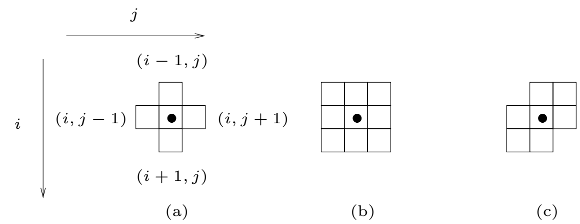

Figure 1: Neighbors of coordinate (i, j) in the (a) $+$ model (b) $\ast$ model (c) hexagonal model.

II. Two-Dimensional Burst-Correcting Codes

Construction A: Let α be a primitive root of $GF(2^m)$. We construct the following $(2^m - 1) \times (2^m - 1) \times (2m + 2)$ check matrix $H_2(m)$: $h_{ij} = [1, i \mod 2, \alpha^{i+j}, \alpha^{i-j}]^T$.

Theorem 1 *$H_2(m)$ is the parity-check matrix of an optimal 2-burst-correcting code for all $m \geq 3$.*

This construction may be generalized to optimal and nearly optimal D-dimensional codes as seen in [1].

Construction B: Let α be a primitive root of $GF(2^m)$, m even, $\log_\alpha(1 + \alpha) \not\equiv 2 \pmod 3$, and β a primitive root of $GF(4)$. We construct the following $(2^m - 1) \times (2^m - 1) \times (2m + 3)$ parity-check matrix $H_2^{\ast}(m)$: $h_{ij} = [j \mod 2, \beta^{i+2j}, \alpha^{i+2j}, \alpha^{i-2j}]^T$.

Theorem 2 *$H_2^{\ast}(m)$ is the parity-check matrix of an optimal 2-burst-correcting code in the $\ast$ model for all even $m \geq 6$.*

Construction C: Let α be a primitive root of $GF(2^m)$, m even, $\log_\alpha(1 + \alpha) \not\equiv 2 \pmod 3$, and β a primitive root of $GF(4)$. We construct the following $(2^m-1)\times(2^m-1)\times(2m+2)$ parity-check matrix $H_2^{\mathrm{hex}}(m)$: $h_{ij} = [\beta^{i-2j}, \alpha^{i+2j}, \alpha^{i-2j}]^T$.

Theorem 3 *$H_2^{\mathrm{hex}}(m)$ is the parity-check matrix of an optimal 2-burst-correcting code in the hexagonal model for all even $m \geq 4$.*

Construction D: Let α be a primitive root of $GF(2^m)$, m even, $\log_\alpha(1 + \alpha) \not\equiv 2 \pmod 3$, and β a primitive root of $GF(4)$. We construct the following $(2^m - 1) \times (2^m - 1) \times (2m + 7)$ parity-check matrix $H_3(m)$: $h_{ij} = [1, \beta^i, \beta^{i+2j}, \beta^{i-2j}, \alpha^{i+2j}, \alpha^{i-2j}]^T$.

Theorem 4 *$H_3(m)$ is the parity-check matrix of a 3-burst-correcting code.*

When the b-bursts are in rows or columns we have:

Construction E: Let $\mathbf{e}_i$, $1 \leq i \leq b-1$, be the i-th length $b-1$ unit vector. By abuse of notation we denote by $\mathbf{e}_0$ the all-zero vector. Take H, a $(m + b - 1) \times (2^m - 1)$ binary parity-check matrix for a one-dimensional b-cyclic-burst correcting code [2]. We denote the columns of H as $h_0, h_1, \dots, h_{2^m-2}$. Also, let α be a primitive root of $GF(2^m)$. We construct the following $(2^m-1)\times(2^m-1)\times(2m+2b-2)$ parity-check matrix $H_b^{(S)}(m)$: $h_{ij} = [\mathbf{e}_{i \bmod b}, h_{i+j}, \alpha^{i-j}]^T$, where the index of h_{i+j} is taken modulo $2^m - 1$.

Theorem 5 *$H_b^{(S)}(m)$ is the parity-check matrix of a b-straight-burst-correcting code.*

III. Singleton-Type Bounds

We give Singleton-type bounds, the strongest of which is:

Theorem 6 *If C is a $[n_1 \times n_2, n_1n_2 - r]$ b-burst-correcting code with $b \geq 4$, $n_1 \geq 3$, $n_2 \geq \lceil \frac{6b-2}{7} \rceil + 1$, and $n_1n_2 \geq 2b + \lfloor \frac{4b-6}{7} \rfloor + 1$, then $r \geq 2b + \lfloor \frac{4b-6}{7} \rfloor$.*

References

[1] M. Schwartz and T. Etzion, "Two-dimensional burst-correcting codes," in preparation.

[2] K. A. S. Abdel-Ghaffar, R. J. McEliece, A. M. Odlyzko, and H. C. A. van Tilborg, "On the existence of optimum cyclic burst-correcting codes," *IEEE Trans. on Inform. Theory*, vol. 32, no. 6, pp. 768–775, Nov. 1986.