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From amplitude-limited vectors to Maxwell's equations

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Summary. — Physical experiments show c, the speed of light *in vacuo* to be a universal constant. Relativistic arguments show it to be also an upper bound on physically measurable velocity. Vector addition of velocities, large or incrementally small, is no longer acceptable as the law of composition of measured velocities. Amplitude-limitation is imposed by using velocity vectors as the defining parameters of sophisticated operators such as four-vectors of unit norm and boost matrices whose interactions never allow it to exceed c. In a previous paper the unit energy-momentum matrix was added and for a given parametrizing velocity was shown to have the structure of the square of the corresponding boost matrix. As a matrix its Lorentz transformation needs pre- and post-multiplication by the appropriate boost. Using the same operator, but mapped into SL(2C), to cover electric and magnetic vectors, the pre- and post multipliers must now be mutually inverse since mere transformation cannot generate an electromagnetic field. The resulting formulae generate Maxwell's equations.

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Notation

Vectors are written $\vec{\mathbf{u}} \equiv (u_1, u_2, u_3)$. Unit vectors or tensors are written $\hat{\mathbf{v}}$, $\hat{\mathbf{f}}$, $\hat{\mathbf{T}}$, etc. Greek symbols are scalars. Velocity is measured in units for which c the speed of light is unity. A vector may also be considered as a column matrix. Subscript s means measured in the observer's framework and m in a framework moving with respect to the observer. $\vec{\mathbf{o}}$ is the zero vector, $\mathbf{I}$ the unit matrix, $\mathbf{i} = (-1)^{1/2}$ and $\boldsymbol{\eta}$ the matrix whose leading diagonal is $(1, -1, -1, -1)$ and is zero elsewhere. All parameters are real unless specifically named complex.

1. – Introduction

The experiments of Michelson and Morley as reported by Moller [1] show c, the speed of light *in vacuo*, to be a universal constant. Moller also shows it to be an upper bound on velocity. By contrast the mathematical concept of a Cartesian velocity vector obeying vector addition is unbounded. It follows that such addition is no longer a viable model for composition of physical velocities. In its place, from a velocity $\vec{\mathbf{v}} = v\hat{\mathbf{v}}$ a FitzGerald coefficient $\gamma = (1 - v^2)^{-1/2}$ is formed and the combination used to parametrize a more sophisticated operator such as the four-vector

$$\mathbf{V}(\gamma, \gamma\vec{\mathbf{v}}) \equiv \{\gamma, \gamma\vec{\mathbf{v}}\} = \{\gamma, \gamma v_1, \gamma v_2, \gamma v_3\} \tag{1.1}$$

of unit norm $\mathbf{V}\boldsymbol{\eta}\mathbf{V} = \gamma^2 - \gamma^2 v^2 = 1$ and the symmetric boost matrix given by Ungar [2]

$$\mathbf{B}(\gamma, \gamma\vec{\mathbf{v}}) \equiv \begin{bmatrix} \gamma & \gamma v_1 & \gamma v_2 & \gamma v_3 \\ \gamma v_1 & 1 + \frac{\gamma^2 {v_1}^2}{1+\gamma} & \frac{\gamma^2 v_1 v_2}{1+\gamma} & \frac{\gamma^2 v_1 v_3}{1+\gamma} \\ \gamma v_2 & \frac{\gamma^2 v_2 v_1}{1+\gamma} & 1 + \frac{\gamma^2 {v_2}^2}{1+\gamma} & \frac{\gamma^2 v_2 v_3}{1+\gamma} \\ \gamma v_3 & \frac{\gamma^2 v_3 v_1}{1+\gamma} & \frac{\gamma^2 v_3 v_2}{1+\gamma} & 1 + \frac{\gamma^2 {v_3}^2}{1+\gamma} \end{bmatrix} \equiv \begin{bmatrix} \gamma & \gamma\vec{\mathbf{v}}^T \\ \gamma\vec{\mathbf{v}} & \mathbf{I} + [(\gamma\vec{\mathbf{v}})(\gamma\vec{\mathbf{v}})^T/(1+\gamma)] \end{bmatrix} \tag{1.2}$$

with $\det \mathbf{B} = \gamma^2 - \gamma^2 v^2 = 1$. For γ to be real in such operators $v < 1$. For clarity this will be termed amplitude-limitation and is imposed by the physics of the variable. In building mathematical models for physical phenomena only similarly restrictive operators may appear. (1.1) and (1.2) are used to model the transformation of measurements by inertial observers S_s and S_m, in relative motion with velocity $\vec{\mathbf{u}}$ as measured by S_s, of the velocities $\vec{\mathbf{v}}_s$ and $\vec{\mathbf{v}}_m$ of some mobile. The equation, Einstein's transformation without rotation, is

$$\mathbf{V}(\gamma_s, \gamma_s\vec{\mathbf{v}}_s) = \mathbf{B}(\gamma_u, \gamma_u\vec{\mathbf{u}})\mathbf{V}(\gamma_m, \gamma_m\vec{\mathbf{v}}_m) \tag{1.3}$$

which written out in full reads

$$\gamma_s = \gamma_u\gamma_m + (\gamma_m\vec{\mathbf{v}}_m)^T(\gamma_u\vec{\mathbf{u}}), \tag{1.4}$$

$$\gamma_s\vec{\mathbf{v}}_s = \gamma_m\vec{\mathbf{v}}_m + \{\gamma_m + (\gamma_m\vec{\mathbf{v}}_m)^T(\gamma_u\vec{\mathbf{u}})/(1+\gamma_u)\}(\gamma_u\vec{\mathbf{u}}). \tag{1.5}$$

2. – The $\mathbf{B}^2$ matrix

Another admissible operator, central to this article, is obtained from (1.2) by direct multiplication

$$\mathbf{B}^2(\gamma, \gamma\vec{\mathbf{v}}) = \begin{bmatrix} \gamma & \gamma\vec{\mathbf{v}}^T \\ \gamma\vec{\mathbf{v}} & \mathbf{I} + [(\gamma\vec{\mathbf{v}})(\gamma\vec{\mathbf{v}})^T/(1+\gamma)] \end{bmatrix}^2 = \begin{bmatrix} 2\gamma^2 - 1 & 2\gamma^2\vec{\mathbf{v}}^T \\ 2\gamma^2\vec{\mathbf{v}} & \mathbf{I} + 2(\gamma\vec{\mathbf{v}})(\gamma\vec{\mathbf{v}})^T \end{bmatrix}, \tag{2.1}$$

which, by inspection, can also be written as

$$\mathbf{B}^2(\gamma,\gamma\vec{\mathbf{v}}) = -\boldsymbol{\eta} + 2\mathbf{V}(\gamma,\gamma\vec{\mathbf{v}})\mathbf{V}^T(\gamma,\gamma\vec{\mathbf{v}}). \tag{2.2}$$

By (1.3)

$$\mathbf{V}(\gamma_s,\gamma_s\vec{\mathbf{v}}_s)\mathbf{V}^T(\gamma_s,\gamma_s\vec{\mathbf{v}}_s) = \mathbf{B}(\gamma_u,\gamma_u\vec{\mathbf{u}})\cdot\mathbf{V}(\gamma_m,\gamma_m\vec{\mathbf{v}}_m)\mathbf{V}^T(\gamma_m,\gamma_m\vec{\mathbf{v}}_m)\cdot\mathbf{B}(\gamma_u,\gamma_u\vec{\mathbf{u}}). \tag{2.3}$$

Add $-\boldsymbol{\eta}$ to twice the left and its equivalent $-\mathbf{B}\boldsymbol{\eta}\mathbf{B}$ to twice the right and rearrange to give

$$\mathbf{B}^2(\gamma_s,\gamma_s\vec{\mathbf{v}}_s) = \mathbf{B}(\gamma_u,\gamma_u\vec{\mathbf{u}})\cdot\mathbf{B}^2(\gamma_m,\gamma_m\vec{\mathbf{v}}_m)\cdot\mathbf{B}(\gamma_u,\gamma_u\vec{\mathbf{u}}). \tag{2.4}$$

Conversely, starting from (2.4) reduce it to (2.3). By (1.3) a velocity $\vec{\mathbf{w}}$ exists such that

$$\mathbf{B}(\gamma_u,\gamma_u\vec{\mathbf{u}})\mathbf{V}(\gamma_m,\gamma_m\vec{\mathbf{v}}_m) = \mathbf{V}(\gamma_w,\gamma_w\vec{\mathbf{w}}). \tag{2.5}$$

Substituting in (2.3) and writing out both sides in full shows $\vec{\mathbf{w}} = \vec{\mathbf{v}}_m$ proving that the well-known theorems of special relativity can be deduced by starting from (2.4), and in particular identification of $\mathbf{V}(\gamma,\gamma\vec{\mathbf{v}})$ as the unit energy-momentum vector whose form invariance under boost transformation ensures preservation of the principles of conservation of relative mass (energy) and momentum. For this reason $\mathbf{B}^2$ is called below the unit energy-momentum matrix. It was originally derived in [3] as the Minkowski metric equivalent of the unit energy-momentum tensor of general relativity $-\mathbf{g}^{ij} + 2\mathbf{U}^i\mathbf{U}^j$. Its composition law was used to replace vector addition in calculating acceleration. The resulting update for Newton's second law correctly predicts perihelion advance and bending of light.

3. – The rotation matrix R

$\mathbf{B}^2$ is parametrized by a polar vector such as velocity or electric intensity $\vec{\mathbf{e}}$. Its partner for axial vectors such as the magnetic $\vec{\mathbf{h}}$ is the rotation matrix $\mathbf{R}$ or its derivative $\mathbf{R}^2$, also a rotation matrix. This operator is built from an axial pseudo-vector $\vec{\mathbf{n}} = n\hat{\mathbf{n}}$ by first constructing the antisymmetrical matrix

$$\mathbf{N} \equiv \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \tag{3.1}$$

and, introducing β such that $(\beta^2 + n^2) = 1$. The corresponding rotation matrix is

$$\mathbf{R}(\beta,\vec{\mathbf{n}}) \equiv \mathbf{I} + \mathbf{N} + \mathbf{N}^2/(\beta+1). \tag{3.2}$$

For any vector $\vec{\mathbf{x}}$

$$\mathbf{N}\vec{\mathbf{x}} \equiv \vec{\mathbf{n}}\times\vec{\mathbf{x}}, \tag{3.3}$$

$$\mathbf{R}(\cos\varphi,\sin\varphi\hat{\mathbf{n}})\vec{\mathbf{x}} = \cos\varphi\vec{\mathbf{x}} + \sin\varphi\hat{\mathbf{n}}\times\vec{\mathbf{x}} + (1-\cos\varphi)(\hat{\mathbf{n}}^T\vec{\mathbf{x}})\hat{\mathbf{n}} \tag{3.4}$$

which is the classical formula for rotation of $\vec{\mathbf{x}}$ through angle φ around $\hat{\mathbf{n}}$. By inspection

$$\mathbf{R}^T(\beta,\vec{\mathbf{n}}) = \mathbf{R}(\beta,-\vec{\mathbf{n}}) = \mathbf{R}^{-1}(\beta,\vec{\mathbf{n}}), \tag{3.5}$$

$$\mathbf{R}^2 = \mathbf{I} + 2\beta\mathbf{N} + 2\mathbf{N}^2\,. \tag{3.6}$$

There is no ambiguity in using the same symbol $\mathbf{R}$ for the four-dimensional matrix

$$\mathbf{R} \equiv \begin{bmatrix} 1 & \cdot \\ \cdot & \mathbf{R} \end{bmatrix} \tag{3.7}$$

with dimensions fixed by the context.

4. – Quinors

Particularly in building models for electromagnetic field interactions the algebra is considerably reduced by working in SL(2C), the group of 2×2 complex matrices homomorphic to the proper Lorentz group of $\mathbf{B}$ and $\mathbf{R}$. This can be carried out by mapping into spinors constructed from Pauli matrices. However, checking their Lorentz equivalents as given by Naimark [4] shows that they imply operation in a left-handed coordinate framework, obtained from the conventional right-handed system by reversing the direction of the y-axis.

To avoid potential errors of interpretation this paper uses an equivalent operator, the quinor. Any matrix of SL(2C) can be rewritten in terms of a complex scalar ψ and a complex vector $\vec{\mathbf{p}}$ in the form

$$\mathbf{Q}(\psi,\vec{\mathbf{p}}) \equiv \begin{bmatrix} \psi + \mathbf{i}p_3 & -p_2 + \mathbf{i}p_1 \\ p_2 + \mathbf{i}p_1 & \psi - \mathbf{i}p_3 \end{bmatrix}. \tag{4.1}$$

$\det \mathbf{Q} = \psi^2 + \vec{\mathbf{p}}^T\vec{\mathbf{p}} = 1$ is the group property. The combination law for complex elements is

$$\mathbf{Q}(\lambda,\vec{\mathbf{x}})\mathbf{Q}(\mu,\vec{\mathbf{y}}) = \mathbf{Q}[(\lambda\mu - \vec{\mathbf{x}}^T\vec{\mathbf{y}}),(\lambda\vec{\mathbf{y}} + \mu\vec{\mathbf{x}} + \vec{\mathbf{x}} \times \vec{\mathbf{y}})]. \tag{4.2}$$

The unit element is $\mathbf{Q}(1,\vec{\mathbf{o}})$ and the inverse of $\mathbf{Q}(\psi,\vec{\mathbf{p}})$ is $\mathbf{Q}(\psi,-\vec{\mathbf{p}})$. The $\mathbf{Q}$ notation is chosen for alignment with the isomorphic group of complex quaternions of unit norm, or quinors. The correspondence with the proper Lorentz group is given in detail in [4]. In particular, for real parameters

$$\pm\, \mathbf{Q}(\gamma,-\mathbf{i}\gamma\vec{\mathbf{v}}) \to \mathbf{B}^2(\gamma,\gamma\vec{\mathbf{v}}), \tag{4.3}$$

$$\pm\, \mathbf{Q}(\beta,\vec{\mathbf{n}}) \to \mathbf{R}^2(\beta,\vec{\mathbf{n}}). \tag{4.4}$$

Only the + sign is used below.

5. – Boost transformation by quinors

(2.4) models boost transformation of $\mathbf{B}^2$. To map it into quinor language write

$$\vec{\mathbf{u}} = \tanh\varphi\hat{\mathbf{u}}, \qquad \gamma_u = \cosh\varphi \qquad \gamma_u\vec{\mathbf{u}} = \sinh\varphi\hat{\mathbf{u}} \qquad c \equiv \cosh(\varphi/2) \qquad s \equiv \sinh(\varphi/2)$$

giving, by (4.3)

$$\mathbf{Q}(\gamma_s, -\mathbf{i}\gamma_s\vec{\mathbf{v}}_s) = \mathbf{Q}(\mathbf{c}, -\mathbf{is}\hat{\mathbf{u}}) \cdot \mathbf{Q}(\gamma_m, -\mathbf{i}\gamma_m\vec{\mathbf{v}}_m) \cdot \mathbf{Q}(\mathbf{c}, -\mathbf{is}\hat{\mathbf{u}}) \tag{5.1}$$

whose solution is therefore still (1.4) and (1.5). In the particular case of the mobile being at rest in S_m, $\vec{\mathbf{v}}_m = \vec{\mathbf{o}}$ and

$$\mathbf{Q}(\gamma_m, -\mathbf{i}\gamma_m\vec{\mathbf{v}}_m) = \mathbf{Q}(1, \vec{\mathbf{o}}) \tag{5.2}$$

and (5.1) reads

$$\mathbf{Q}(\gamma_s, -\mathbf{i}\gamma_s\vec{\mathbf{v}}_s) = \mathbf{Q}(\mathbf{c}, -\mathbf{is}\hat{\mathbf{u}}) \cdot \mathbf{Q}(1, \vec{\mathbf{o}}) \cdot \mathbf{Q}(\mathbf{c}, -\mathbf{is}\hat{\mathbf{u}}) = \mathbf{Q}(\gamma_u, -\mathbf{i}\gamma_u\vec{\mathbf{u}}) \tag{5.3}$$

since the mobile is still in motion with respect to S_s.

6. – The amplitude-limited electromagnetic field in vacuo

Born and Infeld [5] (BI below) introduced the concept of amplitude limitation for the electric $\vec{\mathbf{e}}$ and magnetic $\vec{\mathbf{h}}$ vectors of the electromagnetic field by appeal to "… the principle of finiteness which postulates that a satisfactory theory should avoid letting physical quantities become infinte. … Applying it to the electromagnetic field one is led immediately to the assumption of an upper limit on field strength…".

Supposing the vectors to be measured in a universal system in which the upper limit $\kappa = 1$, $\vec{\mathbf{e}}$ will now enter purely electric mathematical models via $\mathbf{B}^2(\varepsilon, \varepsilon\vec{\mathbf{e}})$. In dynamics $\mathbf{B}^2(\gamma, \gamma\vec{\mathbf{v}})$ introduces the FitzGerald coefficient $\gamma = (1 - v^2)^{-1/2}$ which is then used in $m_0\gamma$ and $m_0\gamma\vec{\mathbf{v}}$ to characterize relative mass and momentum respectively. By contrast to $\vec{\mathbf{v}}$ which is bounded by $v = |\vec{\mathbf{v}}| \le 1$, relative mass and momentum are unbounded mathematically. For this reason the photon is held to have zero m_0.

Restricting further development to electromagnetic theory *in vacuo*, BI is a non-dualistic field theory. This is consistent with the definition of the electron as given in [6] "… no known size, assumed point-like no known structure…". It is therefore a point singularity in a purely magnetic field, and as such if $\vec{\mathbf{e}}$ is unbounded, its self-energy equal to its mass is in classical theory

$$\mathbf{E} = (1/8\pi) \cdot \int_0^\infty \left(\vec{\mathbf{d}}^T\vec{\mathbf{e}}\right) \cdot 4\pi r^2 \mathrm{d}r, \tag{6.1}$$

where $\vec{\mathbf{d}}$ is the classic electric displacement vector, this will, even *in vacuo* be infinite for an inverse-square law.

If $\vec{\mathbf{e}}$ in general units is bounded by κ it is still necessary to find an associated function which is unbounded and can therefore be used, as will be shown immediately, to obey the inverse-square law, and the choice proposed is a non-linear electric displacement.

$$\vec{\mathbf{d}} = \varepsilon\vec{\mathbf{e}} = \vec{\mathbf{e}}/[1 - (e/\kappa)^2]^{1/2} \tag{6.2}$$

which can be inverted to give

$$\vec{\mathbf{e}} = \vec{\mathbf{d}}/[1 + (d/\kappa)^2]^{1/2}\,. \tag{6.3}$$

In the electrostatic field of an isolated electron of charge q

$$\vec{\mathbf{d}} = (q/r^2)\hat{\mathbf{r}} \tag{6.4}$$

and the total energy in the field is

$$\begin{aligned} \mathbf{E} &= (1/8\pi) \cdot \int_0^\infty (\vec{\mathbf{d}}^T\vec{\mathbf{e}}) \cdot 4\pi r^2 \mathrm{d}r = (1/2) \cdot \int_0^\infty q^2/(r^4 + q^2/\kappa^2)^{1/2} \mathrm{d}r \\ &= (1/2) \cdot (\kappa q^3)^{1/2} \cdot \int_0^\infty (1+\rho^4)^{-1/2} \mathrm{d}\rho = (1/2) \cdot (\kappa q^3)^{1/2} \times 1.8541. \end{aligned} \tag{6.5}$$

Equating this to electron mass mc^2 gives $\kappa = 2.112 \times 10^{20}$ v/m. Assuming this to be a universal value and choosing units in which it is unity shows that $e \ll 1$ in all normal fields. The comparable figure for a magnetic vector is $\kappa = 2.112 \times 10^{12}$ oersteds.

7. – Quinor mapping of an electromagnetic field in vacuo

$\vec{\mathbf{e}}$ and $\vec{\mathbf{h}}$ parametrize quinor mapping via $\mathbf{Q}(\varepsilon, -\mathbf{i}\varepsilon\vec{\mathbf{e}})$ and $\mathbf{Q}(\beta, \vec{\mathbf{h}})$, respectively. In the previous section $\varepsilon\vec{\mathbf{e}}$ was identified as $\vec{\mathbf{d}}$ in a purely electric field and in a similar fashion $\vec{\mathbf{h}}/\beta$ will be identified as $\vec{\mathbf{b}}$, the magnetic induction, in a purely magnetic field. The validity of such identification in a mixed field is discussed below.

In $\mathbf{B}$, $\mathbf{R}$ Lorentz group language there is an immediate problem, in a mixed field, of ordered placement, *e.g.*, by $\mathbf{B}^2\mathbf{R}^2$, $\mathbf{R}^2\mathbf{B}^2$, $\mathbf{R}\mathbf{B}^2\mathbf{R}$ or any more convoluted representation since such placements affect the accompanying algebra. The difficulty vanishes in quinor mapping by adopting as standard

$$\mathbf{Q}\big[(\lambda + \mathbf{i}\mu), (\vec{\mathbf{h}} - \mathbf{i}\vec{\mathbf{d}})\big], \tag{7.1}$$

where by the group property

$$\lambda^2 - \mu^2 + h^2 - d^2 = 1, \tag{7.2}$$

$$\lambda\mu = \vec{\mathbf{d}}^T\vec{\mathbf{h}} \tag{7.3}$$

and the absence of an electromagnetic field is modeled by $\mathbf{Q}(1, \vec{\mathbf{o}})$. The μ parameter is needed since otherwise (7.3) would automatically define the $\vec{\mathbf{d}}$ and $\vec{\mathbf{h}}$ components as mutually perpendicular.

In Lorentz transformation of velocity the quinor $\mathbf{Q}(\mathbf{c}, -\mathbf{i}\mathbf{s}\hat{\mathbf{u}})$ defines the transformation from the coordinate system of S_m to that of S_s but has no implicit connection to the operator being transformed. Nevertheless, in the particular case of the mobile being at rest in S_m its velocity in S_s is shown by (5.3) to be other than zero.

This shows that (5.1) and (5.3) cannot be used to model boost transformation of an electric $\mathbf{Q}(\varepsilon, -\mathbf{i}\varepsilon\vec{\mathbf{e}})$, etc. since, if $\vec{\mathbf{e}}_m = \vec{\mathbf{o}}$, mere coordinate transformation cannot generate a field. The replacement meeting that requirement, based on $\mathbf{Q}(\mathbf{c}, +\mathbf{i}\mathbf{s}\hat{\mathbf{u}}) = \mathbf{Q}^{-1}(\mathbf{c}, -\mathbf{i}\mathbf{s}\hat{\mathbf{u}})$, is

$$\begin{aligned} \mathbf{Q}\big[(\lambda_s + \mathbf{i}\mu_s), (\vec{\mathbf{h}}_s - \mathbf{i}\vec{\mathbf{d}}_s)\big] = \mathbf{Q}(\mathbf{c}, +\mathbf{i}\mathbf{s}\hat{\mathbf{u}}) \cdot \mathbf{Q}\big[(\lambda_m + \mathbf{i}\mu_m), (\vec{\mathbf{h}}_m - \mathbf{i}\vec{\mathbf{d}}_m)\big] \\ \cdot \mathbf{Q}(\mathbf{c}, -\mathbf{i}\mathbf{s}\hat{\mathbf{u}}) \end{aligned} \tag{7.4}$$

which by repeated multiplication gives

$$\lambda_s + \mathbf{i}\mu_s = \lambda_m + \mathbf{i}\mu_m \,, \tag{7.5}$$

$$\vec{\mathbf{d}}_s = \gamma_u\big(\vec{\mathbf{d}}_m - \vec{\mathbf{d}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}\big) + \vec{\mathbf{d}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}} - \gamma_u\vec{\mathbf{u}}\times\vec{\mathbf{h}}_m \,, \tag{7.6}$$

$$\vec{\mathbf{h}}_s = \gamma_u\big(\vec{\mathbf{h}}_m - \vec{\mathbf{h}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}\big) + \vec{\mathbf{h}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}} + \gamma_u\vec{\mathbf{u}}\times\vec{\mathbf{d}}_m \,. \tag{7.7}$$

By (7.5) both λ and μ are invariants of the transformation and the two sides of (7.6) and (7.7) can be divided throughout by any function $\mathbf{f}(\lambda,\mu)$ when $\vec{\mathbf{h}} - \mathbf{i}\vec{\mathbf{d}}$ becomes $\vec{\mathbf{h}}/\mathbf{f} - \mathbf{i}\vec{\mathbf{d}}/\mathbf{f}$. Defining by this

$$\vec{\mathbf{e}} = \vec{\mathbf{d}}/\mathbf{f}, \qquad \vec{\mathbf{b}} = \vec{\mathbf{h}}/\mathbf{f} \tag{7.8}$$

implies abandoning the definitions of those relations in the electrostatic and magnetostatic fields in favour of a pragmatic definition for a mixed field. Equations (7.6) and (7.7) then generate similar relations in which $\vec{\mathbf{d}}$ is replaced by $\vec{\mathbf{e}}$ and $\vec{\mathbf{h}}$ by $\vec{\mathbf{b}}$ throughout.

To avoid hybrid polar-axial structures for $\vec{\mathbf{e}}$ and $\vec{\mathbf{b}}$ the arbitrary function $\mathbf{f}(\lambda,\mu)$ must be real. The simplest possibility is $\mathbf{f}(\lambda,\mu) = \lambda$. Substituting in (7.2) and (7.3) and rearranging gives $\lambda = (1+F-G^2)^{-1/2}$ where $F = b^2 - e^2$ and $G = \vec{\mathbf{b}}^T\vec{\mathbf{e}}$ are well-known invariants of classical theory so that in a mixed field

$$\vec{\mathbf{d}} = \vec{\mathbf{e}}/(1+F-G^2)^{1/2} \,, \tag{7.9}$$

$$\vec{\mathbf{h}} = \vec{\mathbf{b}}/(1+F-G^2)^{1/2} \tag{7.10}$$

and the parameters satisfy

$$(\mathrm{d}/e)\cdot(b/h) = 1, \tag{7.11}$$

equivalent to the classical $(\mathrm{d}/e)\cdot(b/h) = 1/c^2$.

Similar relations were given by BI, using a completely different methodology

$$\vec{\mathbf{d}} = \big(\vec{\mathbf{e}} + G\vec{\mathbf{b}}\big)/(1+F-G^2)^{1/2} \,, \tag{7.12}$$

$$\vec{\mathbf{h}} = \big(\vec{\mathbf{b}} - G\vec{\mathbf{e}}\big)/(1+F-G^2)^{1/2} \tag{7.13}$$

but these must be rejected since they feature hybrid polar-axial vectors. They can be reached using the arguments of this paper by choosing the complex

$$\mathbf{f}(\lambda,\mu) = (\lambda^2 - \mu^2 - \lambda^2\mu^2)/(1+\mathbf{i}\lambda\mu). \tag{7.14}$$

8. – Boost transformation of an electrostatic field

In (7.6), (7.7) taking $\vec{\mathbf{h}}_m = \vec{\mathbf{o}}$ they reduce to

$$\vec{\mathbf{d}}_s = \gamma_u\big(\vec{\mathbf{d}}_m - \vec{\mathbf{d}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}\big) + \vec{\mathbf{d}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}, \tag{8.1}$$

$$\begin{aligned}\vec{\mathbf{h}}_s &= \gamma_u\vec{\mathbf{u}}\times\vec{\mathbf{d}}_m \\ &= \vec{\mathbf{u}}\times\vec{\mathbf{d}}_s \,.\end{aligned} \tag{8.2}$$

Suppose an observer in S_s travels with constant velocity $\vec{\mathbf{v}}$ and notes the changing composition of some arbitrary vector $\vec{\mathbf{x}}_s$ whose total change in time t_s is given by the Euler equation as

$$\mathrm{D}\vec{\mathbf{x}}_s/\mathrm{D}t_s = \partial\vec{\mathbf{x}}_s/\partial t_s + (\vec{\mathbf{v}} \bullet \nabla_s)\vec{\mathbf{x}}_s\,. \tag{8.3}$$

By the properties of the operator ∇ and the constancy of $\vec{\mathbf{v}}$

$$\nabla_s \times (\vec{\mathbf{v}} \times \vec{\mathbf{x}}_s) = (\nabla_s \bullet \vec{\mathbf{x}}_s)\vec{\mathbf{v}} - (\vec{\mathbf{v}} \bullet \nabla_s)\vec{\mathbf{x}}_s\,. \tag{8.4}$$

Adding (8.3) and (8.4)

$$\mathrm{D}\vec{\mathbf{x}}_s/\mathrm{D}t_s + \nabla_s \times (\vec{\mathbf{v}} \times \vec{\mathbf{x}}_s) = \partial\vec{\mathbf{x}}_s/\partial t_s + (\nabla_s \bullet \vec{\mathbf{x}}_s)\vec{\mathbf{v}}. \tag{8.5}$$

Take $\vec{\mathbf{x}} = \vec{\mathbf{d}}_s$ and $\vec{\mathbf{v}} = \vec{\mathbf{u}}$. The observer is now keeping pace with S_m so that both $\vec{\mathbf{d}}_s$ and $\vec{\mathbf{h}}_s$ are unchanging and in particular $\mathrm{D}\vec{\mathbf{d}}_s/\mathrm{D}t_s = \vec{\mathbf{o}}$. By (8.2), (8.5) becomes

$$\nabla_s \times \vec{\mathbf{h}}_s - \partial\vec{\mathbf{d}}_s/\partial\mathbf{t}_s = \big(\nabla_s \bullet \vec{\mathbf{d}}_s\big)\vec{\mathbf{u}} \tag{8.6}$$

which is one-half of Maxwell's equations for amplitude-limited vectors. Taking the divergence of both sides

$$\partial\big(\nabla_s \bullet \vec{\mathbf{d}}_s\big)/\partial\mathbf{t}_s + \nabla_s \bullet \big[\big(\nabla_s \bullet \vec{\mathbf{d}}_s\big)\vec{\mathbf{u}}\big] = 0 \tag{8.7}$$

which is the corresponding continuity equation.

9. – Boost transformation of a magnetostatic field

In this case, by the arguments of sect. **7**, (7.6) and (7.7) reduce to

$$\vec{\mathbf{b}}_s = \gamma_u\big(\vec{\mathbf{b}}_m - \vec{\mathbf{b}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}\big) + \vec{\mathbf{b}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}, \tag{9.1}$$

$$\begin{aligned}\vec{\mathbf{e}}_s &= -\gamma_u\vec{\mathbf{u}} \times \vec{\mathbf{b}}_m \\ &= -\vec{\mathbf{u}} \times \vec{\mathbf{b}}_s\,.\end{aligned} \tag{9.2}$$

Repeating the arguments of the previous section gives the other half of Maxwell's equations for amplitude-limited vectors

$$\nabla_s \times \vec{\mathbf{e}}_s + \partial\vec{\mathbf{b}}_s/\partial\mathbf{t}_s = -\big(\nabla_s \bullet \vec{\mathbf{b}}_s\big)\vec{\mathbf{u}} \tag{9.3}$$

and the continuity equation

$$\partial\big(\nabla_s \bullet \vec{\mathbf{b}}_s\big)/\partial\mathbf{t}_s + \nabla_s \bullet \big[\big(\nabla_s \bullet \vec{\mathbf{b}}_s\big)\vec{\mathbf{u}}\big] = 0. \tag{9.4}$$

10. – The radiation field

Boost transformation of a mixed field is given by (7.6) and (7.7). From them

$$\vec{\mathbf{h}}_s - \vec{\mathbf{u}} \times \vec{\mathbf{d}}_s = \gamma_u(1-u^2)\cdot\left(\vec{\mathbf{h}}_m - \vec{\mathbf{h}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}\right) + \vec{\mathbf{h}}_m{}^T\hat{\mathbf{u}}\hat{\mathbf{u}}\,, \tag{10.1}$$

so that if (8.2) is to remain valid, necessary and sufficient conditions are

$$u = 1, \tag{10.2}$$

$$\vec{\mathbf{h}}_m{}^T\hat{\mathbf{u}} = 0. \tag{10.3}$$

From (8.2)

$$\vec{\mathbf{h}}_s{}^T\vec{\mathbf{d}}_s = 0 \tag{10.4}$$

and by (7.3), (7.5) this is an invariant of the transformation so that (10.2) and (10.3) imply

$$\vec{\mathbf{h}}_m{}^T\vec{\mathbf{d}}_m = 0. \tag{10.5}$$

A similar argument based on the corresponding conditions for (9.2) to be valid shows them to be (10.2) coupled with

$$\vec{\mathbf{e}}_m{}^T\hat{\mathbf{u}} = 0. \tag{10.6}$$

The trio $(\vec{\mathbf{u}}, \vec{\mathbf{h}}, \vec{\mathbf{d}})$ are now mutually perpendicular in both S_m and S_s and $u = c$, typical conditions for a radiation field. Moreover by (9.2) $b = e$ and substituting in (7.9), (7.10) gives $\vec{\mathbf{d}} = \vec{\mathbf{e}}$, $\vec{\mathbf{h}} = \vec{\mathbf{b}}$ so that $d = e = b = h$.

With these assumptions both halves of Maxwell's equations for amplitude-limited vectors, (8.6) and (9.3), still apply, if only heuristically since keeping pace at the velocity of light is stretching the argument outside the realm of practical possibility.

11. – An electromagnetic tensor

In $\mathbf{B}^2$, $\mathbf{R}^2$ notation the simplest form for a field containing electric and magnetic vectors without precedence for either is

$$\begin{aligned}\mathbf{F} &= \mathbf{R}(\beta, \vec{\mathbf{h}}) \cdot \mathbf{B}^2(\varepsilon, \varepsilon\vec{\mathbf{e}}) \cdot \mathbf{R}(\beta, \vec{\mathbf{h}}) \\ &= \begin{bmatrix} 2\varepsilon^2 - 1 & 2\varepsilon^2\vec{\mathbf{e}}^T\mathbf{R} \\ 2\varepsilon^2\mathbf{R}\vec{\mathbf{e}} & \mathbf{R}^2 + 2\varepsilon^2\mathbf{R}\vec{\mathbf{e}}\vec{\mathbf{e}}^T\mathbf{R} \end{bmatrix}\end{aligned} \tag{11.1}$$

in which if

$$\mathbf{H} = \begin{bmatrix} \cdot & -h_3 & h_2 \\ h_3 & \cdot & -h_1 \\ -h_2 & h_1 & \cdot \end{bmatrix}, \tag{11.2}$$

then

$$\mathbf{R} = \mathbf{I} + \mathbf{H} + \mathbf{H}^2/(1+\beta), \quad \mathbf{R}^2 = \mathbf{I} + 2\beta\mathbf{H} + 2\mathbf{H}^2\,. \tag{11.3}$$

The transformation law for $\mathbf{F}$ is

$$\mathbf{F}_s = \mathbf{B}(-\gamma\vec{\mathbf{v}}) \cdot \mathbf{F}_m \cdot \mathbf{B}(\gamma\vec{\mathbf{v}}) \tag{11.4}$$

which in tensor language marks it as a mixed tensor $\mathbf{F}^{\mu}{}_{\nu}$.

As noted previously, for practical purposes the electric and magnetic vectors are very small with respect to unity and neglecting second-order quantities β and ε can be taken as unity when (11.1) reduces to

$$\mathbf{F}^{\mu}{}_{\nu} \cong \mathbf{I} + 2\begin{bmatrix} \cdot & \vec{\mathbf{e}}^T \\ \vec{\mathbf{e}} & \mathbf{H} \end{bmatrix}. \tag{11.5}$$

To reach an equivalent covariant form (11.4) can be written

$$\begin{aligned}(\eta\mathbf{F}_s) &= \eta\mathbf{B}(-\gamma\vec{\mathbf{v}})\eta \cdot (\eta\mathbf{F}_m) \cdot \mathbf{B}(\gamma\vec{\mathbf{v}}) \\ &= \mathbf{B}(\gamma\vec{\mathbf{v}}) \cdot (\eta\mathbf{F}_m) \cdot \mathbf{B}(\gamma\vec{\mathbf{v}})\end{aligned} \tag{11.6}$$

and (11.5) becomes

$$\mathbf{F}_{\mu\nu} \cong \boldsymbol{\eta} + 2\begin{bmatrix} \cdot & -\vec{\mathbf{e}}^T \\ \vec{\mathbf{e}} & \mathbf{H} \end{bmatrix}. \tag{11.7}$$

The purpose of creating an electromagnetic tensor is to carry it over as is into general relativity together with Maxwell's equations in their tensor formulation. Whilst the $\mathbf{I}$ of (11.5), which remains in the absence of an electromagnetic field, preserves its scalar form in the guise of $\mathbf{g}^{\mu}{}_{\nu}$ the $\boldsymbol{\eta}$ of (11.7) translates into the metric $\mathbf{g}_{\mu\nu}$.

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