Composition of Velocities

IN pre-relativity kinematics the directed quantity velocity is defined by $\mathbf{v} = d\mathbf{x}/dt$, a postulate which inherently specifies the law of composition as the vector, or parallelogram, law. In relativity kinematics the velocity of a point mobile P is still defined as the directed quantity the components of which are those of $d\mathbf{x}/dt$, but it is no longer a vector obeying the parallelogram law of addition, a point which will be stressed by writing $\hat{v}$.

In equations $\hat{v}$ appears through the matrix $T(\hat{v})$, the pure Lorentz matrix describing the transformation between co-ordinate systems attached to P and to the observer, such that every point of the system attached to P is seen to be moving with velocity $\hat{v}$. In the P system a typical point, such as P itself, being stationary, will observe only passage of time, an observation denoted by $[\mathbf{o}, d\tau]$. The corresponding interval will be seen by the observer as $[d\mathbf{x}, dt]$, where by the very definition of the motion $\hat{v} = d\mathbf{x}/dt$. The connexion between the observations is then

$$[d\mathbf{x}, dt] = T(\hat{v})\,[\mathbf{o}, d\tau] \quad (1)$$

Special relativity, however, in addition to defining velocity through equation (1), also specifies its law of composition by extending equation (1) to read

$$[d\mathbf{x}, dt] = T(\hat{v})\,[d\mathbf{y}, d\tau] \quad (2)$$

a relation holding between space–time intervals as measured in the two systems for a point mobile moving with respect to both of them. The fact that equation (2) is usually deduced from the simpler postulate that the system of equations connecting such observations is linear does not alter the fact that it represents an additional postulate over and above equation (1).

Writing $\hat{w} = d\mathbf{x}/dt$, $\hat{u} = d\mathbf{y}/d\tau$, equation (2) can be shown to be equivalent to the matrix equation[1]

$$T(\hat{v})T(\hat{u}) = T(\hat{w})R(\Omega) \quad (3)$$

which demonstrates explicitly the relativity law of composition of velocity $\hat{u}$ followed by velocity $\hat{v}$ to give a resultant velocity $\hat{w}$. The matrix $R(\Omega)$, having the structure of a spatial rotation operator, is usually interpreted as demonstrating that successive pure Lorentz transformations induce a rotation of the spatial co-ordinate frame followed by a resultant pure Lorentz transformation. Equation (3) has the advantage over equation (2) of showing explicitly the appearance of this operator, necessitated by considerations of mathematical congruity.

Equation (3), as shown by its equivalent equation (2), is sufficient for equation (1), but not necessary. If the assumption of a linear system of equations existing between $[\mathbf{dx}, dt]$ and $[\mathbf{dy}, d\tau]$ is abandoned the way is clear for alternative hypotheses as to the composition of velocities. One in particular, which shows extraordinary promise, is to retain the matrix $T(\hat{v})$ to represent a velocity $\hat{v}$ but to postulate the law of composition of velocities as

$$[T(\hat{v})]^2[T(\hat{u})]^2 = [T(\hat{w})]^2[R(\Omega)]^2 \quad (4)$$

Structurally, equations (3) and (4) are similar and the solution of equation (4), that is, expression of $\hat{w}, \Omega$, in terms of $\hat{v}, \hat{u}$ can be obtained from the known solution of (3) by observing that

$$[T(\hat{v})]^2 = T\left[2\hat{v}\Big/\left(1 + \frac{|\hat{v}|^2}{c^2}\right)\right] \quad (5)$$

a similar relation existing for $R(\Omega)$. Once again the factor $[R(\Omega)]^2$ is needed for mathematical congruity. $\hat{w}$ has been defined as the resultant velocity but this time it is meaningless to explain away $R(\Omega)$ as representing a rotation of some spatial co-ordinate framework, but more logical to suppose Ω to characterize some as yet unidentified property of the mobile itself. Because $R(\Omega)$ has the structure of a rotation matrix, Ω itself is inherently an axial directed quantity, as opposed to the parameters $\hat{u}, \hat{v}, \hat{w}$, associated with T-matrices, which are inherently polar.

Thus the motion of a mobile is described by a polar parameter $\hat{w}$ and an axial parameter Ω through the matrix $[T(\hat{w})]^2[R(\Omega)]^2$. If these parameters change, so that the descriptive matrix becomes $[T(\hat{w}+d\hat{w})]^2[R(\Omega+d\Omega)]^2$ then it is logical to connect the change with the externally applied force. If that force is purely polar, $\hat{f}$, the change matrix multiplier will be of type T and this condition lead to

$$\hat{f} = \frac{1 - \dfrac{|\hat{w}|^2}{c^2}}{1 + \dfrac{|\hat{w}|^2}{c^2}} \frac{d}{dt} \left\{ \frac{\hat{w}}{1 - \dfrac{|\hat{w}|^2}{c^2}} \right\} \quad (6)$$

This equation has been derived from purely kinematical considerations. As usual, equality of inertial and gravitational mass is assumed so that equation (6) is converted into the dynamical generalization of Newton's second law by multiplying each side by the mass of the mobile m_0. The energy integral is then

$$m_0 \int \hat{f} \cdot d\mathbf{x} = m_0 c^2 \log\left(1 - \frac{|\hat{w}|^2}{c^2}\right)^{-\frac{1}{2}}$$
$$= \tfrac{1}{2} m_0 |\hat{w}|^2 + \tfrac{1}{2} m_0 \frac{|\hat{w}|^4}{c^2} + \ldots \quad (7)$$

Calculations based on equations (6) and (7) for $\hat{f}$ obeying the inverse square law give the correct values for the advance of the perihelion and the bending of light.

Use of equation (4) in place of equation (3) originally arose in an as yet unpublished investigation of the possibility of representing an electrostatic field vector by a T-matrix and using equation (3) to depict its transformation by a velocity T-matrix into a field containing an electrokinetic T-matrix and a magnetic R-matrix. Although the structural requirements of the physical problem were met, quantitative agreement was only secured when equation (3) was replaced by equation (4), or rather a slight variant of it, where if $\hat{u}$ characterizes the electric field

$$[T(\hat{v})]^2 [T(\hat{u})]^2 = R(\Omega) [T(\hat{g})]^2 R(\Omega) \quad (8)$$

$T(\hat{g})$ being the electrokinetic operator rather than $T(\hat{w})$.

Physical theory founded on equations (4) and (8) gives results very different to those of special relativity. In particular contraction in dimension is isotropic and charge decreases with motion by the contraction factor

$$\gamma = \left(1 - \frac{|\hat{w}|^2}{c^2}\right)^{-\frac{1}{2}}$$

Thus in applying equation (6) to the motion of a charged particle, rest charge q_0, rest mass m_0, in an electric field $\hat{e}$ the force is $q_0 \gamma^{-1} \hat{e}$ and the energy integral is now

$$q_0 \int \hat{e} \cdot d\mathbf{x} = m_0 c^2 (\gamma - 1) = \tfrac{1}{2} m_0 |\hat{w}|^2 + \tfrac{3}{8} m_0 \frac{|\hat{w}|^4}{c^2} + \ldots \quad (9)$$

which is just the result of special relativity!

A similar calculation for a transverse magnetic field once more yields the relativity formula, so that the new theory is entirely consistent with the formulae used in analysing particle accelerators.

It remains to investigate Ω in the kinematic case. Noting that the left hand sides of equations (4) and (8) are identical suggests that there might be a kinematic significance to equation (8) when $\hat{v}$ and $\hat{u}$ are both velocities. $\hat{w}$ is then $\hat{g}$ rotated as prescribed by $R(\Omega)$, the moduli, or speeds, being equal.

This suggests that the true motion $\hat{w}$ of a particle of non-zero Ω might appear "aberrated" into an apparent velocity $\hat{g}$. If this be so, particles of the same $\hat{w}$ but different Ω would present different $\hat{g}$. Thus for a given $|\Omega|$ but random direction for the axis of Ω, $\hat{g}$ would lie within a cone of half-angle $|\Omega|$ centred on $\hat{w}$, or in other words its direction would be uncertain, although bounded.

Finally, two particles having the same apparent velocity $\hat{g}$ cannot have different Ω because their trajectories would then diverge. This is an exclusion principle.

B. L. Coleman

Givat Avia 33,
Yahud, Israel.

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[1] Macfarlane, A. J., *J. Math. Phys.*, 3, 1125 (1962).