The Capacity of the Trapdoor Channel with Feedback

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Based on work with

Paul Cuff, Benjamin Van Roy and Tsachy Weissman

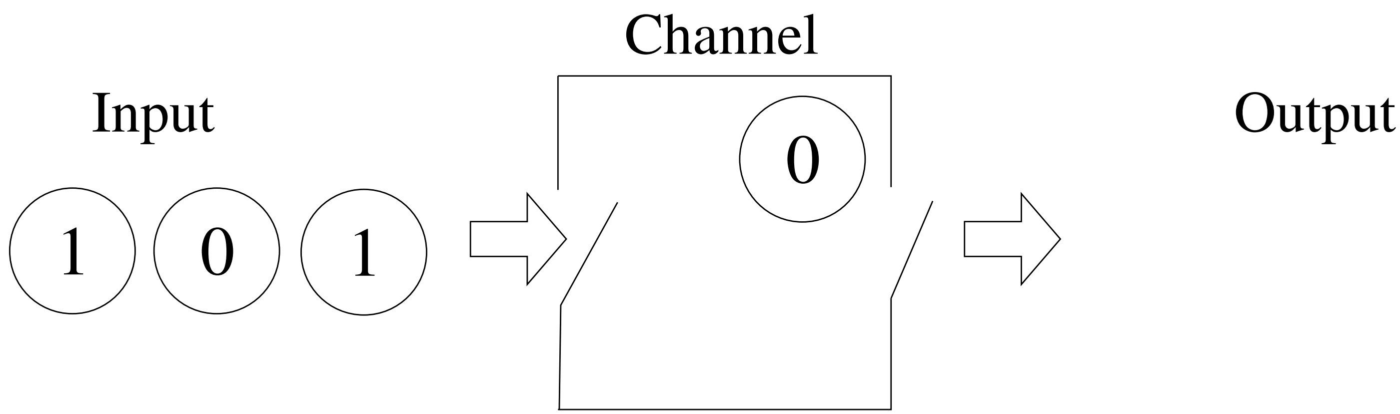
Stanford University

Main Results of the Talk

1. capacity of the trapdoor channel with feedback

2. simple scheme that achieves feedback capacity

The trapdoor channel



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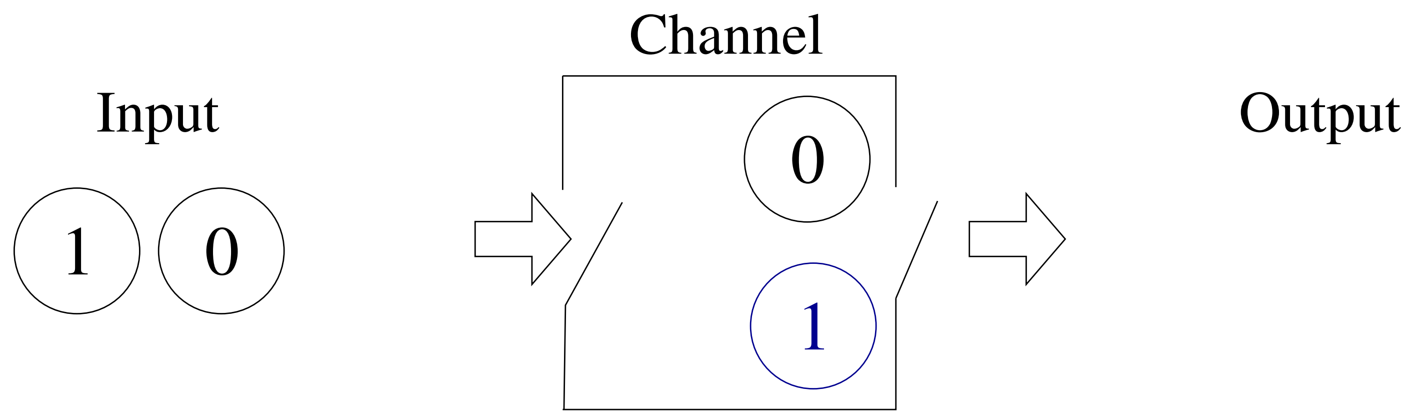
$s_0 = 0$

The trapdoor channel



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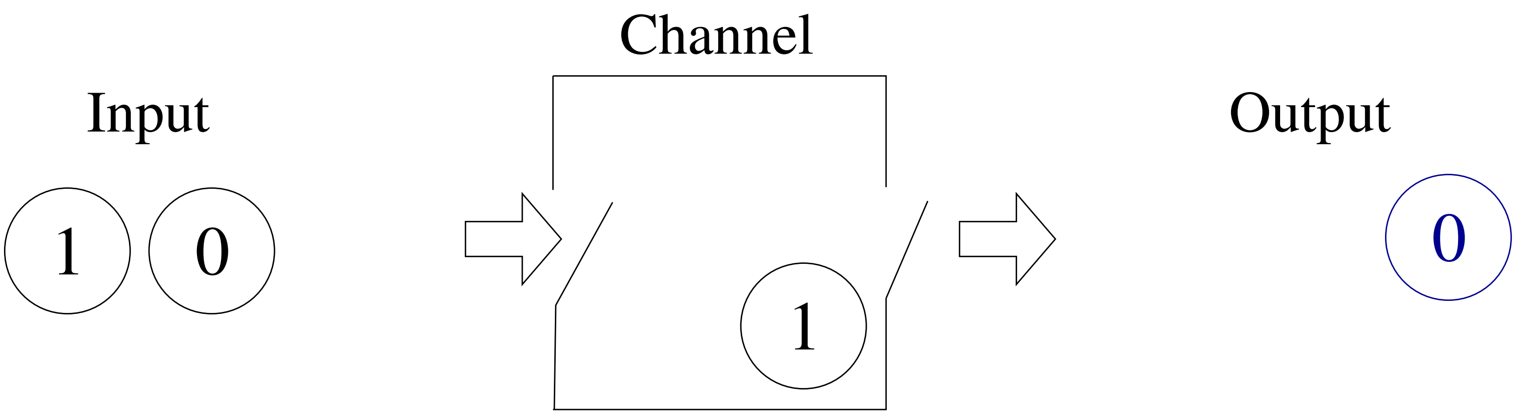

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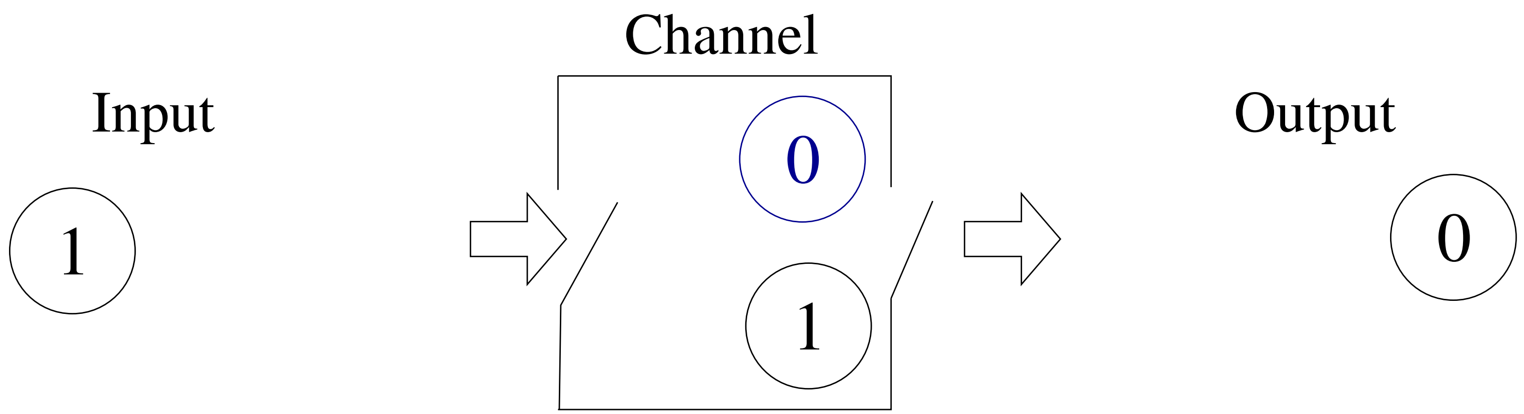

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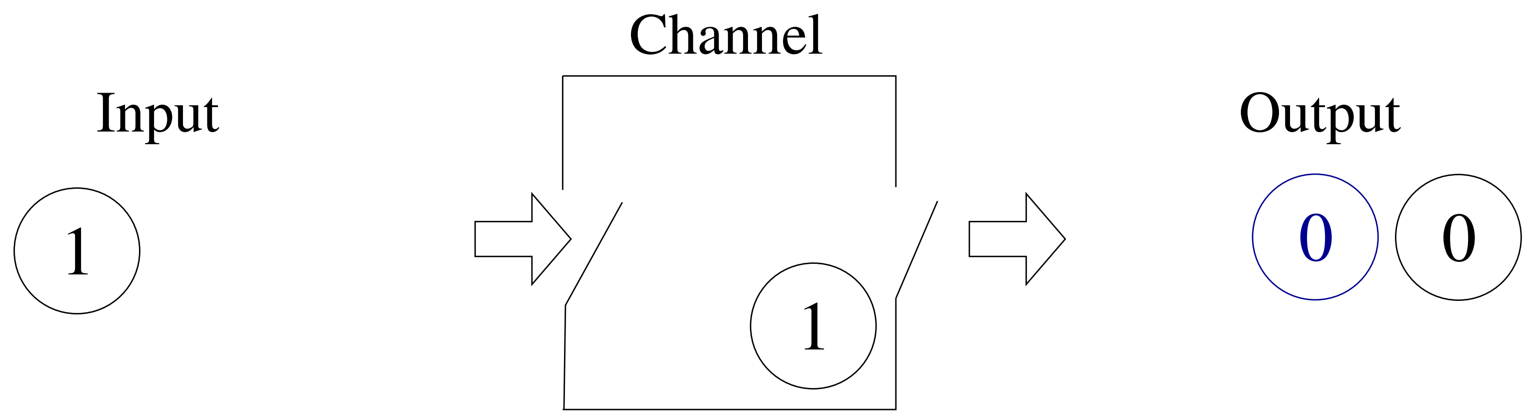

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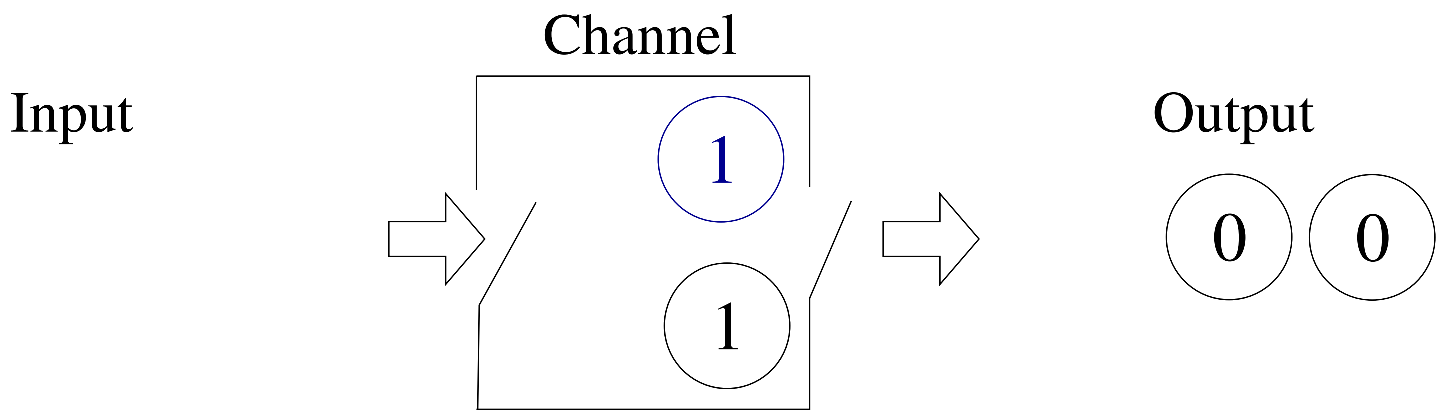

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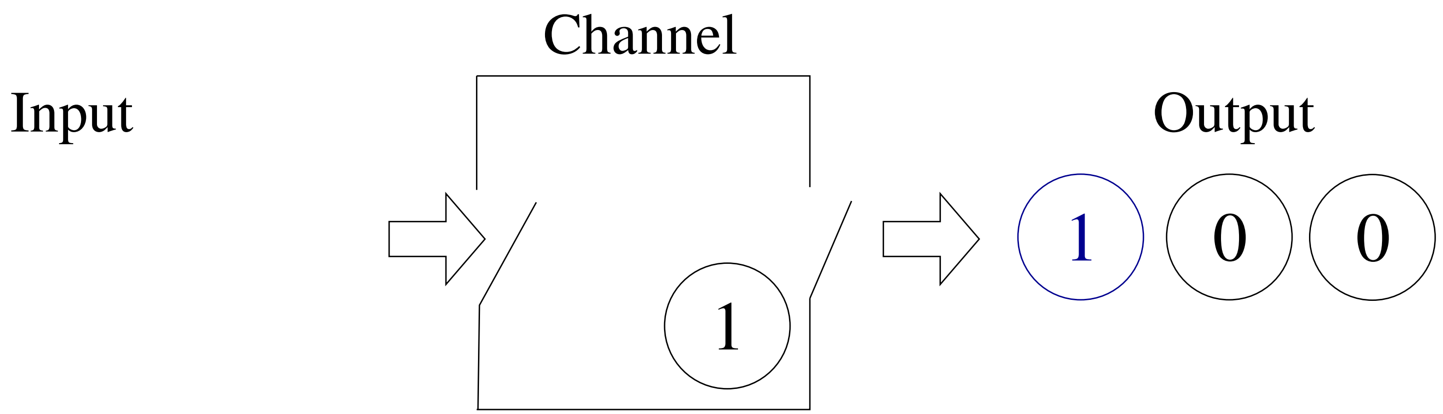

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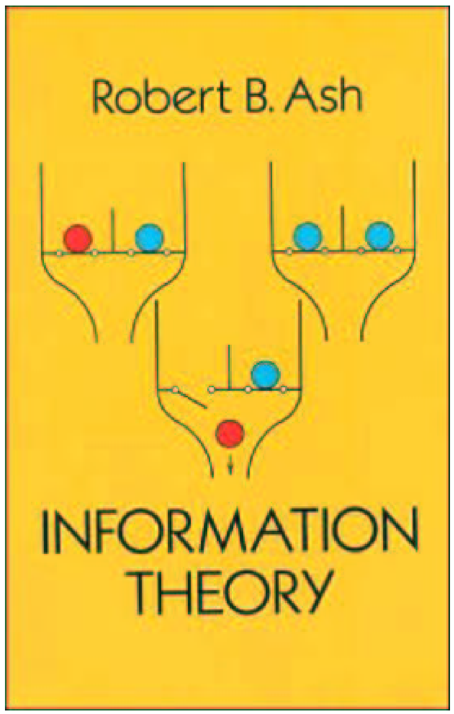
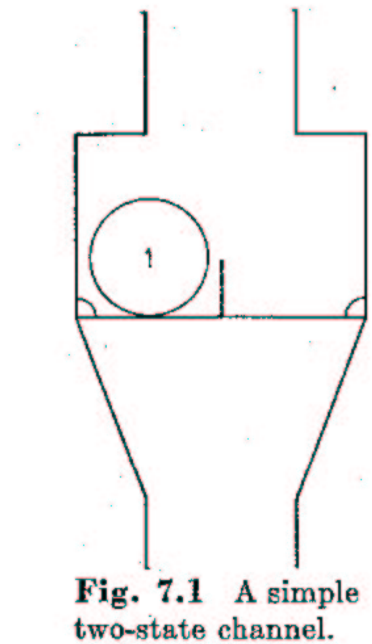
The trapdoor channel

Introduced by David Blackwell in 1961. [Ash65], [Ahlswede & Kaspi 87], [Ahlswede 98], [Kobayashi 02].



(a) Ash book



(b) D. Blackwell

Another appropriate name for this channel is *chemical channel*.

Communication setting

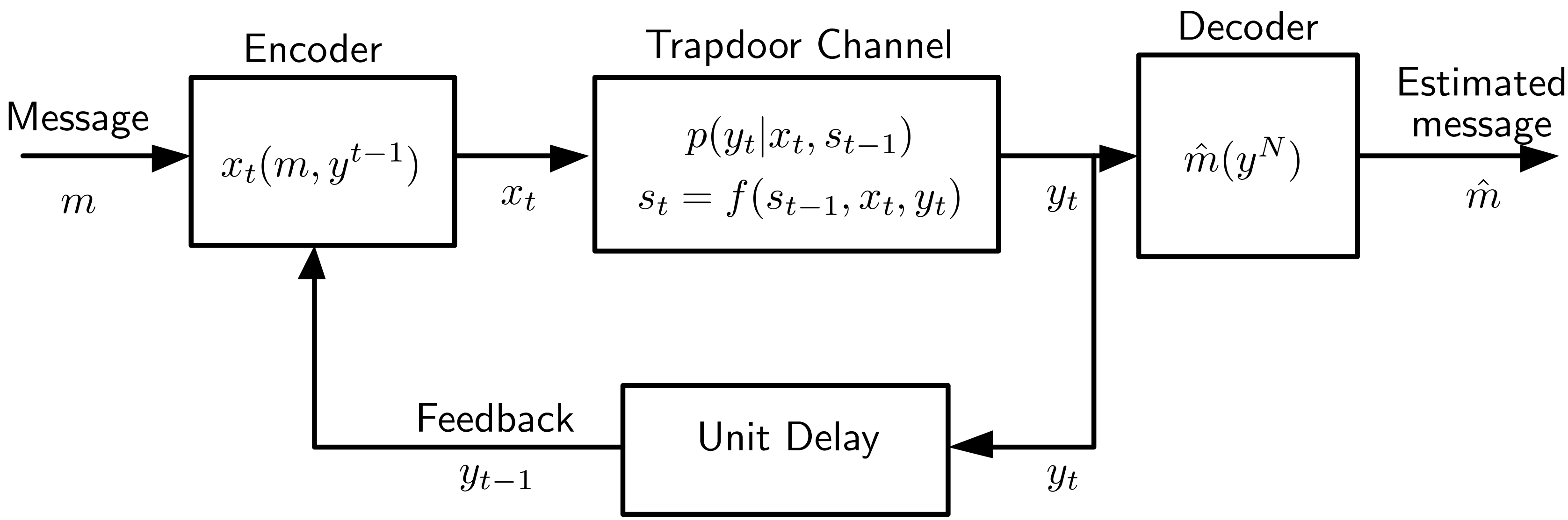


Figure 1: Unifilar FSC with feedback

Finite State Channel(FSC) property: $p(y_i, s_i|x^i, s^{i-1}, y^{i-1}) = p(y_i, s_i|x_i, s_{i-1})$

Unifilar channel [Ziv85]: $s_t = f(s_{t-1}, x_t, y_t)$

Main ingredients

1. Directed information.

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2. Dynamic program average-reward.

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4. Bellman equation.

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1. Directed information.
2. Dynamic program average-reward.
3. Value iteration.
4. Bellman equation.
5. Homework question given by Tom Cover.

Feedback capacity of FSC

Lower and Upper bound

$$C_{FB} \geq \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^N} \min_{s_0} I(X^N \to Y^N | s_0)$$

$$C_{FB} \leq \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^N} \max_{s_0} I(X^N \to Y^N | s_0)$$

[Permuter, Weissman and Goldmith ISIT06]

where

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In the trapdoor channel any state s_t can be reached from any state s_{t-1} with positive probability and hence we get

$$C_{FB} = \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^{N}} I(X^N \to Y^N)$$

Directed information

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\begin{aligned}
I(X^n \to Y^n) &\triangleq \sum_{i=1}^{n} I(X^{i}; Y_i | Y^{i-1}) \\
I(X^n; Y^n) &= \sum_{i=1}^{n} I(X^{n}; Y_i | Y^{i-1})
\end{aligned}
$$

Directed information - intuition

If there is no feedback

$$I(X^n;Y^n) = I(X^n \to Y^n)$$

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Deterministic feedback $k_i(y_i)$

$$I(X^n;Y^n) = I(X^n \to Y^n) + I(K^{n-1} \to X^n)$$

Feedback capacity

$$C_{FB} = \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_t|x^{t-1},y^{t-1})\}_{t=1}^{N}} I(X^N \to Y^N)$$

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&= \sup_{\{p(x_t|s_{t-1},y^{t-1})\}_{t\geq 1}} \liminf_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} I(X_t, S_{t-1}; Y_t | Y^{t-1})
\end{aligned}
$$

Feedback capacity and dynamic programming(DP)

DP consists of of states β_{t-1}, actions $u_t(\beta_{t-1})$, and disturbance w_t.

state:

$$\beta_{t-1} = p(s_{t-1}|y^{t-1}), \quad \beta \in [0,1]$$

action:

$$u_t = p(x_t|s_{t-1}), \quad u_t \in [0,1] \times [0,1]$$

disturbance:

$$w_t = y_{t-1},$$

$$\beta_t = F(\beta_{t-1}, u_t, w_t), \qquad t = 1, 2, 3, \ldots,$$

reward function per unit time

$$g(\beta_{t-1}, u_t) = I(X_t, S_{t-1}; Y_t|\beta_{t-1}).$$

[Tatikonda00], [Yang, Kavčić and Tatikonda05]

Dynamic programing operator, T

The dynamic programming operator T is given by

$$(TJ)(\beta) = \sup_{u \in \mathcal{U}} \left(g(\beta, u) + \int P_w(dw|\beta, u) J(F(\beta, u, w)) \right)$$

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$$\begin{aligned}(TJ)(\beta) \quad = \quad & \sup_{0 \le \delta \le \beta, 0 \le \gamma \le 1-\beta} \left(H\left(\frac{1}{2} + \frac{\delta - \gamma}{2}\right) + \delta + \gamma - 1 + \frac{1 + \delta - \gamma}{2} J\left(\frac{2\delta}{1 + \delta - \gamma}\right) \right. \\ & \left. + \frac{1 - \delta + \gamma}{2} J\left(1 - \frac{2\gamma}{1 - \delta + \gamma}\right)\right)\end{aligned}$$

Properties

- Preservation of *concavity*: if J is concave then TJ is concave.
- Preservation of *continuity*: if J is continuous then TJ is continuous.
- Preservation of *symmetry*: if J is symmetric then TJ is symmetric.

Computational study

Executed 20 value iterations: $J_{k+1}(\beta) = (TJ_k)(\beta)$

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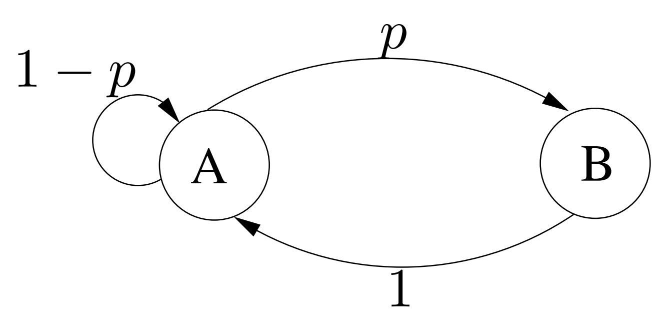
HW question from Prof. Cover class

Entropy rate. Find the maximum entropy rate of the following two-state Markov chain:



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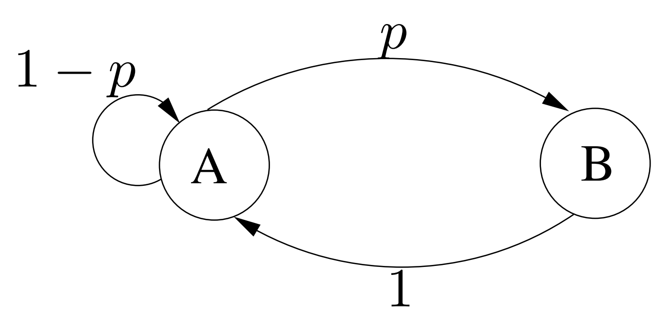
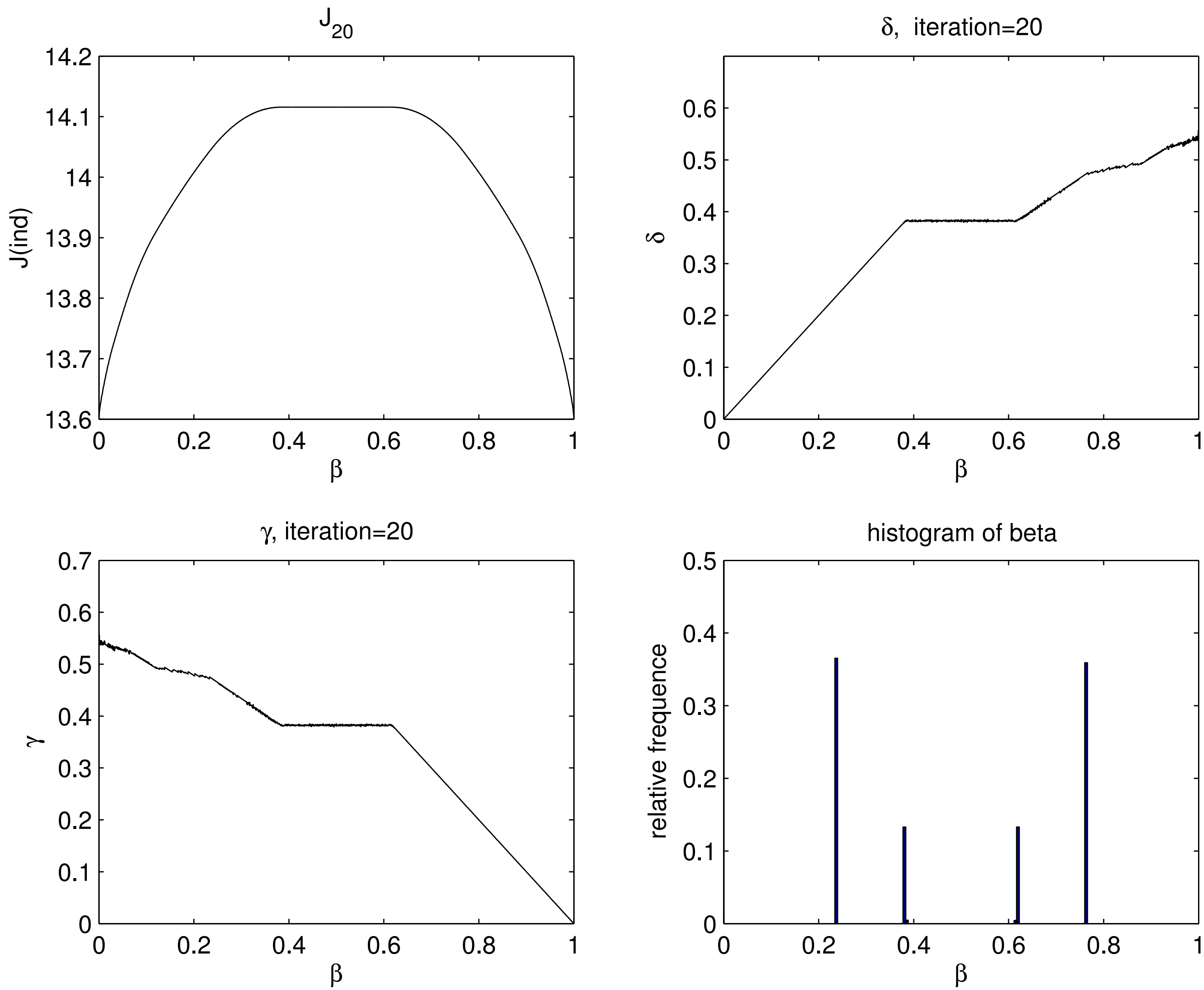
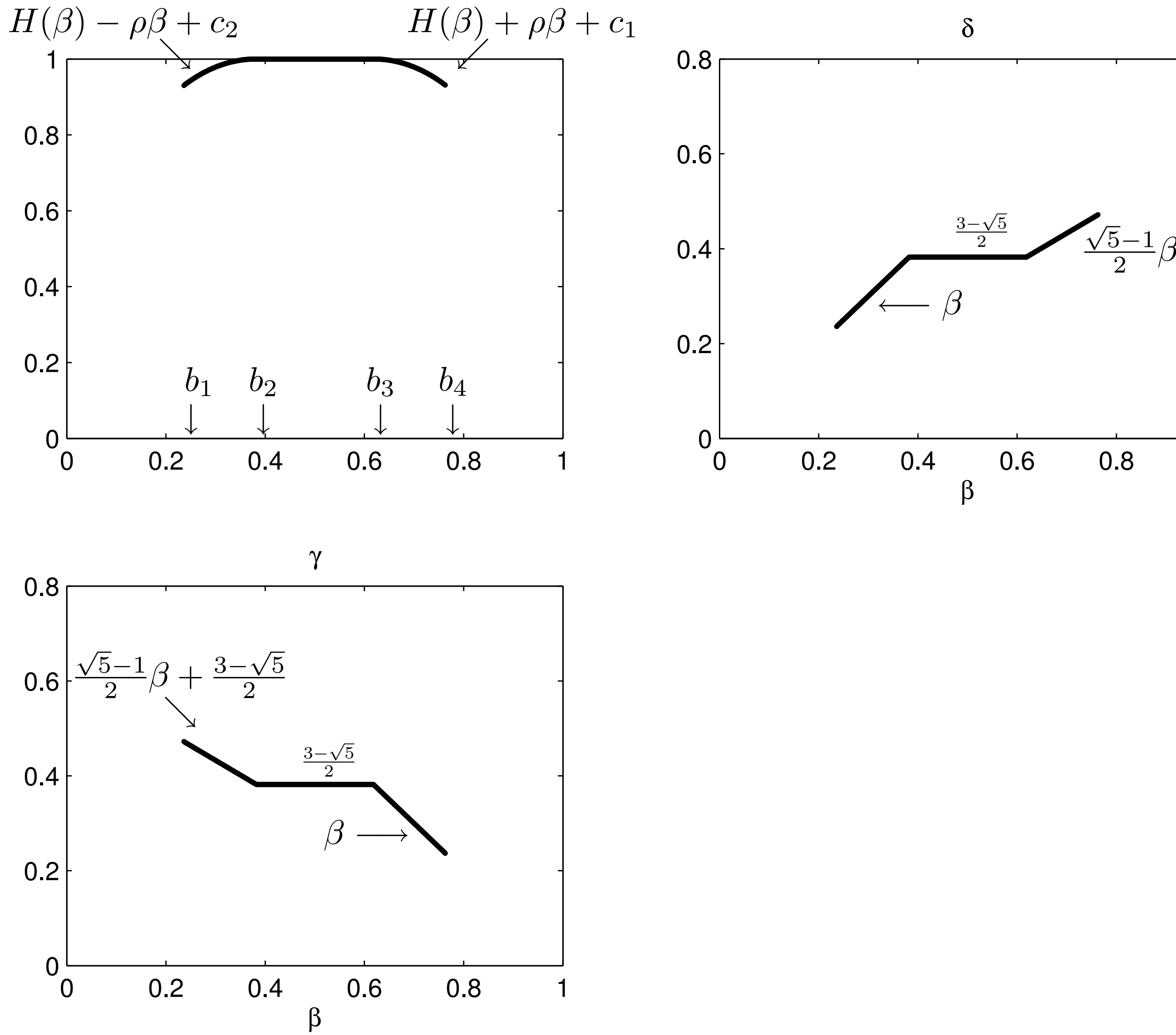


Solution: The entropy rate is $\log_2 \phi = 0.6942...$, where ϕ is the *golden ratio*: $\phi = \frac{\sqrt{5}+1}{2}$.

20th Value iteration



Conjecture of the solution to Bellman equation



Bellman equation

Theorem 1. *If there exists* $(J(\beta), \rho)$ *that satisfies*

$$J(\beta) = (TJ)(\beta) - \rho,$$

then ρ *is the optimal average reward.*

Verifying our conjecture

Construct *value iteration function* $J_k(\beta)$ as follows. Let $J_0(\beta)$ be the pointwise maximum among concave functions satisfying $J_0(\beta) = \tilde{J}(\beta)$ for $\beta \in [b_1, b_4]$

$$J_{k+1}(\beta) = (TJ_k)(\beta) - \tilde{\rho},$$

- concave, continuous and symmetric
- fixed point: for $\beta \in [b_1, b_4]$,$J_k(\beta) = \tilde{J}(\beta)$
- monotonically nonincreasing in k
- converges uniformly to $J^*(\beta)$

Since the sequence $J_{k+1} = TJ_k - \tilde{\rho}\mathbf{1}$ converges uniformly and T is sup-norm continuous, $J^* = TJ^* - \tilde{\rho}\mathbf{1}$.

A scheme that achieves capacity

Question

Number of sequences. To first order in the exponent, what is the number of binary sequences of length n with no two consecutive 1's?

$$00101010100101\ldots$$

A scheme that achieves capacity

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Solution The number of sequences of length n with this property, is the n^{th} Fibonachi number, $f_n \doteq \phi^n$.

The scheme

Let us denote such a sequence by r^n. Map each message m to a sequence $[r^n(m)]$.
encoder: $x_t = s_{t-1} \oplus r_t, t = 1, ..., n$ and $x_{n+1} = s_n$.
decoder: The decoder can decode this sequence error-free!

Conclusions

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Thank You!