Two-way source coding with a common helper

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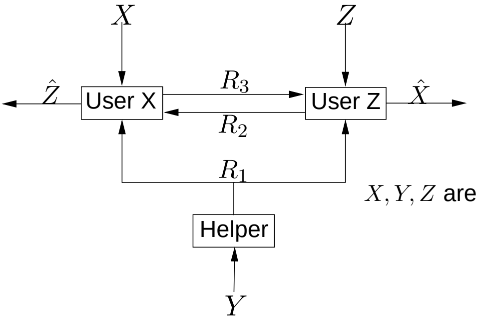
Problem setting

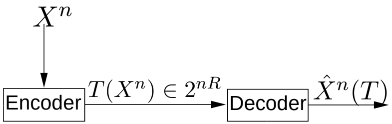


X, Y, Z are i.i.d $\sim P_{X,Y,Z}$

1. Helper Y sends a common message to Users X and Z.
2. User Z and X exchanges messages .

The goal: $\mathbb{E}\left[d_z(Z^n, \hat{Z}^n)\right] \leq D_z, \mathbb{E}\left[d_x(X^n, \hat{X}^n)\right] \leq D_x.$

Background: classical rate distortion problem



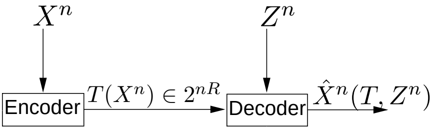
$X_i \sim P_X$, i.i.d.

$\mathbb{E}\left[\sum_{i=1}^{n} \frac{1}{n} d(X_i, \hat{X}_i)\right] \leq D$

Rate-distortion function [Shannon 48]

$$R(D) = \min_{P_{\hat{X}|X}:\mathbb{E}\left[d(X,\hat{X})\right]\leq D} I(X;\hat{X})$$

Rate distortion with side information at the decoder



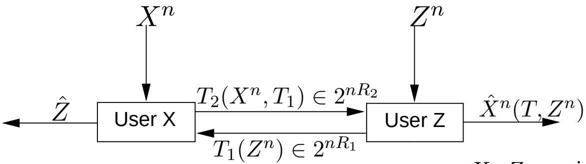
X_i, Y_i are i.i.d $\sim P_{X,Z}$

$\mathbb{E}\left[\sum_{i=1}^{n} \frac{1}{n} d(X_i, \hat{X}_i)\right] \leq D$

Wyner-Ziv rate-distortion function [Wyner/Ziv76]

$$
\begin{aligned}
R(D) &= \min_{U-X-Z:\mathbb{E}\left[d(X,\hat{X}(U,Z))\right]\leq D} I(X;U) - I(U;Z) \\
&= \min_{U-X-Z:\mathbb{E}\left[d(X,\hat{X}(U,Z))\right]\leq D} I(X;U|Z)
\end{aligned}
$$

Two-way rate distortion problem



X_i, Z_i are i.i.d $\sim P_{X,Z}$

$$\mathbb{E}\left[\sum_{i=1}^{n} \frac{1}{n} d_x(X_i, \hat{X}_i)\right] \leq D_x, \quad \mathbb{E}\left[\sum_{i=1}^{n} \frac{1}{n} d_z(Z_i, \hat{Z}_i)\right] \leq D_z$$

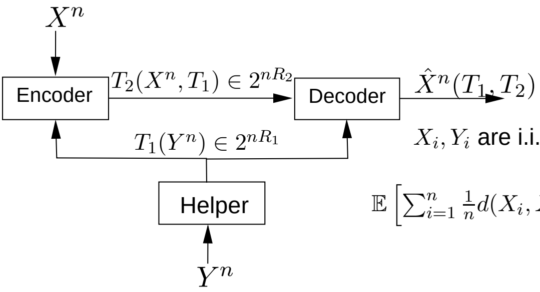
Two-way rate region [Kaspi85]

$$\begin{aligned} R_1 &\geq I(Z;U|X) \\ R_2 &\geq I(X;V|U,Z), \end{aligned}$$

$$U - Z - X, \; V - (U,X) - Z$$

$$\mathbb{E}\left[d_x(X, \hat{X}(V,Z))\right] \leq D_x, \; \mathbb{E}\left[d_z(Z, \hat{Z}(U,X))\right] \leq D_z$$

Rate distortion with a helper



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The achievable region [Vasuadevan/Perron07] [P./Stienberg/Weissman08]

$$\begin{aligned} R_1 &\geq I(Y;U) \\ R_2 &\geq I(X;\hat{X}|U), \end{aligned}$$

$U - Y - X$

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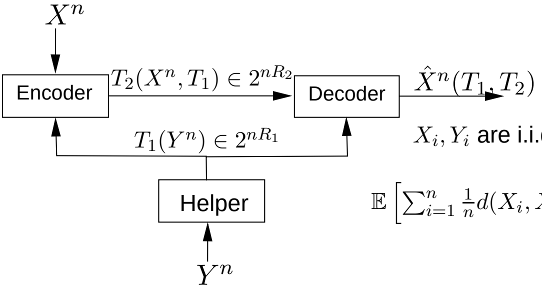

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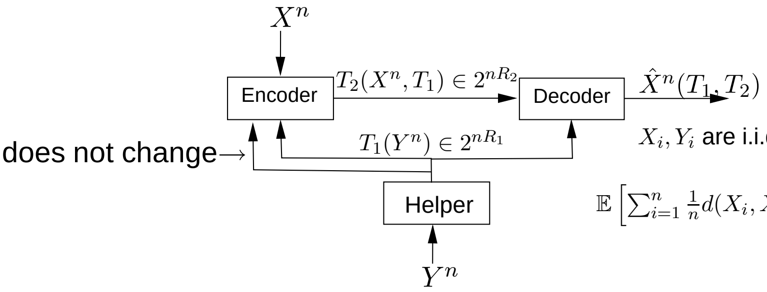
$U - Y - X$ ($\hat{X} - (U, X) - Y$ not needed)

Rate distortion with a helper



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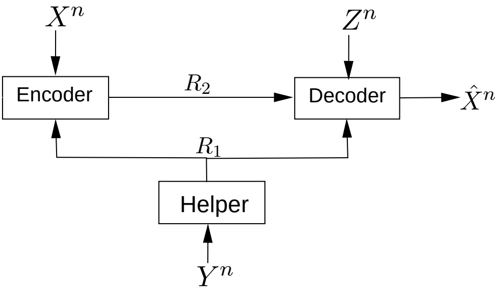
Main results



- Wyner-Ziv with a helper where $Y - X - Z$ or $Y - Z - X$.

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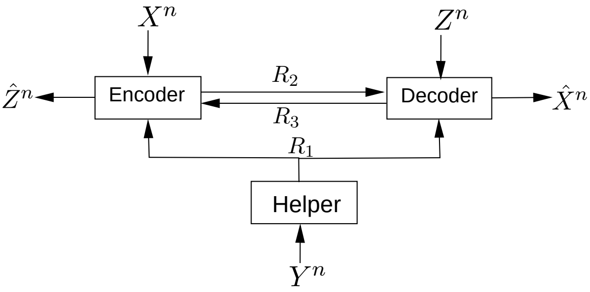

- Wyner-Ziv with a helper where $Y - X - Z$ or $Y - Z - X$.
- Two-way source-coding with a helper where $Y - X - Z$ or $Y - Z - X$.

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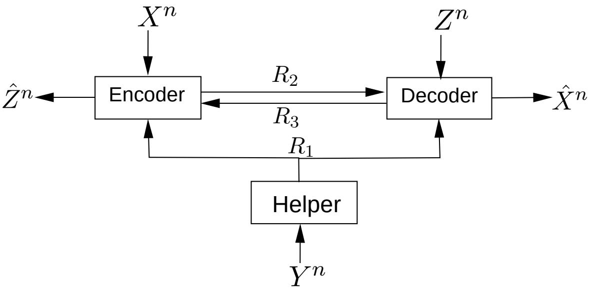

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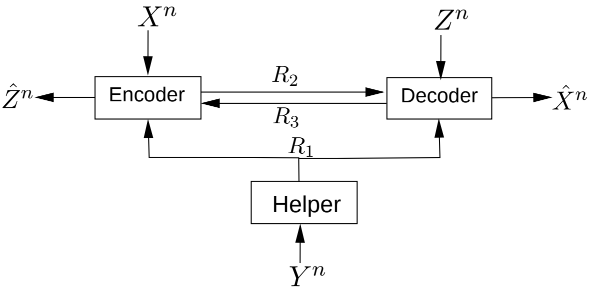
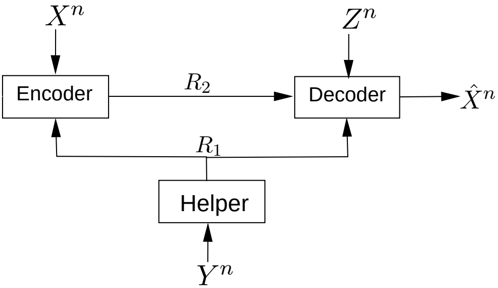


- Wyner-Ziv with a helper where $Y - X - Z$ or $Y - Z - X$.
- Two-way source-coding with a helper where $Y - X - Z$ or $Y - Z - X$.
- Analytical solution for the Gaussian case.
- New tool for checking Markov.

Wyner-Ziv with a helper $Y - X - Z$



Theorem

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\begin{aligned}
R_1 &\geq I(Y;U|Z), \\
R_2 &\geq I(X;W|U,Z),
\end{aligned}
$$

$$U - Y - X - Z, \qquad W - (X,U) - (Z,Y),$$

$$\mathbb{E}\left[d(X,\hat{X}(U,W,Z))\right] \leq D.$$

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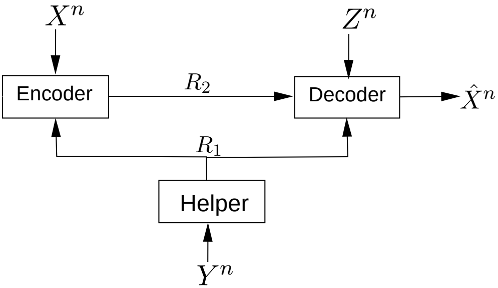

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$$\mathbb{E}\left[d(X,\hat{X}(U,W,Z))\right] \leq D.$$

The region is not enlarged if $W - (X,U,Y) - (Z)$.

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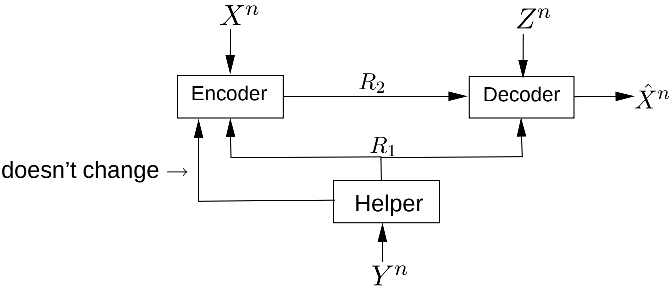

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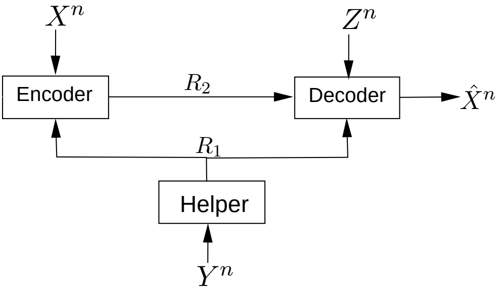

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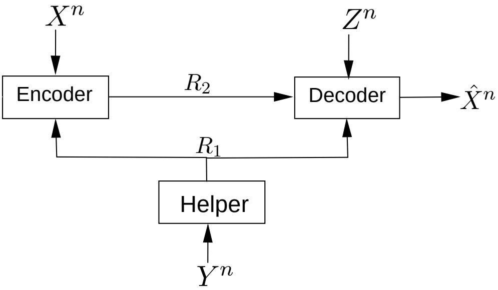

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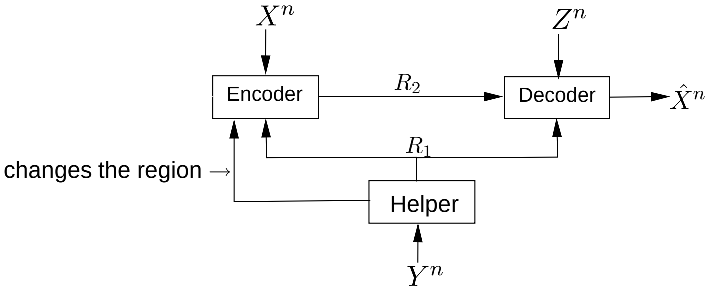

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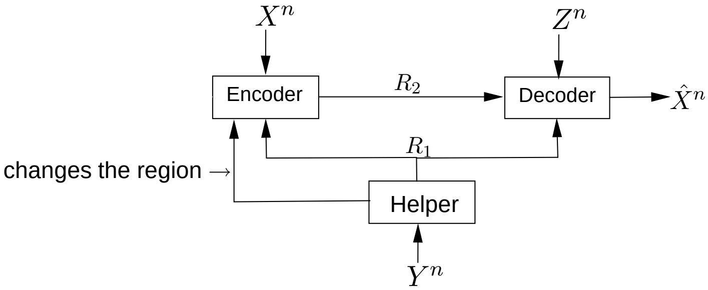

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Converse: Used graphical tools to verify Markov relations.

A technique for checking Markov relations

$X^N = (X_1, X_2, ..., X_N)$- random variables

$$p(x^N) = f(x_{\mathcal{S}_1}) f(x_{\mathcal{S}_2}) \cdots f(x_{\mathcal{S}_K}),$$

where $X_{\mathcal{S}_i} = \{X_j\}_{j \in \mathcal{S}_i}$, and $\mathcal{S}_i$ is a subset of $\{1, 2, \ldots, N\}$.

sufficient condition for $X_{\mathcal{G}_1} - X_{\mathcal{G}_2} - X_{\mathcal{G}_3}$

.

1. Draw an **undirected** graph where all the random variables X^N are nodes in the graph and for all $i = 1, 2, ..K$ draw edges between all the nodes $X_{\mathcal{S}_i}$,
2. Sufficient condition: all paths in the graph from a node in $X_{\mathcal{G}_1}$ to a node in $X_{\mathcal{G}_3}$ pass through a node in $X_{\mathcal{G}_2}$.

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Additional techniques based on **directed** graphs in [Kramer03] [Pearl00].

Example for verifying Markov relation

Consider

$$p(x^2, y^2, z^2) = p(x_1, y_2)p(y_1, x_2)p(z_1|x_1, x_2)p(z_2|y_1).$$

Is $X_1 - X_2 - Z_2$?

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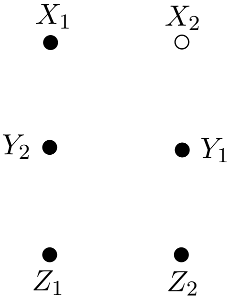

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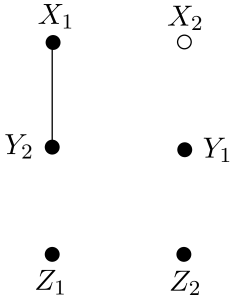

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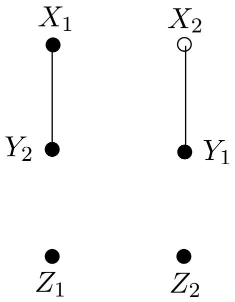

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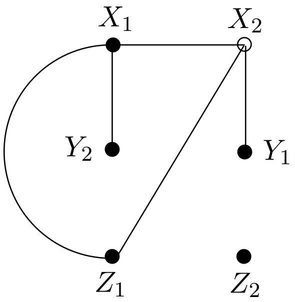

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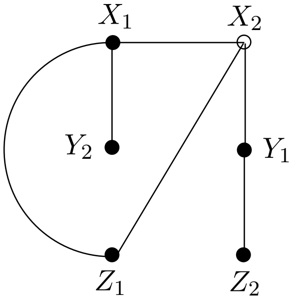

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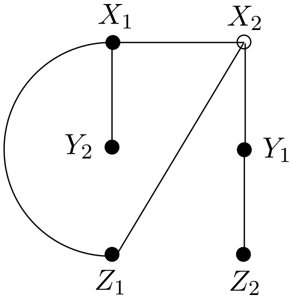

Yes, since all paths from X_1 to Z_2 pass through X_2.

Markov relation in the converse

$$p(x^n, y^n, z^n, t_1, t_2) = p(x^{i-1}, z^{i-1})p(y^{i-1}|z^{i-1})p(x_i, z_i)p(y_i|z_i)\cdot$$
$$p(x_{i+1}^n, z_{i+1}^n)p(y_{i+1}^n|z_{i+1}^n)p(t_1|y^n)p(t_2|x^n, t_1)$$

We need to show

$$X_i - (X_{i+1}^n, T_1, Z^i, T_2) - Z_{i+1}^n$$

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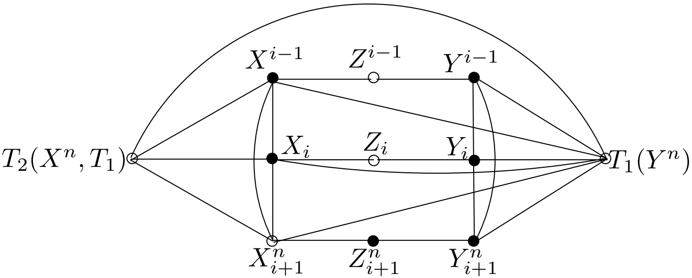

Proof of the technique for verifying Markov relations

Lemma

If

$$p(x, y, z) = f(x, y)f(y, z),$$

then the Markov chain $X - Y - Z$ holds.

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If

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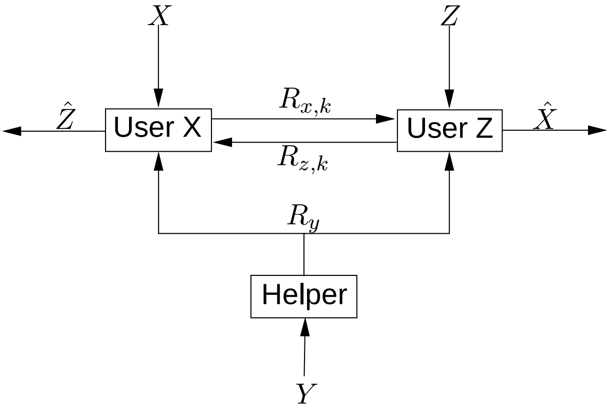
$$\begin{aligned} p(z|y, x) &= \frac{f(x, y)f(y, z)}{f(x, y)\left(\sum_z f(y, z)\right)} \\ &= \frac{f(y, z)}{\sum_z f(y, z)} \\ &= p(z|y) \end{aligned}$$

The two-way multi-stage with a helper $Y - X - Z$



The rate of the code is (R_x, R_y, R_z) where

$$R_x = \sum_{k=1}^{K} R_{x,k}, \qquad R_z = \sum_{k=1}^{K} R_{z,k}.$$

The rate region of the two-way with a helper $Y - X - Z$

Combining Wyner-Ziv with helper results:

$$
\begin{aligned}
R_y &\geq I(U;Y|Z), \\
R_z &\geq \sum_{k=1}^{K} I(Z;V_k|X,U,V^{k-1},W^{k-1}), \\
R_x &\geq \sum_{k=1}^{K} I(X;W_k|Z,U,V^{k},W^{k-1}),
\end{aligned}
$$

for some auxiliary random variables (U,V^K,W^k) that satisfy

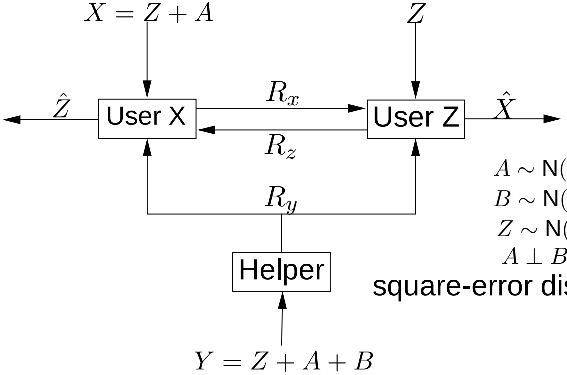
$$U - Y - (X,Z)$$

$$V_k - (Z,U,V^{k-1},W^{k-1}) - (X,Y), \quad k = 1,2,...,K,$$

$$W_k - (X,U,V^{k},W^{k-1}) - (Z,Y), \quad k = 1,2,...,K,$$

$$\mathbb{E}d_x(X,\hat{X}(U,W^K,Z)) \leq D_x, \ \mathbb{E}d_z(Z,\hat{Z}(U,V^K,X)) \leq D_z.$$

Gaussian Case



$A \sim \mathsf{N}(0, \sigma_A^2)$,
$B \sim \mathsf{N}(0, \sigma_B^2)$,
$Z \sim \mathsf{N}(0, \sigma_Z^2)$,
$A \perp B \perp Z$,

square-error distortion D_x, D_z

$$
\begin{aligned}
R_z &\geq \frac{1}{2} \log \frac{\sigma_A^2 \sigma_Z^2}{D_z(\sigma_A^2 + \sigma_Z^2)}, \\
R_x &\geq \frac{1}{2} \log \frac{\sigma_A^2 \left(\sigma_B^2 + \sigma_A^2 2^{-2R_y}\right)}{D_x(\sigma_A^2 + \sigma_B^2)}.
\end{aligned}
$$

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  - graphical technique for checking Markov relations
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- proofs and more...

arXiv:0904.2311

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- proofs and more...

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Thank you very much!