# Capacity Region of the Finite State MAC with Cooperative Encoders and Delayed CSI


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2012 IEEE International Symposium
on Information Theory
July, 2012

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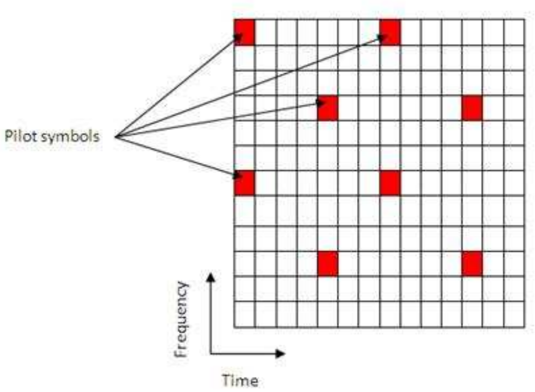
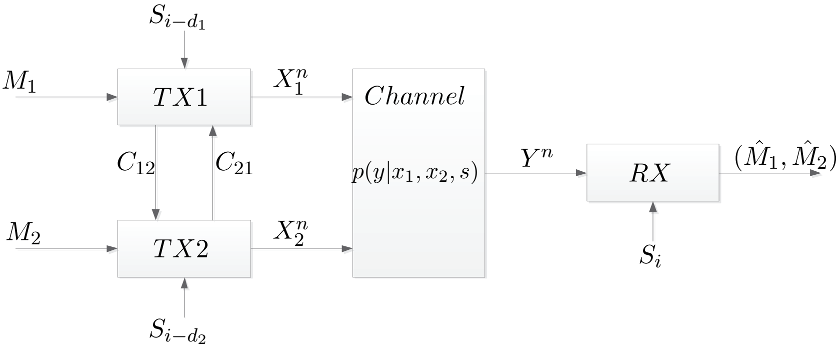
- In LTE uplink standard, pilot signal are sent by the useres

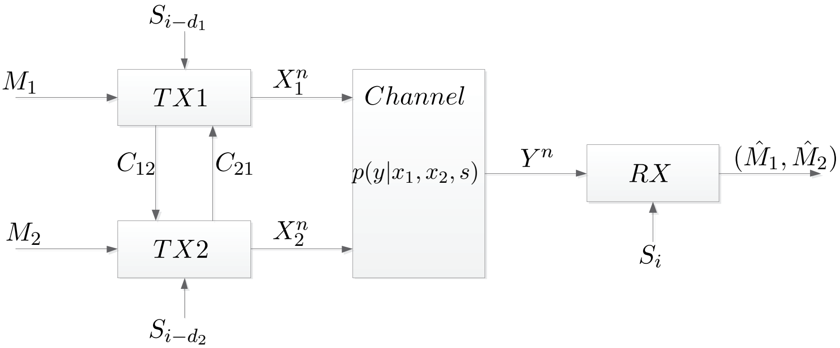


# Uplink with Delayed CSI and conferencing




- CSI known to the Receivers (RX) and delayed CSI known to the Transmitters (TX).
- Conferencing between the TX is possible with limited link.

# Uplink with Delayed CSI and conferencing




Asymmetric/delayed state

- Strictly causal CSI [Steinberg/Lapidoth10][Li/Simeone/Yener10]
- Delayed state for Point-to-point case [Viswanathan99]
- No conferencing [Bashar/Shirazi/P 11]
- Assymetrical state [Sen/Alajaji/uksel/Como12]
- Non-causal state at one encoder [Somekh-Baruch/Shamai/Verdú06] [Kotagiri/Laneman04]

# Channel Model and Notation

- Finite number of states $\mathcal{S} < \infty$.
- Channel state is a stationary Markov process independent of the messages.
- The random variables $S_i$ ,$S_{i-d}$ denote the channel state at time $i$, and $i-d$, respectively.
- The $(S_i, S_{i-d})$ joint distribution is stationary and is given by

$$P(S_i = s_l, S_{i-d} = s_j) = \pi(s_j)K^d(s_l, s_j).$$

- The channel transition probability at time $i$ is given by

$$P(y_i|x_{1,i}, x_{2,i}, s_i)$$

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- The state process is independent of the conference communications,

$$P(s^n, v_1^\ell, v_2^\ell) = P(s^n)P(v_1^\ell, v_2^\ell) = \prod_{i=1}^{n} P(s_i|s_{i-1})P(v_1^\ell, v_2^\ell).$$

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- cooperation link constraint $C_{12}$ and $C_{21}$:

$$\sum_{i=1}^{\ell} \log |\mathcal{V}_{1,i}| \leq nC_{12} \; ; \; \sum_{i=1}^{\ell} \log |\mathcal{V}_{2,i}| \leq nC_{21}.$$

# Code Description

Conferencing encoder

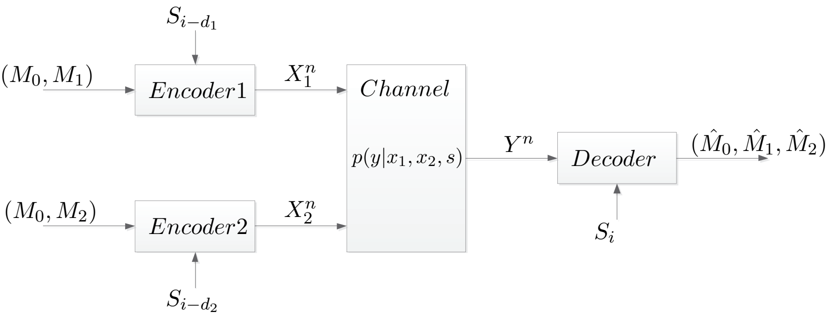
$$V_{1,i} = h_{1,i}(M_1, V_2^{i-1}),$$
$$V_{2,i} = h_{2,i}(M_2, V_1^{i-1}).$$

For each TX an encoding function,

$$X_{1,i} = \left\{ \begin{array}{rc} f_{1,i}(M_1, V_2^{\ell}), & 1 \leq i \leq d_1 \\ f_{1,i}(M_1, V_2^{\ell}, S^{i-d_1}), & d_1 + 1 \leq i \leq n \end{array} \right\}$$

$$X_{2,i} = \left\{ \begin{array}{rc} f_{2,i}(M_2, V_1^{\ell}), & 1 \leq i \leq d_2 \\ f_{2,i}(M_2, V_1^{\ell}, S^{i-d_2}), & d_2 + 1 \leq i \leq n \end{array} \right\}$$

# Common Message Model



# Main Results Common Message with Delayed CSI ($d_1 \geq d_2$)

**Theorem**

*The capacity region of FSM-MAC with a common message, CSI at the decoder and delayed CSI at the encoders with delays $d_1$ and $d_2$, is*

$$\begin{aligned} R_1 &< I(X_1;Y|X_2,U,S,\tilde{S}_1,\tilde{S}_2), \\ R_2 &< I(X_2;Y|X_1,U,S,\tilde{S}_1,\tilde{S}_2), \\ R_1+R_2 &< I(X_1,X_2;Y|U,S,\tilde{S}_1,\tilde{S}_2), \\ R_0+R_1+R_2 &< I(X_1,X_2;Y|S,\tilde{S}_1,\tilde{S}_2), \end{aligned}$$

*for some joint distribution of the form:*

$$P(u|\tilde{s}_1)P(x_1|\tilde{s}_1,u)P(x_2|\tilde{s}_1,\tilde{s}_2,u).$$

The joint distribution $(S,\tilde{S}_1,\tilde{S}_2)$ is the same joint distribution as $(S_i,S_{i-d_1},S_{i-d_2})$.

# Achievability ideas and discussion

- If both the encoder and decoder know the state (with or without delay) one can use MUX-DEMUX scheme. [Goldsmith/varaiya97] [Viswanathan99]

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- Problem 2: Common message generates many corner-points.
- Solution: Encode using MUX, decode using joint-typicality.

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- We need to split $M_2$ into many sub-messages according to $S_2$. Error analysis yield many inequalities.
- The reduction of the inequalities is proved using induction and the Fourier-Motzkin elimination.

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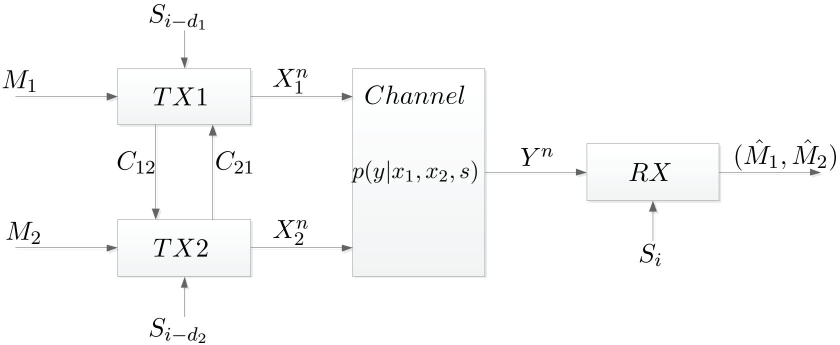
- Auxiliaries can be combined
- Identification of the auxiliary random variable $U$ as the common knowledge of the two encoders.

$$U_i = (M_0, S^{i-d_1-1}).$$

# MAC with conferencing and Delayed CSI



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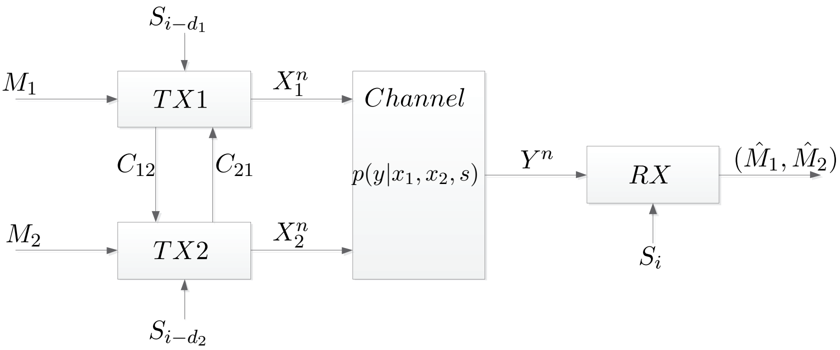


Share as much as possible the massage through the conferencing link.

# Conferencing Setting - Achievability Outline

- Split the original messages $(M_1, M_2)$ into private messages $(M_1', M_2')$ and a common message $(\widetilde{M}_1, \widetilde{M}_2)$

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- Create a common message $M_0$ with rate $\widetilde{R}_1 + \widetilde{R}_2$
- Using common message:

$$
\begin{aligned}
&(R_1 - \widetilde{R}_1) \leq I(X_1; Y | X_2, U, S, \tilde{S}_1, \tilde{S}_2), \\
&(R_2 - \widetilde{R}_2) \leq I(X_2; Y | X_1, U, S, \tilde{S}_1, \tilde{S}_2)), \\
&(R_1 - \widetilde{R}_1) + (R_2 - \widetilde{R}_2) \leq I(X_1, X_2; Y | U, S, \tilde{S}_1, \tilde{S}_2), \\
&(\widetilde{R}_1 + \widetilde{R}_2) + (R_1 - \widetilde{R}_1) + (R_2 - \widetilde{R}_2) \leq I(X_1, X_2; Y | S, \tilde{S}_1, \tilde{S}_2).
\end{aligned}
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# Main Results with conferecing and Delayed CSI ($d_1 \geq d_2$)

## Theorem

*The capacity region of FSM-MAC with partially cooperative encoders, CSI at the decoder and CSI at the encoders with delays* $d_1$ *and* $d_2$*, is*

$$\begin{aligned} R_1 &< I(X_1;Y|X_2,U,S,\tilde{S}_1,\tilde{S}_2) + C_{12}, \\ R_2 &< I(X_2;Y|X_1,U,S,\tilde{S}_1,\tilde{S}_2) + C_{21}, \\ R_1 + R_2 &< \min\left\{ \begin{array}{c} I(X_1,X_2;Y|U,S,\tilde{S}_1,\tilde{S}_2) + C_{12} + C_{21}, \\ I(X_1,X_2;Y|S,\tilde{S}_1,\tilde{S}_2) \end{array} \right\}, \end{aligned}$$

*for some joint distribution of the form:*

$$P(u|\tilde{s}_1)P(x_1|\tilde{s}_1,u)P(x_2|\tilde{s}_1,\tilde{s}_2,u).$$

# Example: Gilbert-Elliot Gaussian MAC

- At any given time $i$ the channel is in one of two possible states $Good$ or $Bad$.
- $\sigma_B^2 > \sigma_G^2$.

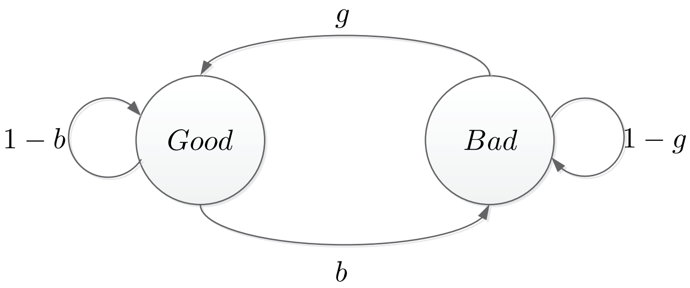


Figure: Two-state AGN channel.

# The Gaussian FSM-MAC

FS additive Gaussian noise (AGN) MAC with partially cooperative encoders and delayed CSI,

$$Y_i = X_{1,i} + X_{2,i} + N_{S_i},$$

- $N_{S_i}$ is a zero-mean Gaussian random variable with variance depending on the state of the channel at time $i$, $S_i$, and denoted by $\sigma_N^2(s)$
- $N_{S_i}$ is independent of $X_{1,2}$ and $X_{2,i}$ for every $i \in \{1, 2, \ldots, n\}$
- The inputs are bounded by the following power constraints:

$$\frac{1}{n}\sum_{i=1}^{n} X_{1,i}^2 \le P_1 \ ; \ \frac{1}{n}\sum_{i=1}^{n} X_{2,i}^2 \le P_2$$

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- Substitute the auxiliary RV $U$, for any given $\widetilde{S}_1 = \tilde{s}_1$ (which we denote by $U(\tilde{s}_1)$), with a $V(\tilde{s}_1) = E[X_1|U(\tilde{s}_1), \tilde{s}_1]$

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- This increases the region and the Markov $X_1^G(\tilde{s}_1) - V^G(\tilde{s}_1) - X_2^G(\tilde{s}_1, \tilde{s}_2)$ holds for any given $(s, \tilde{s}_1, \tilde{s}_2)$.

# Capacity of Gaussian case

$$R_1 < \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1-d_2}(\tilde{s}_2, \tilde{s}_1) \sum_{s} K^{d_2}(s, \tilde{s}_2) \log\left(1 + \frac{\beta_1(\tilde{s}_1) P_1(\tilde{s}_1)}{\sigma_N^2(s)}\right) + C_{12},$$

$$R_2 < \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1-d_2}(\tilde{s}_2, \tilde{s}_1) \sum_{s} K^{d_2}(s, \tilde{s}_2) \log\left(1 + \frac{\beta_2(\tilde{s}_1, \tilde{s}_2) P_2(\tilde{s}_1, \tilde{s}_2)}{\sigma_N^2(s)}\right) + C_{21},$$

$$R_1 + R_2 < \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1-d_2}(\tilde{s}_2, \tilde{s}_1) \sum_{s} K^{d_2}(s, \tilde{s}_2) \times \log\left(1 + \frac{\beta_1(\tilde{s}_1) P_1(\tilde{s}_1) + \beta_2(\tilde{s}_1, \tilde{s}_2) P_2(\tilde{s}_1, \tilde{s}_2)}{\sigma_N^2(s)}\right) + C_{12} + C_{21},$$

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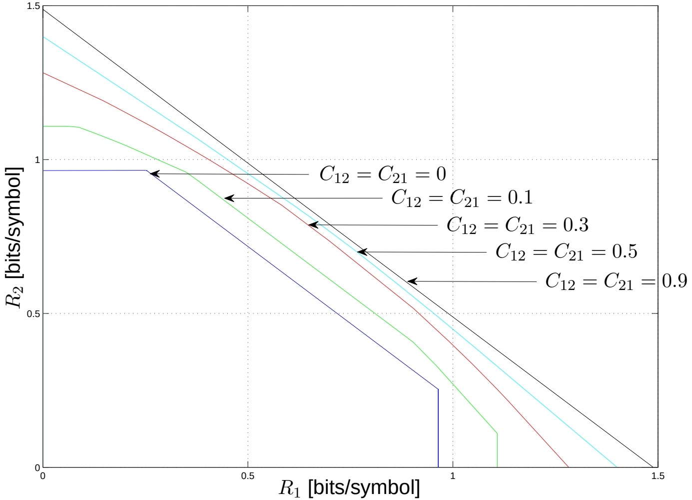
$$\sum_{\tilde{s}_1} \pi(\tilde{s}_1) P_1(\tilde{s}_1) \leq \mathcal{P}_1 \qquad \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} P(\tilde{s}_2|\tilde{s}_1) P_2(\tilde{s}_1, \tilde{s}_2) \leq \mathcal{P}_2,$$

# Capacity region of Two-State AGN MAC Example

Fixed delays $d_1 = d_2 = 2$ and symmetrical con. $C_{12} = C_{21}$



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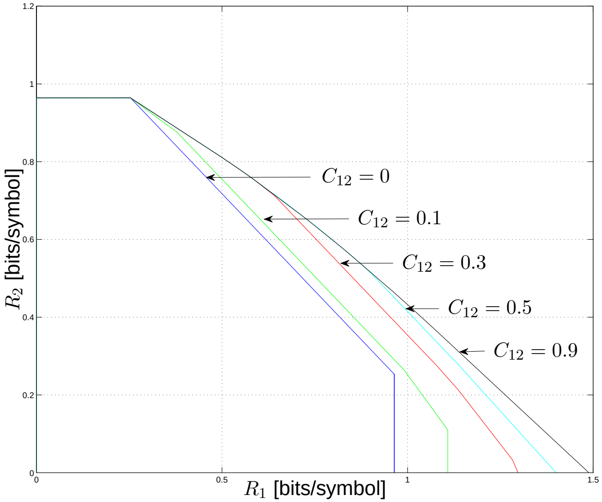
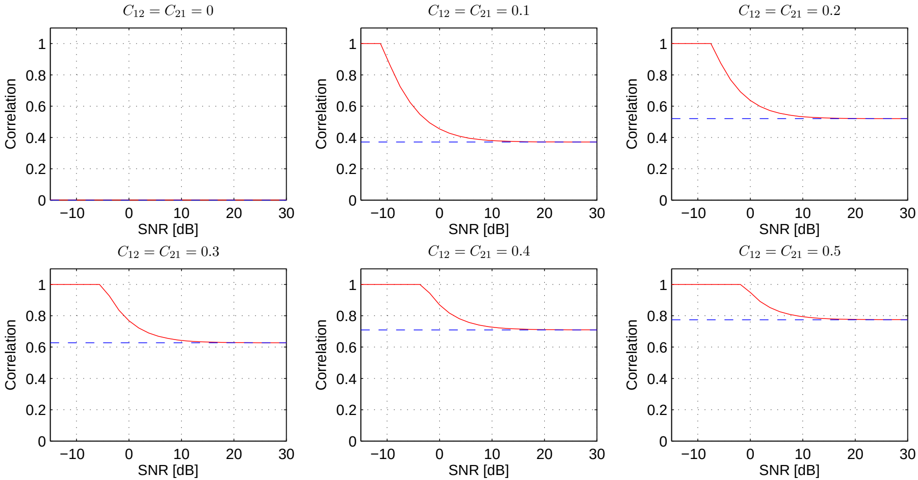
Fixed delays $d_1 = d_2 = 2$ and asymmetrical con. $C_{12} \geq C_{21} = 0$



# Correlation versus SNR



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Thank you!