Toward single-letter feedback capacity via structured auxiliary r.v.

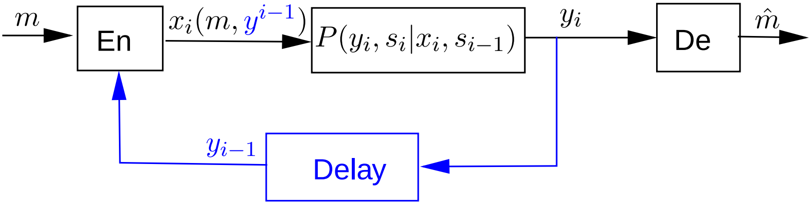
Haim Permuter

Ben-Gurion University

The ISL Colloquium
Feb. 2017

In my Ph.D. ...

considered channel with memory and feedback



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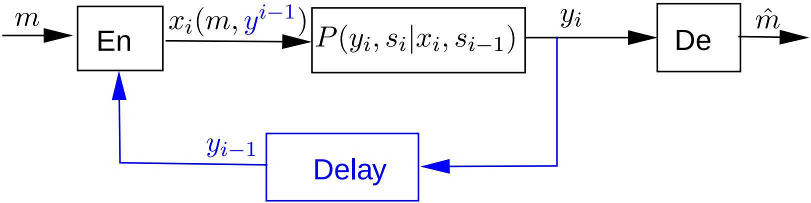

- Finite State Channel (FSC)

$$P(y_i, s_i|x_i, s_{i-1}, x^{i-1}, y^{i-1}, s^{i-2}) = P(y_i, s_i|x_i, s_{i-1})$$

In my Ph.D. ...

Theorem

For any FSC ***with*** *feedback* *[P.&Weissman&Goldsmith09]*

$$C_{FB} \geq \frac{1}{n} \max_{P(x^n||y^{n-1})} \min_{s_0} I(X^n \to Y^n | s_0) - \frac{\log |\mathcal{S}|}{n}$$

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- under mild conditions

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My adviser questions

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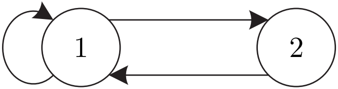

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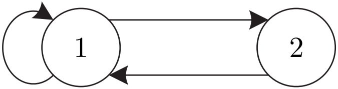
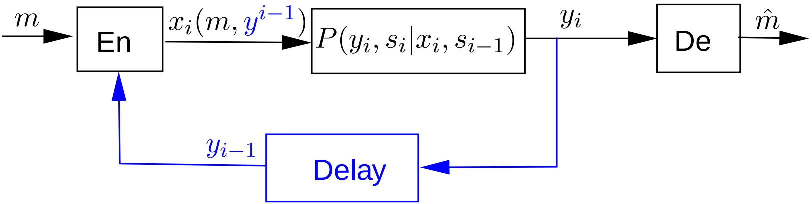


- The graph induces a Markov process
- The single-letter expression is evaluated with the stationary distribution

Unifilar FSC with feedback



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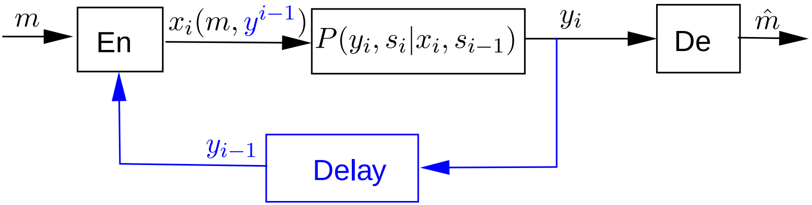

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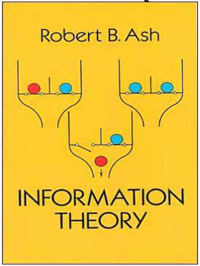
$$S_i = f(S_{i-1}, X_i, Y_i)$$

Trapdoor Channel [Blackwell61]



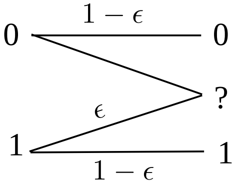
Ising Channel [Berger90]

$$y_i = \begin{cases} x_i, & \text{with prob. } \frac{1}{2} \\ x_{i-1}, & \text{with prob. } \frac{1}{2} \end{cases}$$

Dicode Erasure Channel [Pfister08]

$$y_i = \begin{cases} x_i - x_{i-1}, & \text{with prob. } 1-\epsilon \\ ?, & \text{with prob. } \epsilon \end{cases}$$

Erasure Channel with no repeated 1's



Feedback capacity

Theorem [Sabag/P./Pfiser16]

The feedback capacity of a unifilar FSC is bounded by

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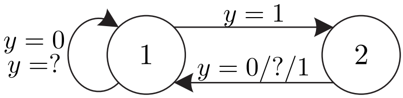

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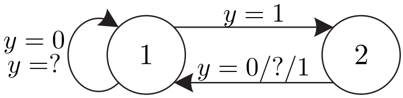

The Q-graph and $P(x|s,q)$ induces

$$p(s,q,x,y) = \pi(s,q)p(x|s,q)p(y|s,x)$$

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- For all known cases the upper bound is tight $|\mathcal{Q}| \leq 4$,
- If $|\mathcal{Q}|$ unbounded then its also achievable, i.e.,

$$C_{fb} = \sup_{Q} \sup_{p(x|s,q)} I(X,S;Y|Q)$$

Sketch Proof

$$C_{fb} = \max_{P(x_i|x^{i-1},y^{i-1})} \frac{1}{n} I(X^n \to Y^n)$$

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Examples

Theorem [Sabag/P./Pfiser16]

$$C_{fb} \leq \sup_{p(x|s,q)} I(X,S;Y|Q), \quad \forall Q\text{-graph}$$

Ex1: Memoryless channel, $|\mathcal{S}| = 1$. Choose Q constant.

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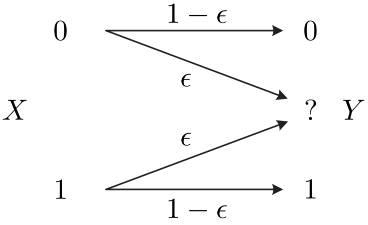
Ex2: State known at the decoder and encoder. Choose $Q = S$

$$C_{fb} \leq \sup_{p(x|s,q)} I(X,S;Y,S|Q) = \sup_{p(x|s)} I(X;Y|S)$$

The input-constrained BEC [Sabag,Permuter,Kashyap 16]

- Binary erasure channel (BEC):



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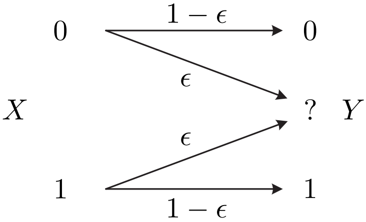

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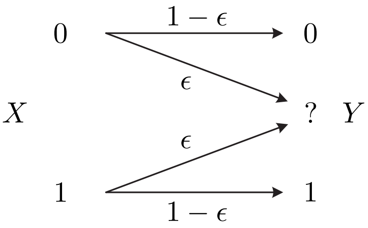
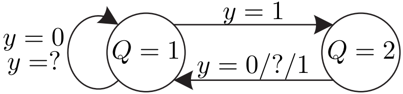


- The codewords contain no-consecutive ones
- The channel state is $S_{i-1} = X_{i-1}$.

Solving BEC with no consecutive '1'

Q-graph



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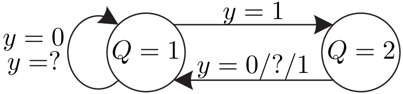

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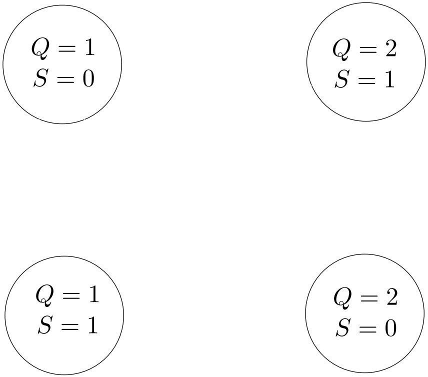

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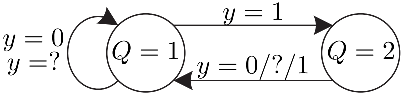

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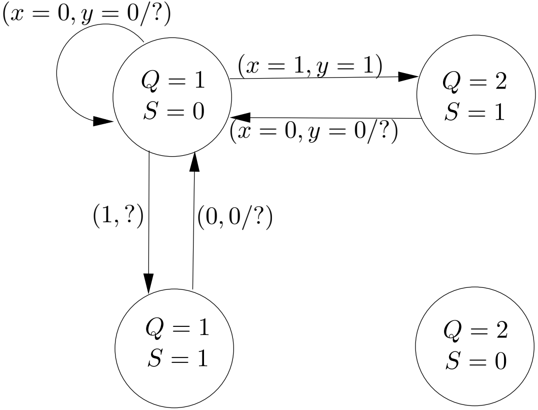

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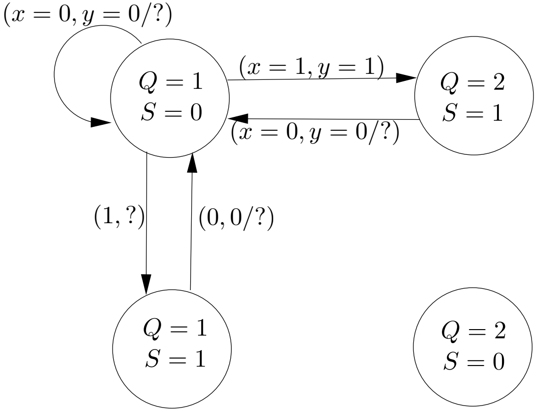

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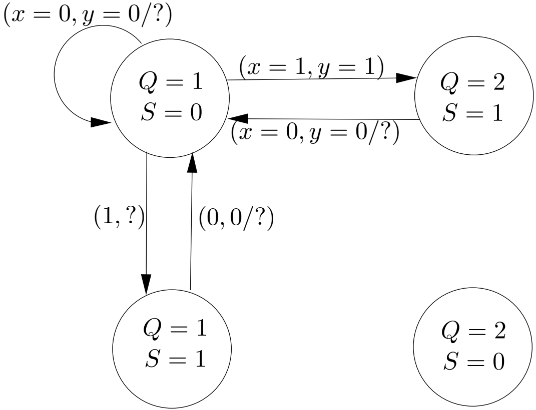

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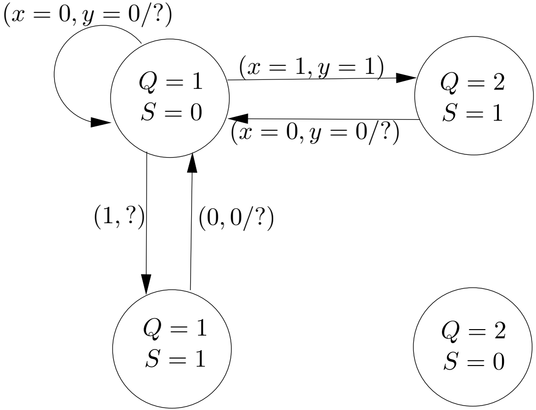

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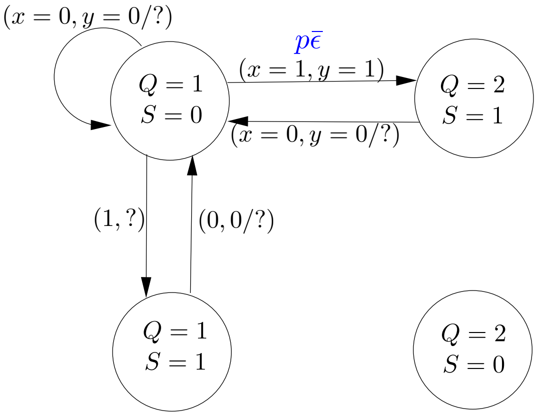

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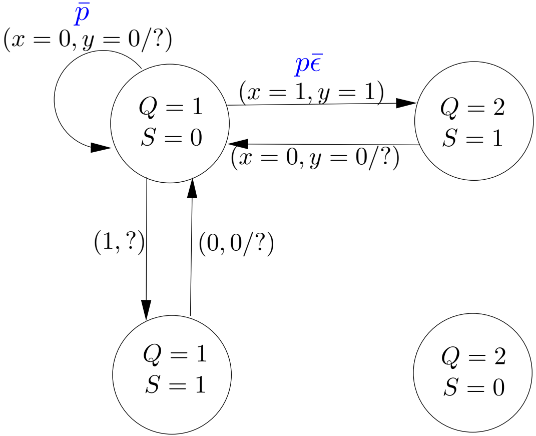

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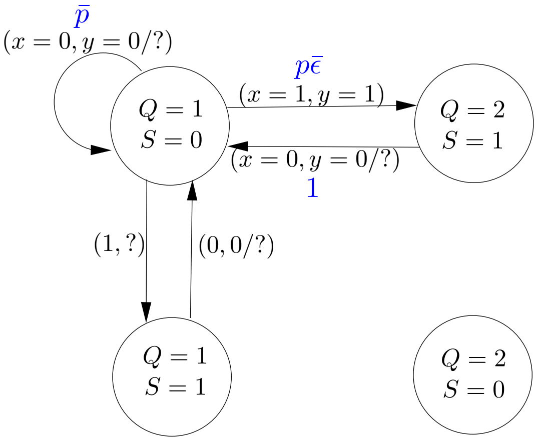

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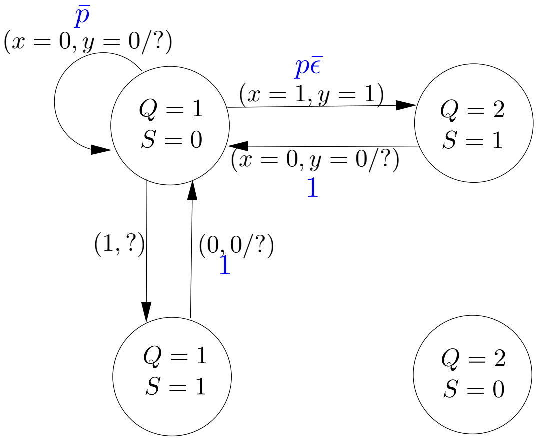

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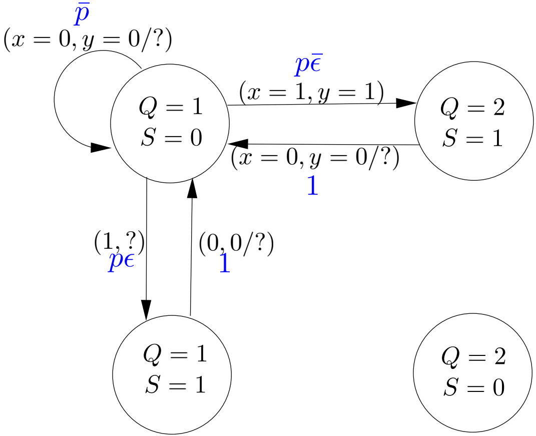

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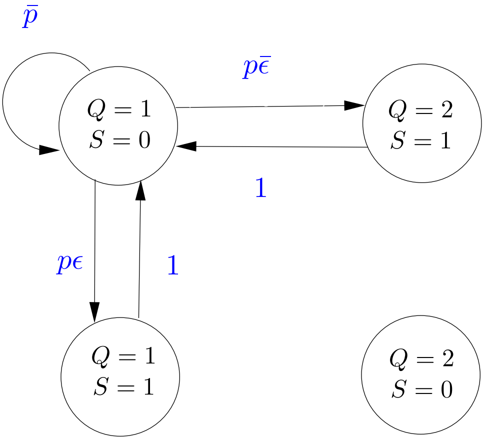

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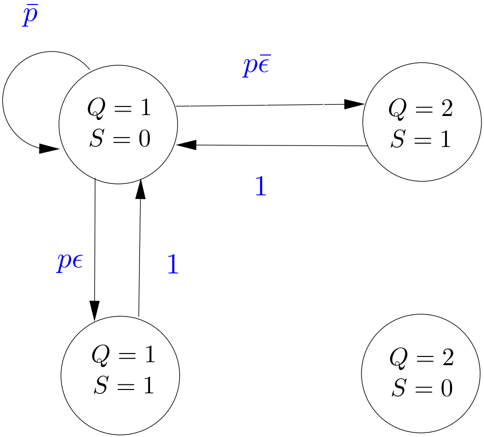

$$[\pi_{0,1}, \pi_{1,1}, \pi_{1,2}] = \left[\frac{1}{1+p}, \frac{p\epsilon}{1+p}, \frac{p\bar{\epsilon}}{1+p}\right]$$

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Evaluate

$$I(X, S; Y|Q)$$

at $\pi(s,q)p(x|s,q)p(y|x,s)$:

$$C_{fb} \leq \max_p \frac{H_2(p)}{p + \frac{1}{1-\epsilon}}.$$

Sketch Proof

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\begin{aligned}
C_{fb} &= \max_{P(x_i|x^{i-1},y^{i-1})} \frac{1}{n} I(X^n \to Y^n) \\
&\triangleq \max_{P(x_i|x^{i-1},y^{i-1})} \frac{1}{n} \sum_{i=1}^{n} I(X^i; Y_i | Y^{i-1}) \\
&= \max_{P(x_i|s_{i-1},y^{i-1})} \frac{1}{n} \sum_{i=1}^{n} I(X_i, S_{i-1}; Y_i | Y^{i-1}) \\
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&= \max_{P(x|s,q)} I(X, S; Y | Q)
\end{aligned}
$$

Sufficient condition

- The channel state estimation:

$$p(s_t|y^t) = \frac{p(s_t, y_t|y^{t-1})}{p(y_t|y^{t-1})}$$

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We show that maximizing input is BCJR, hence

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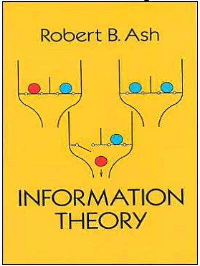
Trapdoor Channel [Blackwell61]



Ising Channel [Berger90]

$$y_i = \begin{cases} x_i, & \text{with prob. } \frac{1}{2} \\ x_{i-1}, & \text{with prob. } \frac{1}{2} \end{cases}$$

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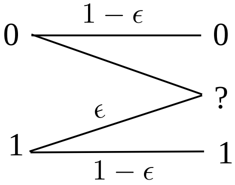

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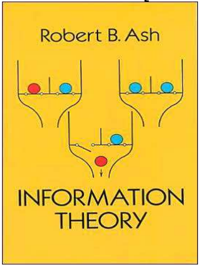

$$C_{fb} = \log \phi, \ \ \phi = \frac{\sqrt{5}+1}{2}$$

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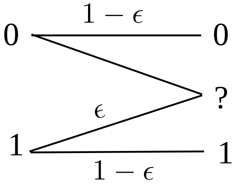

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Matching schemes for memoryless channels

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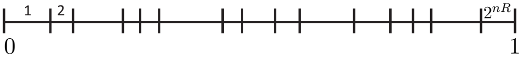

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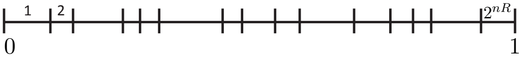

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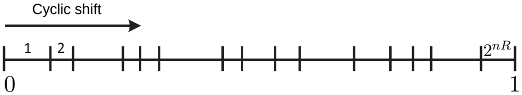

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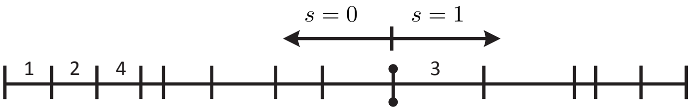

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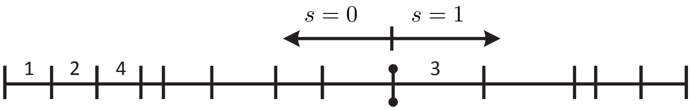

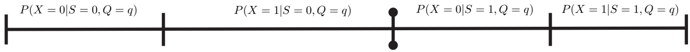

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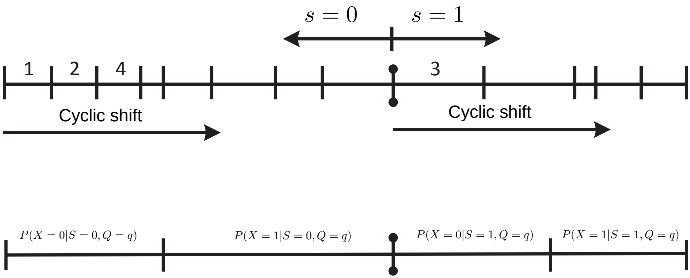

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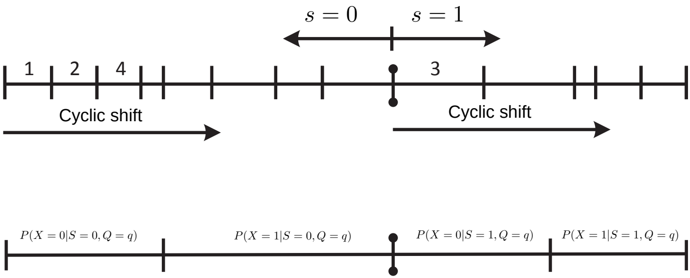

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[Feedback capacity and coding for the BIBO channel with a no-repeated-ones input constraint, Sabag/P/Kashyap, on Arxiv]

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$$p(s_t|y^t) = \frac{p(s_t, y_t|y^{t-1})}{p(y_t|y^{t-1})}$$
$$= \frac{\sum_{x_t, s_{t-1}} p(s_t, y_t, x_t, s_{t-1}|y^{t-1})}{\sum_{x_t, s_{t-1}} p(y_t, x_t, s_{t-1}|y^{t-1})}.$$

- This mapping is denoted by BCJR : $\mathcal{Z} \times \mathcal{Y} \to \mathcal{Z}$.
- If $|\mathcal{Z}| \leq \infty$ and $P_{X|S,Z}$ satisfies the BCJR mapping, we call it *BCJR-invariant*.

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Theorem (Lower bound)

The feedback capacity satisfies

$$C_{fb} \geq I(X, S; Y|Q),$$

for all BCJR-invariant inputs.

Upper bound with sufficient condition

Theorem [Sabag/P./Pfiser16]

The feedback capacity of a unifilar FSC is bounded by

$$C_{fb} \leq \max_{p(x|s,q)} I(X,S;Y|Q), \quad \forall Q\text{-graph}$$

and if $p^*(x|s,q)$ is BCJR-invariant input, equality holds.