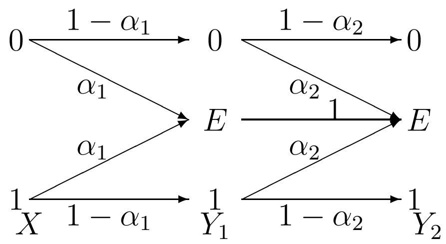
1st Semester 2010/11

Solutions to Homework Set #3
Broadcast channel, Marton's region, Semi-deterministic BC

1. **Degraded Erasure Broadcast Channel.** Consider the following degraded broadcast channel.



   (a) What is the capacity of the channel from X to Y_1?

   (b) From X to Y_2?

   (c) What is the capacity region of all (R_1, R_2) achievable for this broadcast channel? Simplify and sketch.

Solution to Degraded Erasure Broadcast Channel

   (a) The channel from X to Y_1 is a standard erasure channel with probability of erasure $= \alpha_1$, and hence the capacity is $1 - \alpha_1$

   (b) We can show that the effective channel from X to Y_2 is a binary erasure channel with erasure probability $\alpha_1 + \alpha_2 - \alpha_1\alpha_2$, and hence the capacity is $1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 = (1 - \alpha_1)(1 - \alpha_2)$

   (c) As in Problem 15.13, the auxiliary random variable U in the capacity region of the broadcast channel has to be binary. Hence we have the following picture

   We can now evaluate the capacity region for this choice of auxiliary random variable. By symmetry, the best distribution for U is the uniform. Let $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$, and therefore $1 - \alpha = \overline{\alpha} = \overline{\alpha_1\alpha_2}$.

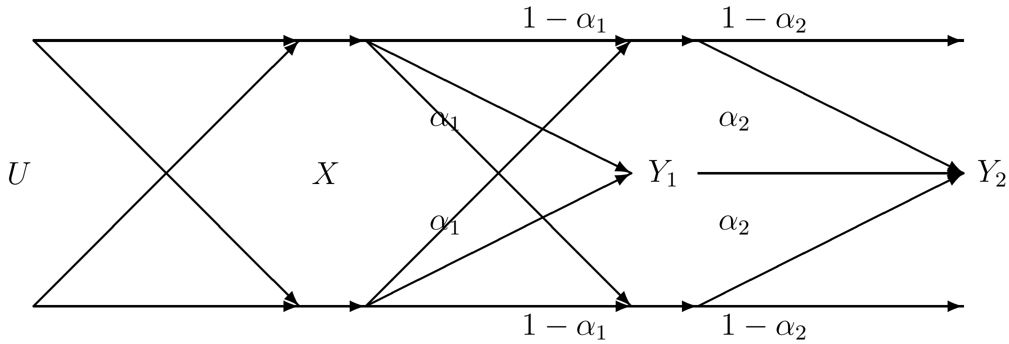


Figure 1: Broadcast channel with auxiliary random variable

Hence

$$\begin{aligned}
R_2 &= I(U;Y_2) && (1)\\
&= H(Y_2) - H(Y_2|U) && (2)\\
&= H\left(\frac{\overline{\alpha}}{2}, \alpha, \frac{\overline{\alpha}}{2}\right) - H((\overline{\beta}\overline{\alpha_1\alpha_2}, \alpha_1 + \overline{\alpha_1}\alpha_2, \beta\overline{\alpha_1\alpha_2}) && (3)\\
&= H(\alpha) + \overline{\alpha}H\left(\frac{1}{2}\right) - H(\alpha) - \overline{\alpha}H(\overline{\beta}, \beta) && (4)\\
&= \overline{\alpha}(1 - H(\beta)). && (5)
\end{aligned}$$

Also

$$\begin{aligned}
R_1 &= I(X;Y_1|U) && (6)\\
&= H(Y_1|U) - H(Y_1|U,X) && (7)\\
&= H(\overline{\beta}\overline{\alpha_1}, \alpha_1, \beta\overline{\alpha_1}) - H(\alpha_1) && (8)\\
&= \overline{\alpha_1}H(\beta) + H(\alpha_1) - H(\alpha_1) && (9)\\
&= \overline{\alpha_1}H(\beta) && (10)
\end{aligned}$$

These two equations characterize the boundary of the capacity region as β varies. When $\beta = 0$, then $R_1 = 0$ and $R_2 = \overline{\alpha}$. When $\beta = \frac{1}{2}$, we have $R_1 = \overline{\alpha_1}$ and $R_2 = 0$.

2. **Deterministic broadcast channel.**

A deterministic broadcast channel is defined by an input X, two outputs, Y_1 and Y_2 which are functions of the input X. Thus $Y_1 = f_1(X)$ and $Y_2 = f_2(X)$. Let R_1 and R_2 be the rates at which information can be sent to the two receivers.

- Prove that

$$\begin{aligned} R_1 &\leq H(Y_1) && (11)\\ R_2 &\leq H(Y_2) && (12)\\ R_1+R_2 &\leq H(Y_1,Y_2) && (13)\end{aligned}$$

- Suggest what would be the capacity region of the deterministic broadcast channel.
- Prove the achievability of the region you have suggested. (Hint: you may use Marton achievable region.)

Solution to Deterministic broadcast channel.

The solution is divided to two parts; Converse and achievability. The achievability proof is given using the Marton region. We begin with the converse. First, we show that $R_1 \leq H(Y_1)$.

$$\begin{aligned} nR_1 &= H(W_1)\\ &\leq I(W_1;Y_1^n)+n\epsilon_n\\ &\leq H(Y_1^n)+n\epsilon_n\\ &= \sum_{i=1}^n H(Y_{1,i}|Y_1^{i-1})+n\epsilon_n\\ &\leq \sum_{i=1}^n H(Y_{1,i})+n\epsilon_n\\ &\overset{(a)}{=} nH(Y_{1,Q}|Q)+n\epsilon_n\\ &\leq nH(Y_{1,Q})+n\epsilon_n\\ &\overset{(b)}{=} nH(Y_1)+n\epsilon_n,\end{aligned}$$

where (a) follows from the definition of $Q \sim U\{1,2,...,n\}$, and (b) follows from the definition of $Y_1 = Y_{1,Q}$. In much the same way we obtain the inequality for R_2, i.e., $R_2 \leq H(Y_2)$.

As for the sum rate,

$$\begin{aligned}
n(R_1+R_2) &= H(W_1,W_2) \\
&\le I(W_1,W_2;Y_1^n,Y_2^n)+n\epsilon_n \\
&\le H(Y_1^n,Y_2^n)+n\epsilon_n \\
&\le \sum_{i=1}^{n} H(Y_{1,i},Y_{2,i})+n\epsilon_n \\
&= nH(Y_{1,Q},Y_{2,Q}|Q)+n\epsilon_n \\
&\le nH(Y_{1,Q},Y_{2,Q})+n\epsilon_n \\
&\stackrel{(a)}{=} nH(Y_1,Y_2)+n\epsilon_n,
\end{aligned}$$

where (a) follows from the definition of $Y_1,\ Y_2$ as in the previous part.

The achievability proof is given using the Marton region, where $U_1 = Y_1$ and $U_2 = Y_2$. Hence,

$$\begin{aligned}
R_1 &\le I(Y_1,Y_1) = (Y_1) \\
R_2 &\le I(Y_2;Y_2) = H(Y_2) \\
R_1+R_2 &\le I(Y_1;Y_1)+I(Y_2;Y_2)+I(Y_2;Y_1) \\
&= H(Y_1)+H(Y_2)-I(Y_2;Y_1) \\
&= H(Y_1,Y_2),
\end{aligned}$$

and the region given above is achievable.

3. **Semi-Deterministic broadcast channel.**

A semi deterministic broadcast channel is defined by an input X, two outputs, Y_1 and Y_2 where Y_1 is function of the input X, i.e., $Y_1 = f_1(X)$, and Y_2 is determined by a memoryless channel $P_{Y_2|X}$. Let R_1 and R_2 be the rates at which information can be sent to the two receivers.

Prove that the capacity region is the set of R_1, R_2 that satisfies

$$\begin{aligned}
R_1 &\le H(Y_1) &\quad (14)\\
R_2 &\le I(U;Y_2) &\quad (15)\\
R_1+R_2 &\le I(U;Y_2)+H(Y_1|U) &\quad (16)
\end{aligned}$$

Solution to Semi-Deterministic broadcast channel.

The solution is divided to two parts; Converse and achievability. The achievability proof is given using the Marton region. We begin with the converse. First, as in (2), it is clear that $R_1 \leq H(Y_1)$. Now,

$$
\begin{aligned}
nR_2 &\leq I(Y_2^n; W_2) + n\epsilon_n \\
&= \sum_{i=1}^{n} \left[H(Y_{2,i}|Y_2^{i-1}) - H(Y_{2,i}|Y_2^{i-1}, W_2) \right] + n\epsilon_n \\
&\leq \sum_{i=1}^{n} \left[H(Y_{2,i}) - H(Y_{2,i}|Y_{1,(i+1)}^n, Y_2^{i-1}, W_2) \right] + n\epsilon_n \\
&\stackrel{(a)}{=} \sum_{i=1}^{n} \left[H(Y_{2,i}) - H(Y_{2,i}|U_i) \right] + n\epsilon_n \\
&= \sum_{i=1}^{n} I(Y_{2,i}; U_i) + n\epsilon_n \\
&\stackrel{(b)}{=} nI(Y_{2,Q}; U_Q|Q) + n\epsilon_n \\
&\leq nI(Y_{2,Q}; U_Q, Q) + n\epsilon_n \\
&\stackrel{(c)}{\leq} nI(Y_2; U) + n\epsilon_n,
\end{aligned}
$$

where (a) follows from the definition for $U_i = \{Y_{1,(i+1)}^n, Y_2^{i-1}, W_2\}$, (b) follows from the definition of Q be a uniformed distributed random variable $Q \sim U\{1, 2, ..., n\}$, and (c) follows from the definition of $Y_2 = Y_{2,Q}$ and $U = \{U_Q, Q\}$.

As for the sum rate,

$$
\begin{aligned}
n(R_1+R_2) &\le I(Y_1^n, Y_2^n; W_1, W_2) + n\epsilon_n \\
&\overset{(a)}{=} I(Y_2^n; W_1, W_2) + I(Y_1^n; W_1, W_2 | Y_2^n) + n\epsilon_n \\
&\overset{(b)}{\le} I(Y_2^n; W_2) + I(Y_1^n; W_2 | Y_2^n) + I(Y_1^n; W_1 | Y_2^n, W_2) + n\epsilon_n \\
&\le nI(Y_2, U) + H(W_2 | Y_2^n) + H(Y_1^n | Y_2^n, W_2) + n\epsilon_n \\
&\le nI(Y_2, U) + \sum_{i=n}^{1} H(Y_{1,i} | Y_{1,(i+1)}^n, Y_2^n, W_2) + n\epsilon_n \\
&\overset{(c)}{\le} nI(Y_2, U) + \sum_{i=n}^{1} H(Y_{1,i} | Y_{1,(i+1)}^n, Y_2^{i-1}, W_2) + n\epsilon_n \\
&= nI(Y_2, U) + \sum_{i=n}^{1} H(Y_{1,i} | U_i) + n\epsilon_n \\
&= nI(Y_2, U) + nH(Y_{1,Q} | U_Q, Q) + n\epsilon_n \\
&= nI(Y_2, U) + nH(Y_1, U) + n\epsilon_n,
\end{aligned}
$$

where (a) follows from the mutual information chain rule, (b) follows from the mutual information chain rule as well as removing W_1 from the conditioning in the first term, (c) follows from removing $Y_{2,i}^n$ from the conditioning in the sum, and we use the definitions above to complete the proof.

Thus, we showed that an upper bound to the rate region is the one given in the question, i.e.,

$$
\begin{aligned}
R_1 &\le H(Y_1) \\
R_2 &\le I(Y_2; U) \\
R_1 + R_2 &\le I(Y_2; U) + H(Y_1 | U).
\end{aligned}
$$

The achievability proof is given using the Marton region, where $U_1 = Y_1$

and $U_2 = U$. Hence,

$$\begin{aligned} R_1 &\leq I(Y_1, Y_1) = (Y_1) \\ R_2 &\leq I(U; Y_2) \\ R_1 + R_2 &\leq I(Y_1; Y_1) + I(U; Y_2) + I(U; Y_1) \\ &= H(Y_1) - I(U; Y_1) + I(U; Y_2) \\ &= H(Y_1|U) + I(U; Y_2), \end{aligned}$$

and the region given above is achievable.

4. **Mutual Covering Lemma**: Prove the following result.

   Let $(U_1, U_2) \sim p(u_1, u_2)$ and $\epsilon > 0$. Let $U_1^n(m_1), m_1 \in [1, ..., 2^{nr_1}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_1}(u_{1,i})$. Similarly, Let $U_2^n(m_2), m_2 \in [1, ..., 2^{nr_2}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_2}(u_{2,i})$. Assume that $U_1^n(m_1) : m_1 \in [1, ..., 2^{nr_1}]$ and $U_2^n(m_2) : m_2 \in [1, ..., 2^{nr_2}]$ are independent.

   Then, there exists $\delta(\epsilon)$ that goes to 0 as $\epsilon \to 0$ such that if

   $$r_1 + r_2 > I(U_1; U_2) + \delta(\epsilon), \tag{17}$$

   then

   $$\lim_{n=\infty} \Pr\{(U_1^n(m_1), U_2^n(m_1)) \notin T_\epsilon^{(n)}(U_1, U_2)\, \forall m_1 \in [1, ..., 2^{nr_1}], m_2 \in [1, ..., 2^{nr_2}]\} = 0 \tag{18}$$

   In addition to the prove, please explain, how it extends the covering lemma.

   Solution to Mutual Covering Lemma

   The solution to the mutual covering lemma is closely related to the regular covering lemma.

$$
\begin{aligned}
\Pr\{\forall m_1, m_2 : (U^n 1(m_1), U_2^n(m_1)) \notin T_\epsilon^{(n)}(U_1, U_2)\} \\
&\overset{(a)}{=} \prod_{(m_1,m_2)=(1,1)}^{(2^{nr_1},2^{nr_2})} \Pr\{(U_1^n(m_1), U_2^n(m_1)) \notin T_\epsilon^{(n)}(U_1, U_2)\} \\
&\overset{(b)}{\leq} \prod_{(m_1,m_2)=(1,1)}^{(2^{nr_1},2^{nr_2})} \left(1 - 2^{-n(I(U_1,U_2)+\epsilon)}\right) \\
&= \left(1 - 2^{-n(I(U_1,U_2)+\epsilon)}\right)^{2^{n(r_1+r_2)}} \\
&\overset{(c)}{\leq} \exp\{-2^{n(r_1+r_2-I(U_1,U_2)-\epsilon)}\} \xrightarrow[r_1+r_2>I(U_1,U_2),\ n\to\infty]{} 0,
\end{aligned}
$$

where (a) follows from the messages m_1, m_2 be independent, and (b) follows from the fact that the probability of two independent sequences u_1^n, u_2^n to be jointly typical is $2^{-n(I(U_1,U_2)+\epsilon)}$. (c) follows from the inequality $(1-y)^n \leq \exp\{-yn\}$.

The extension is given by the fact that now we do not limit our selves to one sequence u_1^n, i.e., $r_1 = 0$. In the covering lemma, where $r_1 = 0$, we need at least $r_2 > I(U_1, U_2)$ sequences from u_2^n to be sure that one of them is jointly typical. Now, we ask for the size of a group containing sequences of $\{u_1^n, u_2^n\}$ such that a pair is jointly typical. Evidently, we need the size to be at least $2^{nI(U_1,U_2)}$, much like in the regular covering lemma for $r_1 = 0$.