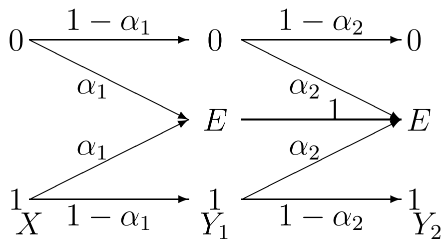
Homework Set #3
Broadcast channel, Marton's region, Semi-deterministic BC

1. **Degraded Erasure Broadcast Channel.** Consider the following degraded broadcast channel.



   (a) What is the capacity of the channel from X to Y_1?

   (b) From X to Y_2?

   (c) What is the capacity region of all (R_1, R_2) achievable for this broadcast channel? Simplify and sketch.

2. **Deterministic broadcast channel.**

   A deterministic broadcast channel is defined by an input X, two outputs, Y_1 and Y_2 which are functions of the input X. Thus $Y_1 = f_1(X)$ and $Y_2 = f_2(X)$. Let R_1 and R_2 be the rates at which information can be sent to the two receivers.

   - Prove that

$$\begin{aligned} R_1 &\leq H(Y_1) && (1)\\ R_2 &\leq H(Y_2) && (2)\\ R_1 + R_2 &\leq H(Y_1, Y_2) && (3) \end{aligned}$$

   - Suggest what would be the capacity region of the deterministic broadcast channel.

- Prove the achievability of the region you have suggested. (Hint: you may use Marton achievable region.)

3. **Semi-Deterministic broadcast channel.**

   A semi deterministic broadcast channel is defined by an input X, two outputs, Y_1 and Y_2 where Y_1 is function of the input X, i.e., $Y_1 = f_1(X)$, and Y_2 is determined by a memoryless channel $P_{Y_2|X}$. Let R_1 and R_2 be the rates at which information can be sent to the two receivers.

   Prove that the capacity region is the set of R_1, R_2 that satisfies

$$\begin{aligned} R_1 &\leq H(Y_1) && (4) \\ R_2 &\leq I(U;Y_2) && (5) \\ R_1 + R_2 &\leq I(U;Y_2) + H(Y_1|U) && (6) \end{aligned}$$

4. **Mutual Covering Lemma**: Prove the following result.

   Let $(U_1, U_2) \sim p(u_1, u_2)$ and $\epsilon > 0$. Let $U_1^n(m_1), m_1 \in [1, ..., 2^{nr_1}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_1}(u_{1,i})$. Similarly, Let $U_2^n(m_2), m_2 \in [1, ..., 2^{nr_2}]$, be pairwise independent random sequences, each distributed according to $\prod_{i=1} P_{U_2}(u_{2,i})$. Assume that $U_1^n(m_1) : m_1 \in [1, ..., 2^{nr_1}]$ and $U_2^n(m_2) : m_2 \in [1, ..., 2^{nr_2}]$ are independent.

   Then, there exists $\delta(\epsilon)$ that goes to 0 as $\epsilon \to 0$ such that if

$$r_1 + r_2 > I(U_1; U_2) + \delta(\epsilon), \tag{7}$$

   then

$$\lim_{n=\infty} \Pr\{(U^n 1(m_1), U_2^n(m_1)) \notin T_\epsilon^{(n)}(U_1, U_2)\, \forall m_1 \in [1, ..., 2^{nr_1}], m_2 \in [1, ..., 2^{nr_2}]\} = 0 \tag{8}$$

   In addition to the prove, please explain, how it extends the covering lemma.