1st Semester 2010/11

Homework Set #2
Broadcast channel, degraded message set, Csiszar Sum Equality

1. **Convexity of capacity region of broadcast channel.** Let $\mathbf{C} \subseteq \mathbf{R}^2$ be the capacity region of all achievable rate pairs $\mathbf{R} = (R_1, R_2)$ for the broadcast channel. Show that $\mathbf{C}$ is a convex set by using a timesharing argument.

   Specifically, show that if $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$ are achievable, then $\lambda \mathbf{R}^{(1)} + (1-\lambda)\mathbf{R}^{(2)}$ is achievable for $0 \le \lambda \le 1$.

2. **Joint typicality** Let x^n, y^n be jointly strong-typical i.e., $(x^n, y^n) \in T_\epsilon^{(n)}(X,Y)$, and let Z^n be distributed according to $\prod_{i=1}^n p_{Z|X}(z_i|x_i)$ (instead of $p_{Z|X,Y}(z_i|x_i,y_i)$). Then, $P\{(x^n, y^n, Z^n) \in T_\epsilon^{(n)}(X,Y,Z)\} \le 2^{-n(I(Y;Z|X)-\delta(\epsilon))}$, where $\delta(\epsilon) \to 0$ when $\epsilon \to 0$.

3. **Broadcast capacity depends only on the conditional marginals.** Consider the general broadcast channel $(X, Y_1 \times Y_2, p(y_1, y_2 \mid x))$. Show that the capacity region depends only on $p(y_1 \mid x)$ and $p(y_2 \mid x)$. To do this, for any given $((2^{nR_1}, 2^{nR_2}), n)$ code, let

$$P_1^{(n)} = P\{\hat{W}_1(\mathbf{Y}_1) \neq W_1\}, \quad (1)$$
$$P_2^{(n)} = P\{\hat{W}_2(\mathbf{Y}_2) \neq W_2\}, \quad (2)$$
$$P^{(n)} = P\{(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\}. \quad (3)$$

   Then show

$$\max\{P_1^{(n)}, P_2^{(n)}\} \le P^{(n)} \le P_1^{(n)} + P_2^{(n)}.$$

   The result now follows by a simple argument.

   Remark: The probability of error $P^{(n)}$ *does* depend on the conditional joint distribution $p(y_1, y_2 \mid x)$. But whether or not $P^{(n)}$ can be driven to zero (at rates (R_1, R_2)) *does not* (except through the conditional marginals $p(y_1 \mid x), p(y_2 \mid x)$).

4. **Degraded broadcast channel.** Find the capacity region for the degraded broadcast channel in following figure.

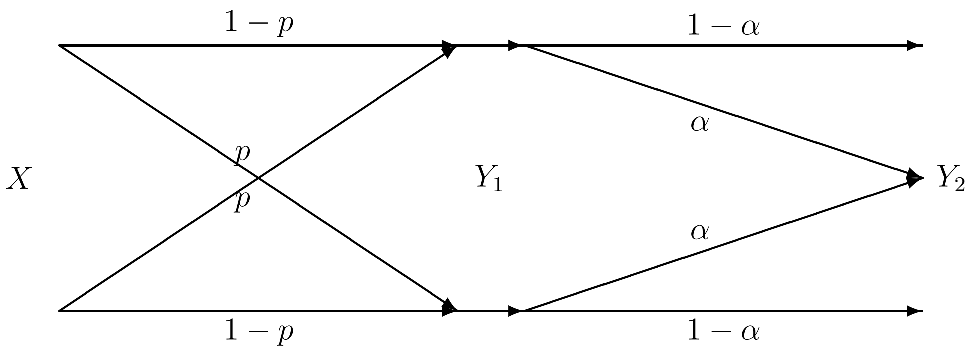

Figure 1: Broadcast channel with a binary symmetric channel and an erasure channel

5. **Csiszar Sum Equality**. Let X^n and Y^n be two random vectors with arbitrary joint probability distribution. Show that:

$$\sum_{i=1}^{n} I(X_{i+1}^n; Y_i | Y^{i-1}) = \sum_{i=1}^{n} I(Y^{i-1}; X_i | X_{i+1}^n) \tag{4}$$

As we shall see this inequality is useful in proving converses to several multiple user channels. (Hint: You can prove this by induction or by expanding the terms on both sides using the chain rule.)

6. **Broadcast Channel with Degraded Message Sets.** Consider a general DM broadcast channel $(\mathcal{X}; p(y_1, y_2|x); \mathcal{Y}_1; \mathcal{Y}_2)$. The sender X encodes two messages $(W_0; W_1)$ uniformly distributed over $\{1, 2..., 2^{nR_0}\}$ and $\{1, 2, ...2^{nR_1}\}$. Message W_0 is to be sent to both receivers, while message W_1 is only intended for receiver Y_1.

The capacity region is given by the set $\mathcal{C}$ of all $(R_0; R_1)$ such that:

$$R_0 \leq I(U; Y_2) \tag{5}$$
$$R_1 \leq I(X; Y_1|U) \tag{6}$$
$$R_0 + R_1 \leq I(X; Y_1) \tag{7}$$

for some $p(u)p(x|u)$.

(a) Show that the set $\mathcal{C}$ is convex.

(b) Provide the achievability proof

(c) Provide a converse proof. You may derive your own converse or use the steps below.

- An alternative characterization of the capacity region is the set $\mathcal{C}'$ of all (R_0, R_1) such that:

$$\begin{aligned} R_0 &\leq \min\{I(U;Y_1); I(U;Y_2)\} && (8) \\ R_0 + R_1 &\leq \min\{I(X;Y_1); I(X;Y_1|U) + I(U;Y_2)\} && (9) \end{aligned}$$

  for some $p(u)p(x|u)$. Show that $\mathcal{C} = \mathcal{C}'$.

- Define $U_i = (W_0; Y_1^{i-1}, Y_{2,i+1}^n)$. Show that

$$n(R_0 + R_1) \leq \sum_{i=1}^{n} (I(X_i; Y_{1,i}|U_i) + I(U_i; Y_{2,i})) + n\epsilon_n \qquad (10)$$

  using the steps

$$\begin{aligned} n(R_0 + R_1) &\leq I(W_1; Y_1^n|W_0) + I(W_0; Y_2^n) + n\epsilon_n \\ &\leq \sum_{i=1}^{n} I(X_i; Y_{1,i}|U_i) + I(Y_{2,i+1}^n; Y_{1,i}|W_0.Y_1^{i-1}) \\ &\quad + I(U_i; Y_{2,i}) - I(Y_1^{i-1}; Y_{2,i}|W_0, Y_{2,(i+1)}^n) + n\epsilon_n. \end{aligned} \qquad (11)$$

  Then use the identity from previous exercise to cancel the second and fourth terms.