Multi-User Information Theory December, 2013

Lecture 3

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I. HAN-KOBAYASHI INNER BOUND

The Han-Kobayashi inner bound is the best-known bound on the capacity region of the discrete memoryless interference channel (DM-IC) [1]. It includes all the inner bounds we discussed so far, and is tight for all interference channels with known capacity regions. We consider the following characterization of this inner bound.

Theorem 1 (Han-Kobayashi Inner Bound) Let $\mathcal{C}$ be the capacity region of the DM-IC $P_{Y_1,Y_2|X_1,X_2}$. Let $\mathcal{R}_{HK}$ be the region defined by the union of all sets of rate pairs $(R_1,R_2)\in\mathbb{R}_+^2$ satisfying:

$$R_1 < I(X_1;Y_1|U_2,Q), \tag{1a}$$

$$R_2 < I(X_2;Y_2|U_1,Q), \tag{1b}$$

$$R_1+R_2 < I(X_1,U_2;Y_1|Q)+I(X_2;Y_2|U_1,U_2,Q), \tag{1c}$$

$$R_1+R_2 < I(X_2,U_1;Y_2|Q)+I(X_1;Y_1|U_1,U_2,Q), \tag{1d}$$

$$R_1+R_2 < I(X_1,U_2;Y_1|U_1,Q)+I(X_2,U_1;Y_2|U_2,Q), \tag{1e}$$

$$2R_1+R_2 < I(X_1,U_2;Y_1|Q)+I(X_1;Y_1|U_1,U_2,Q)+I(X_2,U_1;Y_2|U_2,Q), \tag{1f}$$

$$R_1+2R_2 < I(X_2,U_1;Y_2|Q)+I(X_2;Y_2|U_1,U_2,Q)+I(X_1,U_2;Y_1|U_1,Q), \tag{1g}$$

where the union is taken over all joint distributions of the form $P_Q P_{U_1,X_2|Q} P_{U_2,X_2|Q}$, $|\mathcal{U}_1|\leq|\mathcal{X}_1|+4$, $|\mathcal{U}_2|\leq|\mathcal{X}_2|+4$, and $|\mathcal{Q}|\leq 6$. Then the the following inclusion holds:

$$\mathcal{R}_{HK}\subseteq\mathcal{C}. \tag{2}$$

Remark 1 The Han-Kobayashi inner bound reduces to the interference-as-noise inner bound by setting $U_1=U_2=\emptyset$. At the other extreme, the Han-Kobayashi inner bound reduces to the simultaneous-nonunique-decoding inner bound by setting $U_1 = X_1$ and $U_2 = X_2$. Thus, the bound is tight for the class of DM-ICs with strong interference.

Remark 2 The Han-Kobayashi inner bound can be readily extended to the Gaussian IC with average power constraints and evaluated using Gaussian (U_j, X_j), $j \in \{1, 2\}$. It is not known, however, if the restriction to the Gaussian distribution is sufficient.

Proof: The proof uses rate splitting. We represent each message M_j, $j \in \{1, 2\}$, by independent "public" message M_{j0} at rate R_{j0} and "private" message M_{jj} at rate R_{jj}. Thus, $R_j = R_{j0} + R_{jj}$. These messages are sent via superposition coding, whereby the cloud center U_j represents the public message M_{j0} and the satellite codeword X_j represents the message pair (M_{j0}, M_{jj}). The public messages are to be recovered by both receivers, while each private message is to be recovered only by its intended receiver. We First show that the tuple $(R_{10}, R_{20}, R_{11}, R_{22})$ is achievable if

$$R_{11} < I(X_1; Y_1|U_1, U_2, Q), \tag{3a}$$

$$R_{11} + R_{10} < I(X_1; Y_1|U_2, Q), \tag{3b}$$

$$R_{11} + R_{20} < I(X_1, U_2; Y_1|U_1, Q), \tag{3c}$$

$$R_{11} + R_{10} + R_{20} < I(X_1, U_2; Y_1|Q), \tag{3d}$$

$$R_{22} < I(X_2; Y_2|U_1, U_2, Q), \tag{3e}$$

$$R_{22} + R_{20} < I(X_2; Y_2|U_1, Q), \tag{3f}$$

$$R_{22} + R_{10} < I(X_2, U_1; Y_2|U_2, Q), \tag{3g}$$

$$R_{22} + R_{20} + R_{10} < I(X_2, U_1; Y_2|Q), \tag{3h}$$

for some PMF $P_Q P_{U_1,X_2|Q} P_{U_2,X_2|Q}$.

Throughout this proof we denote a sequence of length n with symbol from the alphabet $\mathcal{X}$ by a boldface letter, i.e., $\mathbf{x}$.

Codebook Generation: Fix a PMF $P_Q P_{U_1,X_2|Q} P_{U_2,X_2|Q}$ and $\epsilon > 0$. Generate a sequence $\mathbf{q}$ in an i.i.d. manner according to P_Q. For $j \in \{1, 2\}$, randomly and conditionally independently generate $2^{nR_{j0}}$ sequences $\mathbf{u}_j(m_{j0})$, $m_{j0} \in \{1, \ldots, 2^{nR_{j0}}\}$,

TABLE I: The joint PMFs induced by different (m_{10}, m_{20}, m_{11}) triples.

	m_{10}	m_{20}	m_{11}	Joint PMF	Rate Bound
1	1	1	1	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1\|\mathbf{x}_1,\mathbf{u}_2)$	–
2	1	1	*	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1\|\mathbf{u}_1,\mathbf{u}_2)$	$R_{11} < I(X_1;Y_1\|U_1,U_2,Q)$
3	*	1	*	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1\|\mathbf{u}_2)$	$R_{10}+R_{11} < I(X_1,U_1;Y_1\|U_2,Q)$
4	*	1	1	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1\|\mathbf{u}_2)$	$R_{10} < I(X_1,U_1;Y_1\|U_2,Q)$
5	1	*	*	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1\|\mathbf{u}_1)$	$R_{20}+R_{11} < I(X_1,U_2;Y_1\|U_1,Q)$
6	*	*	1	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1)$	$R_{10}+R_{20} < I(X_1,U_1,U_2;Y_1\|Q)$
7	*	*	*	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1)$	$R_{10}+R_{20}+R_{11} < I(X_1,U_1,U_2;Y_1\|Q)$
8	1	*	1	$p(\mathbf{u}_1,\mathbf{x}_1)p(\mathbf{u}_2)p(\mathbf{y}_1\|\mathbf{x}_1)$	$R_{20} < I(X_1;Y_1\|U_1,U_2,Q)$

each according to $\prod_{i=1}^{n} P_{U_j|Q}(u_{ji}|q_i)$. For each m_{j0}, randomly and conditionally independently generate $2^{nR_{jj}}$ sequences $\mathbf{x}_j(m_{j0}, m_{jj})$, $m_j j \in \{1,\ldots,2^{nR_{jj}}\}$, each according to $\prod_{i=1}^{n} P_{X_j|U_j,Q}\big(x_{ji}|u_{ji}(m_{j0}), q_i\big)$.

Encoding: To send $m_j = (m_{j0}, m_{jj})$, $j \in \{1,2\}$, Encoder j transmits $\mathbf{x}_j(m_{j0}, m_{jj})$.

Decoding: We use simultaneous nonunique decoding. Upon receiving $\mathbf{y}_1$, Decoder 1 finds the unique message pair $(\hat{m}_{10}, \hat{m}_{11})$ such that $(\mathbf{q}, \mathbf{u}_1(\hat{m}_{10}), \mathbf{u}_2(m_{20}), \mathbf{x}_1(\hat{m}_{10}, \hat{m}_{11}), \mathbf{y}_1) \in \mathcal{T}_\epsilon^{(n)}$, for some $m_{20} \in \{1,\ldots,2^{nR_{20}}\}$; otherwise it declares an error. Decoder 2 finds the message pair $(\hat{m}_{20}, \hat{m}_{22})$ similarly.

Analysis of the Probability of Error: Assume message pair $\big((1,1),(1,1)\big)$ is sent. We bound the average probability of error for each decoder. First consider Decoder 1. As shown in Table I, we have eight cases to consider (here conditioning on $\mathbf{q}$ is suppressed). Cases 3 and 4, and 6 and 7, respectively, share the same PMF, and case 8 does not cause an error. Thus, we are left with only five error events. Accordingly, Decoder 1 makes an error only if one or more of the following events occur:

$$\mathcal{E}_{10} = \Big\{\big(\mathbf{Q}, \mathbf{U}_1(1), \mathbf{U}_2(1), \mathbf{X}_1(1,1), \mathbf{Y}_1\big) \notin \mathcal{T}_\epsilon^{(n)}\Big\}, \tag{4}$$

$$\mathcal{E}_{11} = \Big\{\exists m_{11} \neq 1, \big(\mathbf{Q}, \mathbf{U}_1(1), \mathbf{U}_2(1), \mathbf{X}_1(1,m_{11}), \mathbf{Y}_1\big) \in \mathcal{T}_\epsilon^{(n)}\Big\}, \tag{5}$$

$$\mathcal{E}_{12} = \Big\{\exists m_{10} \neq 1, m_{11}, \big(\mathbf{Q}, \mathbf{U}_1(m_{10}), \mathbf{U}_2(1), \mathbf{X}_1(m_{10},m_{11}), \mathbf{Y}_1\big) \in \mathcal{T}_\epsilon^{(n)}\Big\}, \tag{6}$$

$$\mathcal{E}_{13} = \Big\{\exists m_{20} \neq 1, m_{11} \neq 1, \big(\mathbf{Q}, \mathbf{U}_1(1), \mathbf{U}_2(m_{20}), \mathbf{X}_1(1, m_{11}), \mathbf{Y}_1\big) \in \mathcal{T}_\epsilon^{(n)}\Big\}, \quad (7)$$

$$\mathcal{E}_{14} = \Big\{\exists m_{10} \neq 1, m_{20} \neq 1, m_{11}, \big(\mathbf{Q}, \mathbf{U}_1(m_{10}), \mathbf{U}_2(m_{20}), \mathbf{X}_1(m_{10}, m_{11}), \mathbf{Y}_1\big) \in \mathcal{T}_\epsilon^{(n)}\Big\}. \quad (8)$$

Hence, the average probability of error for Decoder 1 is upper bounded as

$$\mathbb{P}\big[\mathcal{E}_1\big] \leq \sum_{i=0}^{4} \mathbb{P}\big[\mathcal{E}_{1i}\big]. \quad (9)$$

We bound each term. By the LLN, $\mathbb{P}\big[\mathcal{E}_{10}\big]$ tends to zero as $n \to \infty$. By the packing lemma, $\mathbb{P}\big[\mathcal{E}_{11}\big]$ tends to zero as $n \to \infty$ if $R_{11} < I(X_1; Y_1|U_1, U_2, Q) - \delta(\epsilon)$. Similarly, by the packing lemma, $\mathbb{P}\big[\mathcal{E}_{12}\big]$, $\mathbb{P}\big[\mathcal{E}_{13}\big]$ and $\mathbb{P}\big[\mathcal{E}_{14}\big]$ tend to zero as $n \to \infty$ if the conditions $R_{11} + R_{10} < I(X_1; Y_1|U_2, Q) - \delta(\epsilon)$, $R_{11} + R_{20} < I(X_1, U_2; Y_1|U_1, Q) - \delta(\epsilon)$, and $R_{11} + R_{10} + R_{20} < I(X_1, U_2; Y_1|Q) - \delta(\epsilon)$ are satisfied, respectively. The average probability of error for decoder 2 can be bounded similarly.

Finally, substituting $R_{11} = R_1 - R_{10}$ and $R_{22} = R_2 - R_{20}$, and using the Fourier-Motzkin procedure with the constraints $0 \leq R_{j0} \leq R_j$, $j \in \{1, 2\}$, to eliminate R_{10} and R_{20}, we obtain the region given in Theorem 1. Furthermore, the cardinality bound on $\mathcal{Q}$ can be proved using the convex cover method (see [2, Appendix C] for details). This completes the proof of the HanKobayashi inner bound. ■

II. THE SEMI-DETERMINISTIC INJECTIVE INTERFERENCE CHANNEL

Consider the semi-deterministic interference channel depicted in Figure 1. Here the functions y_1 and y_2 satisfy the condition that for every $x_1 \in \mathcal{X}_1$, $y_1(x_1, t_2)$ is a one-to-one function of t_2 and for every $x_2 \in \mathcal{X}_2$, $y_2(x_2, t_1)$ is a one-to-one function of t_1. Note that these conditions imply that $H(Y_1|X_1) = H(T_2)$ and $H(Y_2|X_2) = H(T_1)$. The channel is semi-deterministic in the sense that the mapping from X_i to T_i, where $i \in \{1, 2\}$, is random.

Note that if we assume the channel variables to be real-valued instead of finite, the Gaussian IC becomes a special case of this semi-deterministic IC with by taking $T_1 = g_{21}X_1 + Z_2$ and $T_2 = g_{12}X_2 + Z_1$.

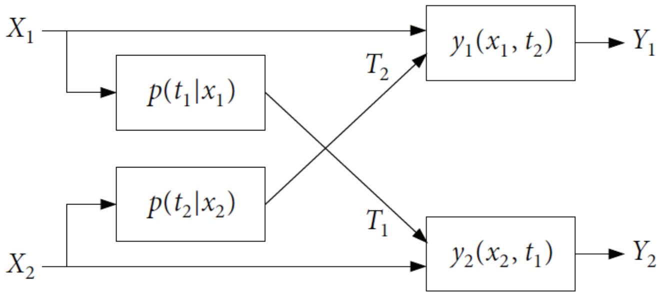

Fig. 1: Semi-deterministic interference channel.

Consider the following bound on the capacity region of the semi-deterministic IC [3].

Theorem 2 (Outer Bound) Let $\mathcal{C}_{SD}$ be the capacity region of the semi-deterministic IC. Let $\mathcal{R}_O$ be the region defined by the union of all sets of rate pairs $(R_1, R_2) \in \mathbb{R}^2_+$ satisfying:

$$R_1 \leq H(Y_1|X_2, Q) - H(T_2|X_2), \tag{10a}$$

$$R_2 \leq H(Y_2|X_1, Q) - H(T_1|X_1), \tag{10b}$$

$$R_1 + R_2 \leq H(Y_1|Q) + H(Y_2|U_2, X_1, Q) - H(T_1|X_1) - H(T_2|X_2), \tag{10c}$$

$$R_1 + R_2 \leq H(Y_1|U_1, X_2, Q) + H(Y_2|Q) - H(T_1|X_1) - H(T_2|X_2), \tag{10d}$$

$$R_1 + R_2 \leq H(Y_1|U_1, Q) + H(Y_2|U_2, Q) - H(T_1|X_1) - H(T2|X2), \tag{10e}$$

$$2R_1 + R_2 \leq H(Y_1|Q) + H(Y_1|U_1, X_2, Q) + H(Y_2|U_2, Q) - H(T_1|X_1) - 2H(T_2|X_2), \tag{10f}$$

$$R_1 + 2R_2 \leq H(Y_2|Q) + H(Y_2|U_2, X_1, Q) + H(Y_1|U_1, Q) - 2H(T_1|X_1) - H(T_2|X_2) \tag{10g}$$

where the union is taken over all joint distributions of the form $P_Q P_{X_1|Q} P_{X_2|Q} P_{U_1|X_1} P_{U_2|X_2}$, where $P_{U_j|X_j} = P_{T_j|X_j}$ for $j \in \{1, 2\}$. Then the the following inclusion holds:

$$\mathcal{C}_{SD} \subseteq \mathcal{R}_O. \tag{11}$$

Proof: Consider a sequence of $(2^{nR_1}, 2^{nR_2})$ codes with $\lim_{n\to\infty} P_e^{(n)} = 0$. Furthermore, let X_1^n, X_2^n, T_1^n, T_2^n, Y_1^n and Y_2^n denote the random variables resulting from encoding and transmitting the independent messages M_1 and M_2. Define the random variables U_1^n and U_2^n such that U_{ji} is jointly distributed with X_{ji} according to $P_{T_j|X_j}(u_{ji}|x_{ji})$, conditionally independent of T_{ji} given X_{ji} for $j \in \{1,2\}$ and $i \in \{1,\ldots,n\}$. By Fanos inequality,

$$\begin{aligned} nR_j &= H(M_j) \\ &\le I(M_j;Y_j^n) + n\epsilon_n \\ &\le I(X_j^n;Y_j^n) + n\epsilon_n \end{aligned} \tag{12}$$

Next, observe that

$$\begin{aligned} I(X_1^n;Y_1^n) &= H(Y_1^n) - H(Y_1^n|X_1^n) \\ &\overset{(a)}{=} H(Y_1^n) - H(T_2^n|X_1^n) \\ &\overset{(b)}{=} H(Y_1^n) - H(T_2^n) \\ &\le \sum_{i=1}^{n} H(Y_{1i}) - H(T_2^n) \end{aligned} \tag{13}$$

where (a) follows from the fact that Y_1^n and T_2^n are one-to-one given X_1^n, while (b) follows from the fact that T_2^n is independent of X_1^n. The second term $H(T_2^n)$, however, is not easily upper-bounded in a single-letter form. Now consider the following augmentation

$$\begin{aligned} I(X_1^n;Y_1^n) &\le I(X_1^n;Y_1^n,U_1^n,X_2^n) \\ &= I(X_1^n;U_1^n) + I(X_1^n;X_2^n|U_1^n) + I(X_1^n;Y_1^n|U_1^n,X_2^n) \\ &\overset{(a)}{=} H(U_1^n) - H(U_1^n|X_1^n) + H(Y_1^n|U_1^n,X_2^n) - H(Y_1^n|X_1^n,U_1^n,X_2^n) \\ &\overset{(b)}{=} H(T_1^n) - H(U_1^n|X_1^n) + H(Y_1^n|U_1^n,X_2^n) - H(T_2^n|X_2^n) \\ &\le H(T_1^n) - \sum_{i=1}^{n} \Big[H(U_{1i}|X_{1i}) + H(Y_{1i}|U_{1i},X_{2i}) - H(T_{2i}|X_{2i}) \Big] \end{aligned} \tag{14}$$

First, note that (a) follows from the fact by the choice of the joint distribution in Theorem 2, T_1^n and U_1^n are identically distributed and the fact that (X_1^n, X_2^n) are independent conditioned on U_1^n. To see that (b) holds consider the fact that the second and fourth terms in (b) represent the output of a memoryless channel given its input; thus, they readily single-letterize with equality. The third term in (b) can be upper-bounded in a single-letter form. The first term $H(T_1^n)$ will be used to cancel terms like $H(T_2^n)$ in (13). Similarly, we can write

$$\begin{aligned}
I(X_1^n;Y_1^n) &\leq I(X_1^n;Y_1^n,U_1^n) \\
&= I(X_1^n;U_1^n) + I(X_1^n;Y_1^n|U_1^n) \\
&= H(U_1^n) - H(U_1^n|X_1^n) + H(Y_1^n|U_1^n) - H(Y_1^n|X_1^n,U_1^n) \\
&= H(T_1^n) - H(U_1^n|X_1^n) + H(Y_1^n|U_1^n) - H(T_2^n) \\
&= H(T_1^n) - H(T_2^n) - \sum_{i=1}^{n} \Big[H(U_{1i}|X_{1i}) + H(Y_{1i}|U_{1i}) \Big] \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
I(X_1^n;Y_1^n) &\leq I(X_1^n;Y_1^n,X_2^n) \\
&= I(X_1^n;X_2^n) + I(X_1^n;Y_1^n|X_2^n) \\
&= H(Y_1^n|X_2^n) - H(Y_1^n|X_1^n,X_2^n) \\
&= H(Y_1^n|X_2^n) - H(T_2^n|X_2^n) \\
&= \sum_{i=1}^{n} \Big[H(Y_{1i}|X_{2i}) + H(T_{2i}|X_{2i}) \Big] \qquad (16)
\end{aligned}$$

By symmetry, similar bounds can be established for $I(X_2^n;Y_2^n)$, namely,

$$I(X_2^n;Y_2^n) \leq \sum_{i=1}^{n} H(Y_{2i}) - H(T_1^n), \qquad (17)$$

$$I(X_2^n;Y_2^n) \leq H(T_2^n) - \sum_{i=1}^{n} \Big[H(U_{2i}|X_{2i}) + H(Y_{2i}|U_{2i},X_{1i}) - H(T_{1i}|X_{1i}) \Big], \qquad (18)$$

$$I(X_2^n;Y_2^n) \leq H(T_2^n) - H(T_1^n) - \sum_{i=1}^{n}\Big[H(U_{2i}|X_{2i}) + H(Y_{2i}|U_{2i})\Big], \tag{19}$$

$$I(X_2^n;Y_2^n) \leq \sum_{i=1}^{n}\Big[H(Y_{2i}|X_{1i}) + H(T_{1i}|X_{1i})\Big]. \tag{20}$$

Finally, consider linear combinations of the inequalities in (13)-(20) where all the multi-letter terms, namely $H(T_1^n)$ and $H(T_2^n)$, are canceled. Combining them with the bounds in (12) and using a time-sharing variable $Q \sim \mathrm{U}\{1,\ldots,n\}$ completes the proof of the outer bound. ■

Having the result of Theorem 2, recall the Han-Kobayashi inner bound. By introducing the restriction that $P_{U_1,U_2|Q,X_1,X_2} = P_{T_1|X_1}P_{T_2|X_2}$, the HK region in (1) reduces to the one presented subsequently, which gives rise to the following corollary.

Corollary 1 (Han-Kobayashi Inner Bound for the Semi-Deterministic IC) Let $\mathcal{C}_{SD}$ be the capacity region of the semi-deterministic IC. Let $\mathcal{R}_I$ be the region defined by the union of all sets of rate pairs $(R_1,R_2) \in \mathbb{R}_+^2$ satisfying:

$$R_1 \leq H(Y_1|U_2,Q) - H(T_2|U_2,Q), \tag{21a}$$

$$R_2 \leq H(Y_2|U_1,Q) - H(T_1|U_1,Q), \tag{21b}$$

$$R_1 + R_2 \leq H(Y_1|Q) + H(Y_2|U_1,U_2,Q) - H(T_1|U_1,Q) - H(T_2|U_2,Q), \tag{21c}$$

$$R_1 + R_2 \leq H(Y_1|U_1,U_2,Q) + H(Y_2|Q) - H(T_1|U_1,Q) - H(T_2|U_2,Q), \tag{21d}$$

$$R_1 + R_2 \leq H(Y_1|U_1,Q) + H(Y_2|U_2,Q) - H(T_1|U_1,Q) - H(T_2|U_2,Q), \tag{21e}$$

$$2R_1 + R_2 \leq H(Y_1|Q) + H(Y_1|U_1,U_2,Q) + H(Y_2|U_2,Q) - H(T_1|U_1,Q) - 2H(T_2|U_2,Q), \tag{21f}$$

$$R_1 + 2R_2 \leq H(Y_2|Q) + H(Y_2|U_1,U_2,Q) + H(Y_1|U_1,Q) - 2H(T_1|U_1,Q) - H(T_2|U_2,Q), \tag{21g}$$

where the union is taken over all joint distributions of the form $P_Q P_{X_1|Q} P_{X_2|Q} P_{U_1|X_1} P_{U_2|X_2}$, where $P_{U_j|X_j} = P_{T_j|X_j}$ for $j \in \{1,2\}$. Then the the following inclusion holds:

$$\mathcal{R}_I \subseteq \mathcal{C}_{SD}. \tag{22}$$

The inner bound in (21) is obtained by substituting the joint distribution

$$P_Q P_{X_1|Q} \underbrace{P_{U_1|X_1}}_{=P_{T_1|X_1}} P_{X_2|Q} \underbrace{P_{U_2|X_2}}_{=P_{T_2|X_2}} P_{T_1|X_1} P_{T_2|X_2} \quad \mathbb{1}_{\{Y_1=y_1(T_1,X_1)\}} \quad \mathbb{1}_{\{Y_2=y_2(T_2,X_2)\}}, \tag{23}$$

with the one stated in Theorem 1.

For a fixed $(Q, X_1, X_2) \sim P_Q P_{X_1|Q} P_{X_2|Q}$, let $\mathcal{R}_O(Q, X_1, X_2)$ be the region defined by the set of inequalities in (10), and let $\mathcal{R}_I(Q, X_1, X_2)$ denote the closure of the region defined by the set of inequalities in (21).

Lemma 1 (Gap Between the Inner and Outer Bounds [3]) If $(R_1, R_2) \in \mathcal{R}_O(Q, X_1, X_2)$, then $\Big(R_1 - I(X_2; T_2|U_2, Q)\,,\, R_2 - I(X_1; T_1|U_1, Q)\Big) \in \mathcal{R}_I(Q, X_1, X_2)$.

The result of lemma 1 straightforwardly follows from the structure of the rate bounds in (10) and the fact that $H(Y_j|U_j, Q) \geq H(Y_j|X_j, Q)$, for $j \in \{1, 2\}$.

A. Half-Bit Theorem for the Gaussian IC

We show that the outer bound in Theorem 2, when specialized to the Gaussian IC, is achievable within half a bit per dimension. For the Gaussian IC, the auxiliary random variables in the outer bound can be expressed as

$$U_1 = g_{21} X_1 + Z_2' \tag{24a}$$

$$U_2 = g_{12} X_2 + Z_1', \tag{24b}$$

where Z_1' and Z_2' are $\mathcal{N}(0, 1)$, independent of each other and of (X_1, X_2, Z_1, Z_2). Substituting in the outer bound in Theorem 2, we obtain an outer bound $\mathcal{R}_O^G$ on the capacity region of the Gaussian IC that consists of all rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ such that

$$R_1 \leq C(S_1), \tag{25a}$$

$$R_2 \leq C(S_2), \tag{25b}$$

$$R_1 + R_2 \leq C\left(\frac{S_1}{1 + I_2}\right) + C(I_2 + S_2), \tag{25c}$$

$$R_1 + R_2 \leq C\left(\frac{S_2}{1 + I_1}\right) + C(I_1 + S_1), \tag{25d}$$

$$R_1 + R_2 \le C\left(\frac{S_1 + I_1 + I_1 I_2}{1 + I_2}\right) + C\left(\frac{S_2 + I_2 + I_1 I_2}{1 + I_1}\right), \tag{25e}$$

$$2R_1 + R_2 \le C\left(\frac{S_1}{1 + I_2}\right) + C(I_1 + S_1) C\left(\frac{S_2 + I_2 + I_1 I_2}{1 + I_1}\right), \tag{25f}$$

$$R_1 + 2R_2 \le C\left(\frac{S_2}{1 + I_1}\right) + C(I_2 + S_2) C\left(\frac{S_1 + I_1 + I_1 I_2}{1 + I_2}\right), \tag{25g}$$

where $C(x) = \frac{1}{2}\log(1 + x)$.

Now we show that $\mathcal{R}_O^G$ is achievable with half a bit.

Theorem 3 (Half-Bit Theorem [4]) For the Gaussian IC, if $(R_1, R_2) \in \mathcal{R}_O^G$, then $\left(R_1 - \frac{1}{2}, R_2 - \frac{1}{2}\right)$ is achievable.

Proof: To prove Theorem 3, consider Lemma 1 for the Gaussian IC with the auxiliary random variables in (24). Then, for $j \in \{1, 2\}$, consider

$$\begin{aligned} I(X_j; T_j | U_j, Q) &= h(T_j | U_j, Q) - h(T_j | U_j, X_j, Q) \\ &= h(T_j | U_j) - h(T_j | X_j) \\ &= h(T_j | U_j) - h(Z_j) \\ &\overset{(a)}{\le} h(T_j - U_j) - h(Z_j) \\ &= h(Z_j - Z_j') - h(Z_j) \\ &= \frac{1}{2} \end{aligned}$$

where (a) follows from the fact that conditioning reduces entropy. ■

III. DEGREE OF FREEDOM

Consider the symmetric Gaussian IC with $S_1 = S_2 = S$ and $I_1 = I_2 = I$. Note that S and I fully characterize the channel. Define the *symmetric capacity* of the channel as $\mathcal{C}_{sym} = \max\left\{R : (R, R) \in \mathcal{C}\right\}$ and the *normalized symmetric capacity* as

$$d_{sym} = \frac{\mathcal{C}_{sym}}{C(S)}.$$

We find the *symmetric degrees of freedom* (DoF) d^*_{sym}, which is the limit of d_{sym} as the SNR and INR approach infinity. Note that in taking the limit, we are considering a sequence of channels rather than any particular channel. This limit, however, sheds light on the optimal coding strategies under different regimes of high SNR/INR.

Specializing the outer bound $\mathcal{R}_O^G$ in (25) to the symmetric case yields

$$\begin{aligned}\mathcal{C}_{sym} &\leq \bar{\mathcal{C}}_{sym} \\ &= \min\left\{C(S), \frac{1}{2}C\left(\frac{S}{1+I}\right) + \frac{1}{2}C(S+I), C\left(\frac{S+I+I^2}{1+I}\right), \frac{2}{3}C\left(\frac{S}{1+I}\right) + \frac{1}{3}C(S+2I+I^2)\right\}.\end{aligned} \tag{26}$$

By the half-bit theorem,

$$\frac{\bar{\mathcal{C}}_{sym}}{C(S)} - \frac{1}{2} \leq d_{sym} \leq \frac{\bar{\mathcal{C}}_{sym}}{C(S)}. \tag{27}$$

Thus, the difference between the upper and lower bounds converges to zero as $S \to \infty$, and the normalized symmetric capacity converges to the degrees of freedom d^*_{sym}. This limit, however, depends on how I scales as $S \to \infty$. Since it is customary to measure SNR and INR in decibels (dBs), we consider the limit for a constant ratio between the logarithms of the INR and SNR

$$\alpha = \frac{\log I}{\log S}, \tag{28}$$

or equivalently, $I = S^\alpha$. Then, as $S \to \infty$, the normalized symmetric capacity d_{sym} converges to

$$\begin{aligned}d^*_{sym}(\alpha) &= \lim_{S\to\infty} \frac{\bar{\mathcal{C}}_{sym}\Big|_{I=S^\alpha}}{C(S)} \\ &= \min\left\{1, \max\left\{\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right\}, \max\left\{\alpha, 1-\alpha\right\}, \max\left\{\frac{2}{3}, \frac{2\alpha}{3}\right\} + \max\left\{\frac{1}{3}, \frac{2\alpha}{3}\right\} - \frac{2\alpha}{3}\right\}.\end{aligned}$$

Since the fourth bound inside the minimum is redundant, we have

$$d^*_{sym}(\alpha) = \min\left\{1, \max\left\{\frac{\alpha}{2}, 1-\frac{\alpha}{2}\right\}, \max\left\{\alpha, 1-\alpha\right\}\right\}. \tag{29}$$

The symmetric DoF as a function of α is plotted in Figure 2. Note the unexpected W (instead of V) shape of the DoF curve. When interference is negligible ($\alpha \leq 1/2$), the DoF is $1-\alpha$ and corresponds to the limit of the normalized rates achieved by treating interference as noise. For strong interference ($\alpha \geq 1$), the DoF is $\min\left\{1, \frac{\alpha}{2}\right\}$ and corresponds to simultaneous decoding. In particular, when interference is very strong ($\alpha \geq 2$), it does not impair the DoF. For moderate interference ($1\frac{1}{2} \leq \alpha \leq 1$), the DoF corresponds to the Han-Kobayashi rate splitting. However, the DoF first increases until $\alpha = \frac{2}{3}$ and then decreases to $\frac{1}{2}$ as α is increased to 1. Note that for $\alpha = \frac{1}{2}$ and $\alpha = 1$, time division is also optimal.

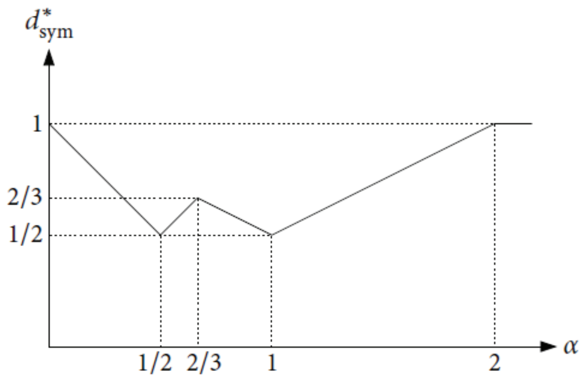

Fig. 2: Degrees of freedom for symmetric Gaussian IC versus $\alpha = \frac{\log I}{\log S}$.

Remark 3 In the above analysis, we scaled the channel gains under a fixed power constraint. Alternatively, we can fix the channel gains and scale the power P to infinity. It is not difficult to see that under this high power regime, $\lim_{P\to\infty} d^* = \frac{1}{2}$, regardless of the values of the channel gains. Thus time division is asymptotically optimal.

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