Multi-User Information Theory	December, 2013
Lecture 2	
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I. Strong and Very Strong Interference

In this lecture we continue to study the interference channel (IC). We address two cases of interference: the strong interference and the very strong interference.

A. Strong Interference

The IC with strong interference is defined as follows.

Definition 1 (Strong Interference) A discrete memoryless IC is said to have strong interference if

$$I(X_1;Y_1|X_2) \leq I(X_1;Y_2|X_2) \tag{1}$$

$$I(X_2;Y_2|X_1) \leq I(X_2;Y_1|X_1) \tag{2}$$

for all $p(x_1)p(x_2)$.

Theorem 1 (Capacity region of the IC with strong interference) The capacity region of the IC with strong interference, as defined in Def. 1, is the set of rate pairs (R_1, R_2) s.t.

$$R_1 \leq I(X_1;Y_1|X_2,Q) \tag{3}$$

$$R_2 \leq I(X_2;Y_2|X_1,Q) \tag{4}$$

$$R_1+R_2 \leq \min\{I(X_1,X_2;Y_1|Q), I(X_1,X_2;Y_2|Q)\} \tag{5}$$

for some pmf $p(q)p(x_1|q)p(x_2|q)p(y_1,y_2|x_1,x_2)$.

Proof: **Achievability:** For the achievability proof we present the simultaneous decoding method.

Simultaneous decoding: In this decoding method, both messages are decoded at the two decoders. Fix a joint distribution $p(q)p(x_1|q)p(x_2|q)p(y_1, y_2|x_1, x_2)$.

Code Design: Generate q^n i.i.d. using $P(q^n) = \Pi_{i=1}^n P(q_i)$ and inform it to the two encoders and decoders. Generate 2^{nR_1} codewords x_1^n i.i.d. using $P(x_1^n|q^n) = \Pi_{i=1}^n P(x_{1,i}|q_i)$ and 2^{nR_2} codewords x_2^n i.i.d. using $P(x_2^n|q^n) = \Pi_{i=1}^n P(x_{2,i}|q_i)$.

Encoding: Encode message m_1 using $x_1^n(m_1, q^n)$ and message m_2 using $x_2^n(m_2, q^n)$. Send x_1^n and x_2^n over the channel.

Decoding at Decoder 1: Decoder 1 looks for $\hat{m}_1$ s.t.

$$(x_1^n(\hat{m}_1, q^n), x_2^n(m_2, q^n), y_1^n, q^n) \in T_\epsilon^{(n)}(X_1, X_2, Y_1, Q). \tag{6}$$

If no such message, or more than one such message, was found, an error is declared.

Decoding at Decoder 2: Decoder 2 looks for $\hat{m}_2$ s.t.

$$(x_1^n(m_1, q^n), x_2^n(\hat{m}_2, q^n), y_2^n, q^n) \in T_\epsilon^{(n)}(X_1, X_2, Y_2, Q). \tag{7}$$

If no such message, or more than one such message, was found, an error is declared.

Error Analysis: We derive the following constraints from decoding at Decoder 1.

m_1 decoded correctly	m_2 decoded correctly	Constraint
✓	$\times$	No constraint needed
$\times$	✓	$R_1 \leq I(X_1; Y_1|X_2, Q)$
$\times$	$\times$	$R_1 + R_2 \leq I(X_1, X_2; Y_1|Q)$

Similarly, from decoding at Decoder 2 we obtain

m_1 decoded correctly	m_2 decoded correctly	Constraint
✓	$\times$	$R_2 \leq I(X_2; Y_2|X_1, Q)$
$\times$	✓	No constraint needed
$\times$	$\times$	$R_1 + R_2 \leq I(X_1, X_2; Y_2|Q)$

That completes the achievability.

Converse: Given an achievable rate-pair (R_1, R_2) we need to show that there exists a joint distribution of the form $p(q)p(x_1|q)p(x_2|q)p(y_1, y_2|x_1, x_2)$ such that the inequalities

Theorem 1 are satisfied. Since (R_1, R_2) is an achievable rate-pair, there exists a $(2^{nR_1}, 2^{nR_2}, n)$ code with an arbitrarily small error probability $P_e^{(n)}$. By Fano's inequality,

$$H(M_1|Y_1^n) \leq nR_1 P_e^{(n)} + H(P_e^{(n)}). \tag{8}$$

We set

$$R_1 P_e^{(n)} + \frac{1}{n} H(P_e^{(n)}) \triangleq \epsilon_n, \tag{9}$$

where $\epsilon_n \to 0$ as $P_e^{(n)} \to 0$. Hence,

$$H(M_1|Y_1^n, M_2) \leq H(M_1|Y_1^n) \leq n\epsilon_n. \tag{10}$$

For R_1 we have the following:

$$\begin{aligned}
nR_1 &= H(M_1) && (11)\\
&= H(M_1|M_2) && (12)\\
&\overset{(a)}{\leq} I(M_1; Y_1^n|M_2) + 2n\epsilon_n && (13)\\
&\overset{(b)}{\leq} I(X_1^n; Y_1^n|X_2^n) + 2n\epsilon_n && (14)\\
&\leq \sum_{i=1}^{n} I(X_{1,i}; Y_{1,i}|X_{2,i}) + 2n\epsilon_n && (15)\\
&= nI(X_1; Y_1|X_2, Q) + n\epsilon_n && (16)
\end{aligned}$$

where (a) follows from Fano's inequality and (b) follows from encoding relations. By symmetry, the same can be derived for R_2. For the sum-rate we have

$$\begin{aligned}
n(R_1 + R_2) &= H(M_1, M_2) && (17)\\
&= H(M_2) + H(M_1|M_2) && (18)\\
&\overset{(a)}{\leq} I(M_2; Y_2^n) + I(M_1; Y_1^n|M_2) + 2n\epsilon_n && (19)\\
&\overset{(b)}{\leq} I(X_2^n; Y_2^n) + I(X_1^n; Y_1^n|X_2^n) + 2n\epsilon_n && (20)\\
&\overset{(c)}{\leq} I(X_2^n; Y_2^n) + I(X_1^n; Y_2^n|X_2^n) + 2n\epsilon_n && (21)\\
&= I(X_1^n, X_2^n; Y_2^n) + 2n\epsilon_n && (22)\\
&\leq \sum_{i=1}^{n} I(X_{1,i}, X_{2,i}; Y_{2,i}) + 2n\epsilon_n && (23)
\end{aligned}$$

$$= \quad nI(X_1, X_2; Y_2|Q) + n\epsilon_n \tag{24}$$

where (a) follows from Fano's inequality, (b) follows from encoding relations, and (c) follows from the following lemma.

Lemma 1 For a DM-IC $p(y_1, y_2|x_1, x_2)$ with strong interference, the inequality

$$I(X_1^n; Y_1^n|X_2^n) \leq I(X_1^n; Y_2^n|X_2^n) \tag{25}$$

holds for all $(X_1^n, X_2^n) \sim p(x_1^n)p(x_2^n)$ and all $n \geq 1$.

The proof is given in Appendix A. The second bound for the sum-rate can be obtained similarly to the first bound. This completes the proof. ■

B. Very Strong Interference

The interference channel with very strong interference is defined as follows.

Definition 2 (Very Strong Interference) A discrete memoryless interference channel is said to have very strong interference if

$$I(X_1; Y_1|X_2) \quad \leq \quad I(X_1; Y_2) \tag{26}$$

$$I(X_2; Y_2|X_1) \quad \leq \quad I(X_2; Y_1) \tag{27}$$

for all $p(x_1)p(x_2)$.

From definitions 1 and 2 we can see that

$$\text{very strong interference} \quad \Rightarrow \quad \text{strong interference}$$

$$\text{very strong interference} \quad \not\Leftarrow \quad \text{strong interference}$$

Theorem 2 (Capacity region of the IC with very strong interference) The capacity region of the IC with very strong interference, as defined in Def. 1, is the set of rate pairs (R_1, R_2) s.t.

$$R_1 \quad \leq \quad I(X_1; Y_1|X_2, Q) \tag{28}$$

$$R_2 \quad \leq \quad I(X_2; Y_2|X_1, Q) \tag{29}$$

for some pmf $p(q)p(x_1|q)p(x_2|q)p(y_1, y_2|x_1, x_2)$.

Sketch of Proof: **Achievability:** We use simultaneous decoding as in the strong interference case. Under the constraints in Def. 2, the region in Theorem 1 reduces to the one in Theorem 2.

Converse: The two inequalities in (2) are identical to the ones in the strong interference case. ■

II. Gaussian Interference Channel

We now address the Gaussian IC depicted in Fig. 1.

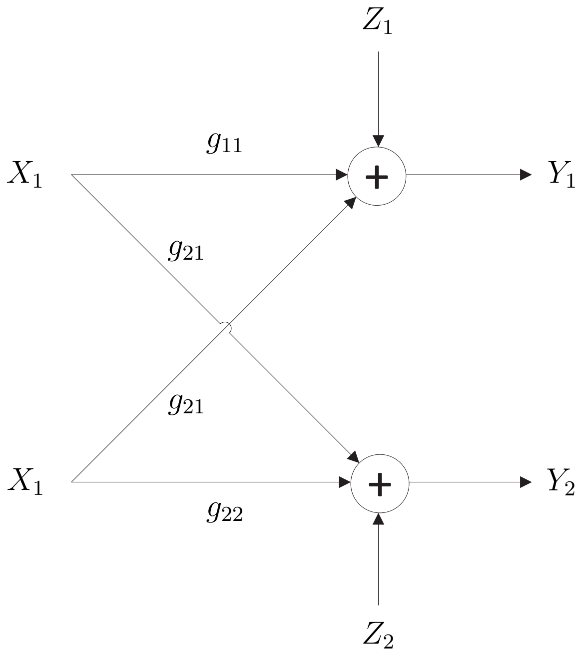

Fig. 1. The Gaussian IC. The input to Decoder 1 is the sum of the output of Encoder 1 with fading factor g_{11}, the output of Encoder 2 with fading factor g_{21}, and a Gaussian noise Z_1. The input to Decoder 2 is the sum of the output of Encoder 2 with fading factor g_{22}, the output of Encoder 1 with fading factor g_{12}, and a Gaussian noise Z_2.

The Gaussian channel is quite popular since it provides a simple model for several real-world communication channels, such as wireless and digital subscriber line (DSL) channels, or in this case, a simple wireless interference channel or a DSL cable bundle. The channel outputs corresponding to the inputs X_1 and X_2 are

$$Y_1 = g_{11}X_1 + g_{21}X_2 + Z_1 \tag{30}$$

$$Y_2 = g_{12}X_1 + g_{22}X_2 + Z_2 \tag{31}$$

where $g_{jk}, j, k = 1, 2$, is the channel gain from sender j to receiver k, and $Z_1 \sim N(0, \sigma_0^2)$ and $Z_2 \sim N(0, \sigma_0^2)$ are noise components. Assume average power constraint P on each of X_1 and X_2. We assume without loss of generality that $\sigma_0^2 = 1$ and define the received SNRs as $S_1 = g_{11}^2 P$ and $S_2 = g_{22}^2 P$ and the received interference-to-noise ratios (INRs) as $I_1 = g_{21}^2 P$ and $I_2 = g_{12}^2 P$.

We present several inner bounds for the gaussian IC:

1) **Time division with power control.**

   If the senders are allowed to use higher powers during their transmission periods (without violating the power constraint over the entire transmission block), strictly higher rates can be achieved. We divide the transmission block into two subblocks, one of length αn and the other of length $\overline{\alpha} n$ (assuming αn is an integer). During the first subblock, sender 1 transmits using Gaussian random codes at average power P/α (rather than P) and sender 2 does not transmit. During the second subblock, sender 2 transmits at average power $P/\overline{\alpha}$ and sender 1 does not transmit. Note that the average power constraints are satis4ed. This scheme achieves the set of rate pairs (R_1, R_2) such that

$$R_1 \leq \alpha C(S_1/\alpha) \tag{32}$$

$$R_2 \leq \overline{\alpha} C(S_2/\overline{\alpha}) \tag{33}$$

   for some $\alpha \in [0, 1]$ where $C(x) = \frac{1}{2}\log(1 + x)$

2) **Treating interference as noise.**

   If each encoder treats the other encoder's transmission as noise, we obtain the inner bound consisting of all rate pairs (R_1, R_2) such that

$$R_1 \leq I(X_1; Y_1|Q) \tag{34}$$

$$R_2 \leq I(X_2; Y_2|Q) \tag{35}$$

   for some pmf $p(q)p(x_1|q)p(x_2|q)p(y_1, y_2|x_1, x_2)$. By using our notations for the gaussian interference channel and setting $Q = \varnothing$, we obtain

$$R_1 \leq C(S_1/(1 + I_1)) \tag{36}$$

$$R_2 \leq C(S_2/(1+I_2)) \tag{37}$$

3) **Simultaneous decoding.**

If we use our notations for the gaussian interference channel and set $Q = \varnothing$ in Theorem 1, we obtain

$$R_1 \leq C(S_1) \tag{38}$$

$$R_2 \leq C(S_2) \tag{39}$$

$$R_1 + R_2 \leq \min\{C(S_1+I_1), C(S_2+I_2)\} \tag{40}$$

Let us compare the performances of the inner bounds for different interferences using Fig. 2. Without loss of generality, we set $S = S_1 = S_2 = 1$ and $I = I_1 = I_2$.

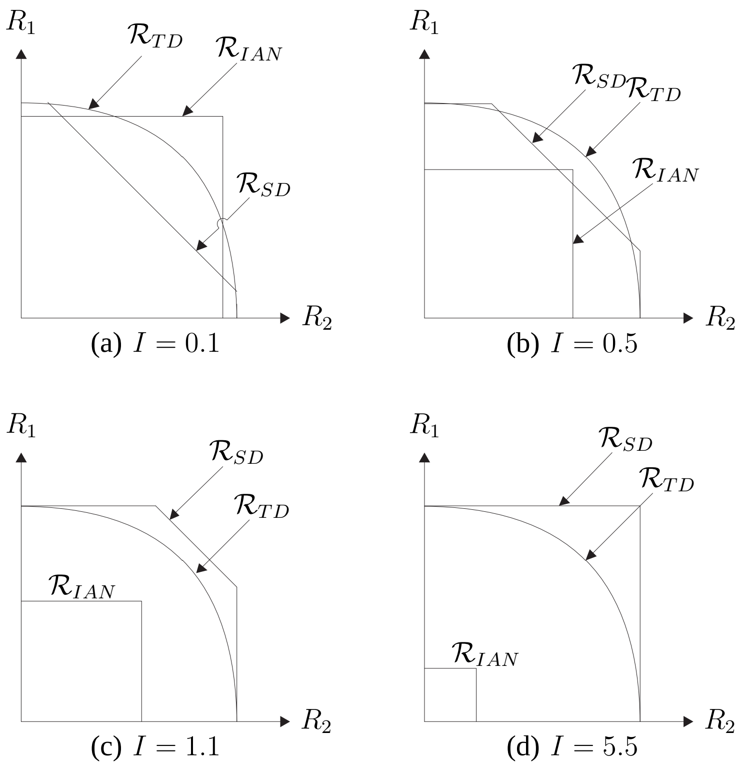

Fig. 2. Comparing the performance of the inner bounds. TD - Time Division, IAN - Interference as Noise, SD - Simultaneous Decoding.

When interference is weak (Fig. 2a), treating interference as noise can outperform time division and simultaneous decoding. As interference becomes stronger (Fig. 2b), simultaneous decoding and time division begin to outperform treating interference as noise. As interference becomes even stronger, simultaneous decoding outperforms the other two coding schemes (Fig. 2c,d), ultimately achieving the interference-free rate region consisting of all rate pairs (R_1, R_2) such that $R_1 < C_1$ and $R_2 < C_2$ (Fig. 2d).

APPENDIX A
PROOF OF LEMMA 1

To prove the lemma we first note that the hypothesis implies that $I(X_1; Y_1|X_2, U) \leq I(X_1; Y_2|X_2, U)$, where $U - (X_1, X_2) - (Y_1, Y_2)$ and $X_1 - U - X_2$ form Markov chains. Using this inequality we derive the following

$$\begin{aligned}
&I(X_1^n; Y_2^n|X_2^n) - I(X_1^n; Y_1^n|X_2^n) \\
&\quad\stackrel{(a)}{=} I(X_1^n; Y_2^{n-1}|X_2^n) + I(X_1^n; Y_{2,n}|X_2^n, Y_2^{n-1}) - I(X_1^n; Y_{1,n}|X_2^n) - I(X_1^n; Y_1^{n-1}|X_2^n, Y_{1,n}) \\
&\quad\stackrel{(b)}{=} I(X_1^n, Y_{1,n}; Y_2^{n-1}|X_2^n) + I(X_1^n; Y_{2,n}|X_2^n, Y_2^{n-1}) - I(X_1^n, Y_2^{n-1}; Y_{1,n}|X_2^n) \\
&\qquad - I(X_1^n; Y_1^{n-1}|X_2^n, Y_{1,n}) \\
&\quad\stackrel{(c)}{=} I(Y_{1,n}; Y_2^{n-1}|X_2^n) + I(X_1^{n-1}; Y_2^{n-1}|X_2^n, Y_{1,n}) + I(X_{1,n}; Y_2^{n-1}|X_2^n, Y_1^n, X_1^{n-1}) \\
&\qquad + I(X_{1,n}; Y_{2,n}|X_2^n, Y_2^{n-1}) + I(X_1^{n-1}; Y_{2,n}|X_2^n, Y_2^{n-1}, X_{1,n}) - I(Y_2^{n-1}; Y_{1,n}|X_2^n) \\
&\qquad - I(X_{1,n}; Y_{1,n}|X_2^n, Y_2^{n-1}) - I(X_1^{n-1}; Y_{1,n}|X_2^n, Y_2^{n-1}, X_{1,n}) \\
&\qquad - I(X_1^{n-1}; Y_1^{n-1}|X_2^n, Y_{1,n}) - I(X_{1,n}; Y_1^{n-1}|X_2^n, Y_{1,n}, X_1^{n-1}) && (41) \\
&\quad\stackrel{(d)}{=} I(X_{1,n}; Y_{2,n}|X_2^n, Y_2^{n-1}) - I(X_{1,n}; Y_{1,n}|X_2^n, Y_2^{n-1}) \\
&\qquad + I(X_1^{n-1}; Y_2^{n-1}|X_2^n, Y_{1,n}) - I(X_1^{n-1}; Y_1^{n-1}|X_2^n, Y_{1,n}) && (42)
\end{aligned}$$

where (a), (b), and (c) follow from the chain rule, and step (d) follows from the memorylessness of the channel (the 3rd, 5th, 8th, and 10th terms in (41) are null). If we set $U_n = (X_2^{n-1}, Y_2^{n-1})$, we see that the following Markov chains hold

$$\begin{array}{ccccc}
(X_2^{n-1}, Y_2^{n-1}) & - & (X_{1,n}, X_{2,n}) & - & (Y_{1,n}, Y_{2,n}), \\
X_{1,n} & - & (X_2^{n-1}, Y_2^{n-1}) & - & X_{2,n}
\end{array} \qquad (43)$$

The sum of the first two terms in (42) is greater or equal to zero. By induction, the 3rd and 4th terms in (42) are also greater or equal to zero since the following Markov chains hold

$$\begin{array}{ccccc} (X_{2,n}, Y_{1,n}) & - & (X_1^{n-1}, X_2^{n-1}) & - & (Y_1^{n-1}, Y_2^{n-1}), \\ X_1^{n-1} & - & (X_{2,n}, Y_{1,n}) & - & X_2^{n-1} \end{array} \tag{44}$$

and thus

$$I(X_1^n; Y_2^n | X_2^n) - I(X_1^n; Y_1^n | X_2^n) \geq 0 \tag{45}$$

which concludes the proof. ■

References

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