Lecture 8

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I. Causal state information of the encoder

The system illustrated in Figure 1 shows a channel that has a state which distributes according to $P(s)$. The state S is known causally to the decoder, meaning, if the encoder is currently looking at state i, the channel states known to the encoder are $\{S_i, S_{i-1}, \ldots, S_0\}$ or S^i.

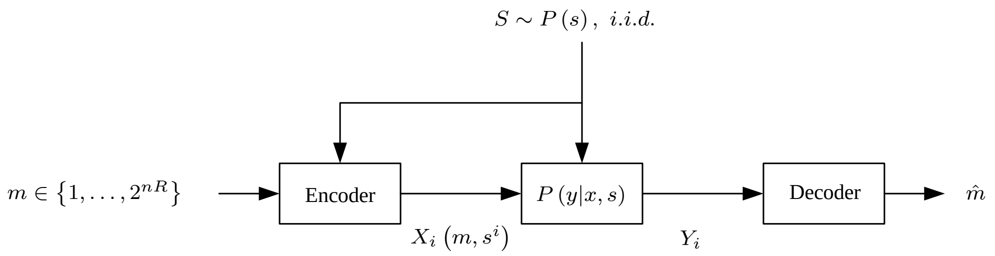

Fig. 1. System illustration - causal information is known to the encoder

Theorem 1 (Channel capacity when state information is known at the encoder [1])

$$C = \max_{P(u),\ x=f(u,s)} I(U;Y) \tag{1}$$

$$P(u,s,x,y) = P(s)\,P(u)\,P\underbrace{(x|u,s)}_{x=f(u,s)}P(y|x,s) \tag{2}$$

Where $P(s)$ and $P(y|x,s)$ are given by the communication problem.

Proof:

$$\begin{aligned} nR &= H(M) \\ &= H(M) + H(M|Y^n) - H(M|Y^n) \\ &= I(M;Y^n) + H(M|Y^n) \\ &\overset{(a)}{\leq} \sum_{i=1}^{n} I\left(M;Y_i,Y^{i-1}\right) + n\varepsilon_n \\ &= \sum_{i=1}^{n} \left[H\left(Y_i|Y^{i-1}\right) - H\left(Y_i|Y^{i-1},M\right)\right] + n\varepsilon_n \end{aligned} \tag{3}$$

$$
\begin{aligned}
&\overset{(b)}{\leq} \sum_{i=1}^{n} \left[H\left(Y_i\right) - H\left(Y_i|Y^{i-1}, M\right)\right] + n\varepsilon_n \\
&= \sum_{i=1}^{n} I\left(Y^{i-1}, M; Y_i\right) + n\varepsilon_n \\
&\overset{(c)}{\leq} \sum_{i=1}^{n} I\left(\underbrace{Y^{i-1}, M, S^{i-1}}_{U_i}; Y_i\right) + n\varepsilon_n \\
&= \sum_{i=1}^{n} I\left(U_i; Y_i\right) + n\varepsilon_n
\end{aligned} \tag{4}
$$

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Where:

(a) Fano's inequality and memoryless channel.

(b) $H\left(X\right) \geq H\left(X|Y\right)$.

(c) $I\left(X; Y, Z\right) \geq I\left(X; Y\right)$.

Some remarks about (c):

$$X_i = f\left(M, S^i\right) = \tilde{f}\left(M, S^i, Y^{i-1}\right)$$

Where $\tilde{f}$ is a dummy function, since $\tilde{f}$ doesn't really depends on Y^{i-1}.

$$
\begin{aligned}
R &\overset{(a)}{\leq} \frac{1}{n}\sum_{i=1}^{n} I\left(U_i; Y_i\right) && (5) \\
&= \frac{1}{n}\sum_{i=1}^{n} I\left(U_q; Y_q | q = i\right) \\
&\overset{(b)}{=} I\left(U_Q; Y_Q | Q\right) \\
&\leq I\left(U_Q, Q; Y_Q\right) \\
&\overset{(c)}{=} I\left(U; Y\right) && (6)
\end{aligned}
$$

Where:

(a) Fano's inequality.

(b) $P\left(q\right) = \frac{1}{n}$, $Q \sim U\left\{1, \ldots, n\right\}$.

(c) $U \triangleq U_Q, Q,\ Y = Y_Q$.

Does the equation $x_i = f\left(U_i, S_i\right)$ holds?

$$
\begin{aligned}
X_i &= f\left(U_i, S_i\right) \\
&= f\left(M, S^{i-1}, Y^{i-1}, S_i\right)
\end{aligned}
$$

$$= \quad f\left(M, S^i, Y^{i-1}\right)$$

Yes, it holds.

Does the relation $P\left(u_i, s_i\right) = P\left(u_i\right) P\left(s_i\right)$ holds?

$$\begin{aligned} P\left(m, s^{i-1}, y^{i-1}, s_i\right) &\stackrel{(a)}{=} P\left(m\right) P\left(s^{i-1}\right) P\left(y^{i-1} | m, s^{i-1}\right) P\left(s_i\right) \\ &= P\left(\underbrace{m, s^{i-1}, y^{i-1}}_{u_i}\right) P\left(s_i\right) \end{aligned} \tag{7}$$

(a) Using chain rule, M,S are independent, S is i.i.d.. The following Markov chain holds: $Y_i - \left(Y_i, S_i\right) - \left(M, S^{i-1}, Y^{i-1}\right)$

II. Achievability of the capacity

We would like to check if

$$C = \max_{P(u), x=f(U,S)} I(U;Y) \tag{8}$$

is achievable.

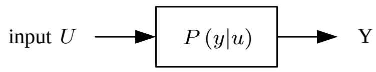

Fig. 2. New channel

Assuming U^n is i.i.d, is the new channel is memoryless?

Yes.

<u>Explenation:</u>

$$P\left(y_i|u_i, y^{i-1}, u^{i-1}\right) = P(y_i|u_i) \tag{9}$$

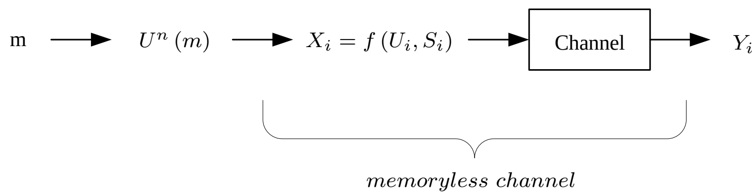

Fig. 3. Memoryless channel

- Fix $P(u)$, $x = f(U,S)$.
- Code design: Generate $2^{n(I(U;Y)-\varepsilon)} = 2^{nR}$ codewords using $P(u)$ i.i.d.
- Encoder: $x_i = f(U_i(m), S_i)$
- Decoder: Find $U^n(\hat{m})$ that is jointly typical with Y^n. If such an $\hat{m}$ does not exists, the decoder declares on error.

<u>Analysis of error:</u>

Assume we send m=1

E1: $\left(U^n\left(1\right),Y^n\right)\in A_n^{(\varepsilon)}\left(U,Y\right)$

E2: There exists a $j\neq 1$ such that $\left(U^n\left(j\right),Y^n\right)\in A_n^{(\varepsilon)}\left(U,Y\right)$

$$P\left(E_1\right)\stackrel{L.L.N}{\rightarrow}0 \tag{10}$$

$$P\left(E_{2,j}\right)=2^{-n(I(U;Y))} \tag{11}$$

$$\begin{aligned} P\left(\bigcup_{j=1}^{2^{nR}}E_{2,j}\right) &\leq \sum_{j=2}^{2^{nR}}2^{-nI(U;Y)} \qquad (12)\\ &= 2^{n(I(U;Y)-\varepsilon)}2^{-nI(U;Y)} \end{aligned}$$

III. STATE IS KNOWN TO ENCODER AND DECODER

In the system illustrated in Figure 4, the causal channel state S^i is known both to the encoder and the decoder.

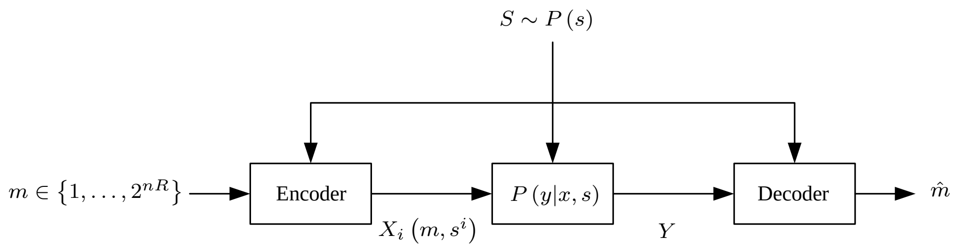

Fig. 4. System illustration - Channel state is known both to the encoder and the decoder

Theorem 2 (Channel capacity for a system in which the channel state known for the encoder and the decoder)

$$C = \max_{P(x|s)} I(X;Y|S) \tag{13}$$

Show that:

$$\begin{aligned} C &= \max_{P(u), x=f(U,S)} I(U;Y,S) \\ &= \max_{P(x|s)} I(X;Y|S) \end{aligned} \tag{14}$$

Proof:

$$\begin{aligned} C &= \max_{P(u), x=f(U,S)} I(U;Y,S) & (15) \\ &= \max_{P(u), x=f(U,S)} \left[I(U;S) + I(U;Y|S)\right] \\ &= \max_{P(u), x=f(U,S)} I(U;Y|S) \\ &= \max_{P(u), x=f(U,S)} \left[H(Y|S) - H(Y|S,U)\right] \\ &= \max_{P(u), x=f(U,S)} \left[H(Y|S) - H(Y|S,X)\right] \\ &= \max_{P(u), x=f(U,S)} I(Y;X|S) & (16) \end{aligned}$$

Since $P(x|s)$ can be represented as a function of U,S where, U and S are independent, we can now write:

$$C = \max_{P(x|s)} I(Y;X|S) \tag{17}$$

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Achievability proof can also be found at [2].

IV. Non-causal state information at the decoder

State sequence S is known only at the decoder.

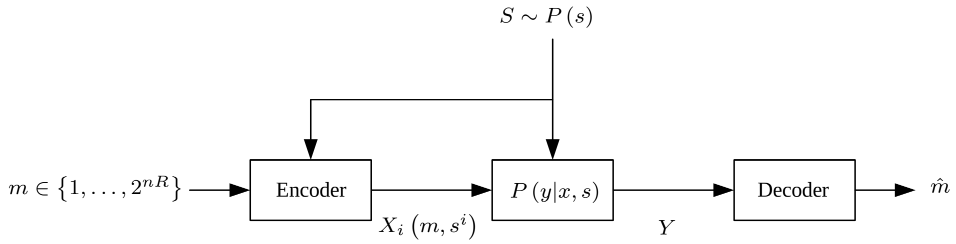

Fig. 5. System illustration

$$m \in \left\{1, 2, \ldots, 2^{nR}\right\} \tag{18}$$

- Encoder: $x_i = f(m, S^n),\ f : M \times S^n \longmapsto X^n$
- Decoder: $Y^n \longmapsto \hat{M}$
- Probability of error: $P_e^{(n)} = P\left(M \neq \hat{M}\right)$
- R achievable rate, if there exists a sequence of codes $\left(2^{nR}, n\right)$ such that $P_e^{(n)} \to 0$
- Capacity is the suprimum

Example 1 [3]

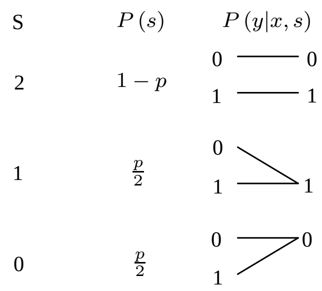

Fig. 6. Memory with defect example

1) If the state is known at the encoder and decoder, at what rate one can write?

   Answer: $1 - P$
2) If the state is not known at all, what one can write?

   Answer: $1 - H\left(\frac{p}{2}\right)$, see Figure 7
3) If the state is known only at the encoder (non-casual), at what rate one can write?

   Answer: $1 - P$, see Equation 22

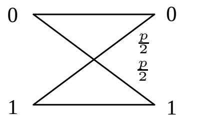

Fig. 7. Channel with error

Theorem 3 (Gelfand - Pinsker [4])

$$C_{non\ causal} \quad = \quad \max_{P(u,s),x=f(U,S),P(y|x,s)} I\left(U;Y\right) - I\left(U;S\right) \tag{19}$$

One result is:

$$\begin{aligned} I\left(U;Y\right) - I\left(U;S\right) &= H\left(U\right) - H\left(H|Y\right) - H\left(U\right) + H\left(U|S\right) \\ &\stackrel{(a)}{=} H\left(U|S\right) - H\left(U|Y\right) \end{aligned} \tag{20}$$

(a) U and S are dependent

Reminder for the causal case:

$$C_{causal} = \max_{P(u),x=f(U,S)} I\left(U;Y\right) - I\left(U;S\right) \tag{21}$$

Solving the memory example

Note that:

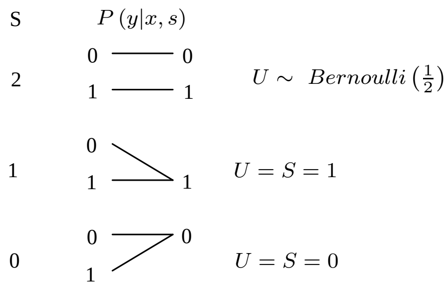

Fig. 8. Memory with defect example

$$\begin{aligned} C &= H\left(U|S\right) - H\left(U|Y\right) \\ &\stackrel{(a)}{=} H\left(U|S\right) \\ &\stackrel{(b)}{=} 1 \cdot \left(1 - P\right) \end{aligned} \tag{22}$$

(a) $H\left(U|Y\right) = 0$

(b) $H\left(U|S=2\right) = 1,\ P\left(S=2\right) = 1 - P$

Min-Max relation

$$\max_a \min_b f(a,b) \leq \min_b \max_a f(a,b) \tag{23}$$

Proof:

$$\min_{b'} f(a,b') \leq f(a,b) \, \forall a,b \tag{24}$$

$$\max_a \min_{b'} f(a,b') \leq \max_a f(a,b) \, \forall b \tag{25}$$

In particular it is true for

$$b^* = \arg\min_b \max_a f(a,b) \tag{26}$$

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[1] C. E. Shannon, “Channels with side information at the transmitter,” *IBM Journal of Research and Development*, vol. 2, no. 4, p. 289, 1958.

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[3] K. A. Vasil’evich and T. B. Solomonovich, “Coding in a memory with defective cells,” *Probl. Peredachi Inf.*, vol. 10, no. 2, pp. 52–60, 1974.

[4] S. I. Gelfand and M. S. Pinsker, “Coding for channel with random parameters,” *Probl. Control Inf. Theory*, vol. 9, no. 1, pp. 19–31, 1980.