Multi-User Information Theory October 26th, 2009

Lecture 2

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I. NOTATION

- $H_b(p) = -plogp - (1-p)log(1-p)$

II. MARKOV CHAINS

A. Markov Process

Definition 1 A discrete stochastic process $\{X_i\}_{i\geq 1}$ is said to be *Markov Process* if

$$P(X_{n+1}|X^n) = P(X_{n+1}|X_n), \quad \forall n \tag{1}$$

In this case, the joint probability mass function of the random variables can be written as

$$p(x_1, x_2, ..., x_n) = p(x_1)p(x_2|x_1)p(x_3|x_2)\cdots p(x_n|x_{n-1}) \tag{2}$$

Definition 2 The Markov Process is said to be *time invariant* if the conditional probability $p(x_{n+1}|x_n)$ does not depend on n, that is

$$P(X_{n+1} = i|X^n = j) = P(X_2 = i|X_1 = j) = p_{ij}, \quad \forall n \ \ and \ \ \forall i, j \in \mathcal{X}. \tag{3}$$

B. Markov Chain

Definition 3 A Markov Chain is finite *Markov Process*. If $\{X_i\}$ is a Markov Chain, X_n is called state at time *n*. A time invariant(stationary) Markov Chain is characterized by a *transition matrix*,

$$\Pi = P_{i,j}, \quad i, j \in \{1, 2, ..., m\}. \tag{4}$$

The initial state probability is $P_0(i) = P_r(X_0 = 1)$.

Let $P_t = [P_r(X_t = 1), P_r(X_t = 2), ..., P_r(X_t = m)]$ be a probability vector, and Π the transition matrix of the stationary Markov Chain, thus we can write:

$$P_t = P_{t-1} \cdot \Pi \tag{5}$$

$$P_t = P_0 \cdot \Pi^t. \tag{6}$$

Also note that the probability of X_t to be equal to j is defined by:

$$P(X_t = j) = \sum_{i=1}^{m} P(X_t = j, X_{t-1} = i) = \sum_{i=1}^{m} P(X_{t-1} = i)P(X_t = j|X_{t-1} = i) = \sum_{i=1}^{m} P_{t-1}(i)P_{i,j} \tag{7}$$

Properties of Markov Chain:

1) *Irreducible* There exists a positive probability of getting to any state from any state. That is, all the states are connected. This can be demonstrated by the following figure:

TBD - ADD FIGURE FROM CLASS LECTURE @@@.

2) *Aperiodic* The largest common factor (GCD) of all possible loops in state is 1, i.e. returns to state i can occur at irregular times. This can be demonstrated by the following figure:

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Only when the Markov Chain is irreducible, the aperiodic property is defined. Also, when the Markov Chain is irreducible, then if of the states is aperiodic, then all the states are aperiodic.

Theorem 1 If a finite-state Markov Chain is aperiodic and irreducible there exists an unique stationary distribution. That is $P_{X_n} = P_{X_{n+1}}$.

Definition 4 μ is *stationary distribution* if exists μ such that $\mu\Pi = \mu$. That is for each initial state we start from, we will get μ after finite number of steps.

Example 1 ([1]) Consider a two-state Markov chain that is irreducible and aperiodic with a probability transition matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix},$$

as shown in the following figure:

TBD - ADD a figure here

Let the stationary distribution be represented by the vector $\mu = [\mu_1 \quad \mu_2]$, such that each component of the vector is the stationary probability of states 1 and 2. We can find the stationary probability by solving $\mu P = \mu$. From the fact that $\mu_1 + \mu_2 = 1$ we get:

$$\mu_1 = \frac{\beta}{\alpha+\beta}, \tag{8}$$

$$\mu_1 = \frac{\alpha}{\alpha+\beta}. \tag{9}$$

Recall that the *entropy rate* of a stationary process is,

$$H'(\mathcal{X}) = \lim_{n\to\infty} H(X_n|X^{n-1}). \tag{10}$$

Now we compute $H(X_n|X^{n-1})$ for stationary Markov process and by so we will get the entropy rate of stationary Markov process,

$$H(X_n|X^{n-1}) \stackrel{(a)}{=} H(X_n|X_{n-1}) \tag{11}$$

$$\begin{aligned}
&\overset{(b)}{=} \quad H(X_1|X_0) \\
&= \quad \sum_{i=1}^{m} p(i)H(X_1|X_0 = i) \\
&= \quad -\sum_{i=1}^{m}\sum_{j=1}^{m} p(i)P_{ij}log(P_{ij})
\end{aligned}$$

where

(a) follows from Markovity.

(b) follows from stationary process.

The entropy rate of example 1 is,

$$H(X) = \frac{\beta}{\alpha+\beta}H_b(\alpha) + \frac{\alpha}{\alpha+\beta}H_b(\beta). \tag{12}$$

TBD-consider adding material on HMM

III. Gambling

In this section we will show that there is strong duality between the growth rate of investment in a horse race and the entropy rate of the horse race.

A. The horse race

In order to describe the horse race let us define the following:

- m - Number of horses.
- X_i - Random variable that tells us which horse wins at time i. $X_i = \{1, 2, ..., m\}$.
- P_X - The pmf of the winning horse.
- $o(X)$ - The amount of money we get, for each dollar we put, if horse X wins.
- $b(X)$ - Betting strategy on horse X. $b(X) \geq 0 \;\; \forall X$ and $\sum_x b(X) = 1$.
- S - Money.

- $S_0 = 1$ - Money the gambler has in time 0.
- $S_1 = b(X_1) \cdot o(X_1) \cdot S_0$ - Money the gambler has in time 1.
- $S_2 = b(X_2) \cdot o(X_2) \cdot S_1$ - Money the gambler has in time 2.
- $S_n = b(X_n) \cdot o(X_n) \cdot S_{n-1} = \Pi_{i=1}^{n} b(X_i) \cdot o(X_i)$ - Money the gambler has in time n.

In the horse race we assume that:

- The gambler distributes all of his wealth across the horses.
- The winning probability is time invariant.
- The wealth at the end of the race is a random variable.

- The gambler wishes to maximize the value of this random variable.

The objective goal will be to find

$$b = \arg\max_{b(\cdot)} \mathbb{E}[logS_n] \tag{13}$$

$$\begin{aligned}
\max_{b(x)} \mathbb{E}[logS_n] &= \max_{b(x)} \mathbb{E}[\log \Pi_{i=1}^{n} b(x_i) + \log \Pi_{i=1}^{n} O(x_i)] \qquad (14)\\
&= \max_{b(x)} \sum_{i=1}^{n} (\mathbb{E}[\log b(x_i)] + \mathbb{E}[\log O(x_i)])\\
&\overset{(a)}{=} \max_{b(x)} \sum_{i=1}^{n} (\mathbb{E}[\log b(x_i)])\\
&\overset{(b)}{=} n \cdot \arg\max_{b(x)} \mathbb{E}[\log b(x_i)]\\
&= n \cdot \max_{b(x)} \sum_{x} p(x) \log b(x)\\
&= n \cdot \max_{b(x)} \sum_{x} (p(x) \log \frac{b(x)}{p(x)} + p(x) \log p(x))\\
&\overset{(c)}{=} n \cdot \max_{b(x)} [-D(p(x)||b(x)) - H(X)]\\
&\overset{(d)}{\leq} -n \cdot H(X)
\end{aligned}$$

Thus,

$$b(x) = p(x) \tag{15}$$

where

(a) follows from the face that $O(x_i)$ does not depend on $b(x_i)$.

(b) maximizing over the same argument.

(c) follows from definition of divergence and entropy.

(d) follows from the fact that D is non-negative.

B. Gambling with causal side information [2]

Assume there are m racing horses where X_i denotes the horse that wins at time i, i.e., $X_i \in \mathcal{X} := [1, 2, ..., m]$. At time i, the gambler knows some side information which we denote as Y_i. We assume that the gambler invests all his capital in the horse race as a function of the information that he knows at time i, i.e., the previous horse race outcomes X^{i-1} and side information Y^i up to time i. Let $b(x_i|x^{i-1}, y^i)$ be the portion of wealth that the gambler bets on horse x_i given $X^{i-1} = x^{i-1}$ and $Y^i = y^i$. Obviously, the betting scheme should satisfy $b(x_i|x^{i-1}, y^i) \geq 0$ and $\sum_{x_i} b(x_i|x^{i-1}, y^i) = 1$ for any history x^{i-1}, y^i.

Let $o(x_i|x^{i-1})$ denote the odds of a horse x_i given the previous outcomes x^{i-1}, which is the amount of capital that the gambler gets for each unit capital that the gambler invested in the horse. We denote by $S(x^n||y^n)$ the gambler's wealth after n races where the race outcomes were x^n and the side information that was causally available was y^n. We assume that the gambler wishes to maximize his wealth which is a random variable.

Here is a summary of the notation:

- X_i is the outcome of the horse race at time i.
- Y_i is the the side information at time i.
- $o(X_i|X^{i-1})$ is the payoffs at time i for horse X_i given that in the previous race the horses X^{i-1} won.
- $b(X_i|Y^i, X^{i-1})$ betting strategy - the fractions of the gambler's wealth invested in horse X_i at time i given that the outcome of the previous races are X^{i-1} and the side information available time i is Y^i.
- $S(X^n||Y^n)$ the gambler's wealth after n races when the outcomes of the races are X^n and the side information Y^n is causally available.

Without loss of generality, we assume that, initially, the gambler's capital is 1; therefore $S_0 = 1$. We assume that at any time n the gambler invests all his capital and therefore we have

$$S(X^n||Y^n) = b(X_n|X^{n-1}, Y^n)o(X_n|X^{n-1})S(X^{n-1}||Y^{n-1}). \tag{16}$$

This also implies that

$$S(X^n||Y^n) = \prod_{i=1}^{n} b(X_i|X^{i-1}, Y^i)o(X_i|X^{i-1}). \tag{17}$$

The objective goal will be to find

$$b = \arg \max_{b(X_i|X^{i-1},Y^i)} \mathbb{E}[logS(X^n||Y^n)] \tag{18}$$

$$\begin{aligned}
\max_{b(X_i|X^{i-1},Y^i)} \mathbb{E}[logS(X^n||Y^n)] &\overset{(a)}{=} \max_{b(x^n||y^n)} \mathbb{E}[\log b(x^n||y^n) + \log o(X_i|X^{i-1})] \\
&\overset{(b)}{=} \max_{b(x^n||y^n)} \mathbb{E}[\log b(x^n||y^n)) \\
&= \max_{b(x^n||y^n)} \sum_{x^n,y^n} p(x^n, y^n) \log b(x^n||y^n) \\
&= \max_{b(x^n||y^n)} \sum_{x^n,y^n} p(x^n, y^n) \log[b(x^n||y^n)\frac{p(x^n||y^n)}{p(x^n||y^n)}] \\
&= \max_{b(x^n||y^n)} \sum_{x^n,y^n} p(x^n, y^n) \log p(x^n||y^n) + \sum_{x^n,y^n} p(x^n, y^n) \log \frac{b(x^n||y^n)}{p(x^n||y^n)}
\end{aligned} \tag{19}$$

$$
\begin{aligned}
&\overset{(c)}{=} \max_{b(x^n||y^n)} -H(X^n||Y^n) + \sum_{x^n,y^n} p(x^n,y^n)\log\frac{b(x^n||y^n)}{p(x^n||y^n)} \\
&\overset{(d)}{\leq} -H(X^n||Y^n) \qquad (20)
\end{aligned}
$$

Thus,

$$b(x^n||y^n) = p(x^n||y^n) \tag{21}$$

where

(a) follows from the fact that $b(x_i|x^{i-1},y^i)$ uniquely determines $b(x^n||y^n)$.

(b) follows from the face that $o(X_i|X^{i-1})$ does not depend on $b(x_i|x^{i-1},y^i)$.

(c) follows from the fact that $\sum_{x^n,y^n} p(x^n,y^n)\log p(x^n||y^n) == -H(X^n||Y^n)$.

(d) follows from:

$$
\begin{aligned}
\sum_{x^n,y^n} p(x^n,y^n)\log\frac{b(x^n||y^n)}{p(x^n||y^n)} &\leq \log[\sum_{x^n,y^n}\frac{p(x^n,y^n)b(x^n||y^n)}{p(x^n||y^n)}] \\
&= \log[\sum_{x^n,y^n}\frac{p(x^n||y^n)p(y^n||x^{n-1})b(x^n||y^n)}{p(x^n||y^n)}] \\
&= \log[\sum_{x^n,y^n} p(y^n||x^{n-1})b(x^n||y^n)] \\
&= \log 1 = 0
\end{aligned}
$$

Note that since $\{p(x_i|x^{i-1},y^i)\}_{i=1}^n$ uniquely determines $p(x^n||y^n)$, and since $\{b(x_i|x^{i-1},y^i)\}_{i=1}^n$ uniquely determines $b(x^n||y^n)$, then ($b(x^n||y^n) = p(x^n||y^n)$) is equivalent to

$$b(x_i|x^{i-1},y^i) = p(x_i|x^{i-1},y^{i-1}). \tag{22}$$

and so in order to maximize the gambler wealth the betting strategy will be,

$$b(x_i|x^{i-1},y^i) = p(x_i|x^{i-1},y^{i-1}), \quad \forall i \in [1,...,n], x^i \in \mathcal{X}^i, y^i \in \mathcal{Y}^i \tag{23}$$

If we wish the evaluate the value of side information in gambling we need to compute the following,

$$
\begin{aligned}
\mathbb{E}[logS(X^n||Y^n)] - \mathbb{E}[logS(X^n)] &= -H(X^n||Y^n) + H(X^n) \qquad (24) \\
&= I(Y^n \to X^n)
\end{aligned}
$$

References

[1] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. New-York: Wiley, 2006.

[2] H. Permuter, Y. H. Kim and T. Weissman, *On Directed Information and Gambling*, ISIT 2008, Toronto, Canada.