Conservation of Mutual and Directed Information

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***Abstract*— Two conservation laws for mutual information in terms of directed informations between two synchronized sequences of random variables are derived, the first for the case of no conditioning and the second for the case of causal conditioning on a third synchronized sequence. As a byproduct of the derivation of the first conservation law, the directed information specifying the feedback flowing from the second sequence to the first sequence is identified, which leads to a simple proof that a previously known sufficient condition for equality of mutual and directed information is also a necessary condition.**

I. Introduction

We will write X^N, for instance, to denote the length-N sequence of random variables $X_1,\ X_2,\ \cdots,\ X_N$. We will use an asterisk to denote concatenation of sequences so that, for instance, $0 * Y^{N-1}$ denotes the length N sequence of random variables whose first component is the constant random variable 0 and whose second component is Y_1. We will further assume that all such sequences are synchronized in the sense that the first component occurs at time instant 1, the second component at time instant 2, etc., on a common clock. Thus, the component of $0 * Y^{N-1}$ at time instant 1 is the constant random variable 0, the component at time instant 2 is Y_1, etc. Hence $0 * Y^{N-1}$ is just the delay by one time instant of the length-$(N-1)$ sequence Y^{N-1} with a “dummy" 0 inserted at time 1.

Following up on pioneering work by Marko [1], Massey [2] introduced in 1990 a measure of the information that flows *from* one sequence of random variables *to* a second synchronized sequence of random variables, which he called *directed information*, and proved some of its key properties. Kramer [3],[4] recently further developed these ideas and confirmed their usefulness in treating systems with feedback.

The purpose of this paper is to prove “conservation laws" for directed information that may provide additional insight into its understanding. More precisely, we demonstrate “conservation laws" for mutual information in terms of directed informations. These “conservation laws" provide an additional basis for the interpretation of directed information as the information flowing from one sequence of random variables to a second sequence synchronized with the first, possibly with conditioning on a third synchronized sequence.

We begin by recalling the definition [2] of the directed information $I(X^N \to Y^N)$ from the sequence X^N of random variables to the sequence Y^N:

$$I(X^N \to Y^N) = \sum_{n=1}^{N} I(X^n; Y_n | Y^{n-1}) \tag{1}$$

where $I(X^n; Y_n|Y^{n-1})$ is the mutual information between X^n and Y_n conditioned on knowledge of Y^{n-1}, which implies that $I(X^N \to Y^N)$ is nonnegative. We note that if X^n were to be replaced by X^N in the sum on the right of (1), then this sum, which could not have decreased, would be the mutual information $I(X^N; Y^N)$. It follows that

$$0 \le I(X^N \to Y^N) \le I(X^N; Y^N). \tag{2}$$

It was shown in [2] that equality holds in the second inequality if there is no feedback from the sequence Y^N to the sequence X^N, which condition was defined to mean

$$H(X_n|X^{n-1}Y^{n-1}) = H(X_n|X^{n-1}), 2 \le n \le N, \tag{3}$$

where $H(\cdot|\cdot)$ denotes conditional entropy. We now prove a proposition characterizing this no–feeback condition in terms of directed information.

Proposition 1: Equation (3) holds, i.e., there is no feedback from the sequence Y^N to the sequence X^N, if and only if $I(0 * Y^{N-1} \to X^N) = 0$.

From definition (1) and the fact that $0 * Y^{n-1}$ and Y^{n-1} are equivalent sequences in mutual information expressions, we have

$$\begin{aligned} I(0 * Y^{N-1} \to X^N) &= \sum_{n=2}^{N} I(Y^{n-1}; X_n|X^{n-1}) \\ &= \sum_{n=2}^{N} \left[H(X_n|X^{n-1}) - H(X_n|X^{n-1}Y^{n-1}) \right]. \end{aligned}$$

It follows that $I(0 * Y^{N-1} \to X^N) \ge 0$ with equality if and only if (3) holds.

II. The Conservation Law

We now prove our first conservation law.

Proposition 2 (Conservation of Information): For the synchronized sequences of random variables X^N and Y^N,

$$I(X^N \to Y^N) + I(0 * Y^{N-1} \to X^N) = I(X^N; Y^N). \tag{4}$$

We prove (4) by induction on N. We first note that $I(X_1 \to Y_1) + I(0 \to X_1) = I(X_1 \to Y_1) = I(X_1; Y_1)$ so that (4) is satisfied for $N = 1$. Suppose then that (4) is satisfied for sequence lengths smaller than N where $N \geq 2$. From the definition (1), we have

$$I(X^N \to Y^N) = I(X^{N-1} \to Y^{N-1}) + I(X^N; Y_N | Y^{N-1}).$$

Similarly,

$$I(0 * Y^{N-1} \to X^N) =$$
$$I(0 * Y^{N-2} \to X^{N-1}) + I(0 * Y^{N-1}; X_N | X^{N-1}).$$

Combining these last two equations, noting that

$$I(0 * Y^{N-1}; X_N | X^{N-1}) = I(X_N; Y^{N-1} | X^{N-1}),$$

and using the induction hypothesis, we obtain

$$\begin{aligned} & I(X^N \to Y^N) + I(0 * Y^{N-1} \to X^N) = \\ & I(X^{N-1}; Y^{N-1}) + I(X_N; Y^{N-1} | X^{N-1}) \\ & + I(X^N; Y_N | Y^{N-1}) = \\ & I(X^N; Y^{N-1}) + I(X^N; Y_N | Y^{N-1}) = I(X^N; Y^N) \end{aligned}$$

so that (4) indeed holds for N, which completes the inductive proof.

Proposition 1 together with the conservation law (4) immediately imply:

Corollary 1: $I(X^N \to Y^N) \leq I(X^N; Y^N)$ with equality if and only if there is no feedback from the sequence Y^N to the sequence X^N.

This corollary strengthens the result in [2] where it was shown only that no feedback was a sufficient condition for $I(X^N \to Y^N) = I(X^N; Y^N)$.

When X^N and Y^N are the input and output sequences, respectively, of a discrete (not necessarily memoryless) channel, then $I(X^N \to Y^N)$ has a natural interpretation as the information flowing through the channel and, in the case of noiseless feedback of the channel output, $I(0 * Y^{N-1} \to X^N)$ has the natural interpretation as the feedback information exploited by the channel encoder.

III. Causal Conditioning

Kramer [3],[4] introduced the notion of "causal conditioning" into the study of directed information. He defined the entropy of Y^N *causally conditioned* on X^N to be the quantity

$$H(Y^N \| X^N) = \sum_{n=1}^{N} H(Y_n | Y^{n-1} X^n)$$

where we note again that if X^n were replaced by X^N on the right, then the sum would be the usual conditional entropy $H(Y^N | X^N)$. In terms of causal conditioning, the definition of directed information becomes simply

$$I(X^N \to Y^N) = H(Y^N) - H(Y^N \| X^N).$$

Kramer also introduced the notion of *causally conditioned directed information*, which he defined as

$$I(X^N \to Y^N \| Z^N) = H(Y^N \| Z^N) - H(Y^N \| X^N Z^N) \quad (5)$$

where X^N, Y^N and Z^N are three synchronized sequences of random variables.

Equation (4) can be written in terms of causally conditioned entropy as

$$I(X^N; Y^N) = H(Y^N) - H(Y^N \| X^N) + I(0 * Y^{N-1} \to X^N).$$

Substituting the definition $I(X^N; Y^N) = H(Y^N) - H(Y^N | X^N)$ into this equation and solving gives the useful relation

$$I(0 * Y^{N-1} \to X^N) = H(Y^N \| X^N) - H(Y^N | X^N), \quad (6)$$

which also implies

$$\begin{aligned} & I(0 * Y^{N-1} \to X^N Z^N) \\ & = H(Y^N \| X^N Z^N) - H(Y^N | X^N Z^N). \quad (7) \end{aligned}$$

Using (6) and (7) in the definition (5) of causally conditioned directed information and then using the definition $I(X^N; Y^N | Z^N) = H(Y^N | Z^N) - H(Y^N | X^N Z^N)$ of conditional mutual information, we obtain the following conservation law.

Proposition 3: (Conservation of Causally Conditioned Information) For the synchronized sequences of random variables X^N, Y^N and Z^N,

$$\begin{aligned} & I(X^N \to Y^N \| Z^N) + I(0 * Y^{N-1} \to X^N Z^N) \\ & - I(0 * Y^{N-1} \to Z^N) = I(X^N; Y^N | Z^N). \quad (8) \end{aligned}$$

This conservation law also admits a natural interpretation when X^N and Y^N are the input and output sequences, respectively, of a discrete (not necessarily memoryless) channel and Z^N is an auxiliary sequence elsewhere in the communication system. Then $I(X^N \to Y^N \| Z^N)$ has a natural interpretation as the information flowing through the channel causally conditioned on knowledge of the auxiliary sequence, and, in the case of noiseless feedback of the channel output, $I(0 * Y^{N-1} \to X^N Z^N) - I(0 * Y^{N-1} \to Z^N)$ has the natural interpretation as the incremental information about X^N contained in the information provided by the feedback sequence $0 * Y^{N-1}$ about the pair X^N and Z^N. This incremental information about X^N is the feedback information exploited by the channel encoder.

References

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