**Homework Set #5**
**Question 3 may increase your final grade by 10 points.**

1. **MAC with common information**. Consider a DM-MAC $P_{Y|X_1,X_2}$ with three independent uniformly distributed messages $W_0 \in [1, ..., 2^{nR0}]$, $W_1 \in [1, ..., 2^{nR1}]$, and $W_2 \in [1, ..., 2^{nR2}]$. The first encoder maps each pair $(w_0, w_1)$ into a codeword $x_1^n(w_0, w_1)$ and the second maps each pair $(w_0, w_2)$ into a codeword $x_2^n(w_0, w_2)$. The decoder upon receiving $y^n$, finds an estimate $(\hat{w}_0, \hat{w}_1, \hat{w}_2)$ of the messages sent. The probability of decoding error is:

$$P^(n)_e = \Pr\left((\hat{W}_0, \hat{W}_1, \hat{W}_2) \neq (W_0, W_1, W_2)\right). \quad (1)$$

Show that the capacity region for this channel is given by the set of rate triples $(R_0, R_1, R_2)$ such that

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2, U), \\ R_2 &\leq I(X_2; Y|X1, U), \\ R_1 + R_2 &\leq I(X_1, X_2; Y|U), \\ R_0 + R_1 + R_2 &\leq I(X_1, X_2; Y), \end{aligned} \quad (2)$$

for some $p(u)p(x_1|u)p(x_2|u)$. You need to prove achievability and converse. (Hint: In proving the converse you may use the identification $U_i = W_0$.)

2. **Strong $\epsilon$-typicality**. Achievability proofs involving *covering*, e.g., for the rate distortion theorem, require that we find a good lower bound on the probability that one specific typical sequence $x^n$ is jointly typical with a randomly drawn sequence $Y^n$. Using strong typicality, the desired lower bound can be established. Let $(X_i, Y_i)$ be drawn i.i.d. $\sim P(x, y)$ and assume that the cardinalities $\mathcal{X}, \mathcal{Y}$ are finite. Let the marginals of $X$ and $Y$ be $P(x)$ and $P(y)$, respectively (you may use ideas and results from methods of types to solve the exercise). We use in this exercise a specific notation $\delta(\epsilon)$ that implies that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.

(a) Show that if $x^n \in T_\epsilon^{(n)}(X)$, then

$$p(x^n) \doteq 2^{n(H(X)\pm\delta(\epsilon))}. \tag{3}$$

(b) Show that $\Pr(X^n \in T_\epsilon^{(n)}(X)) \to 1$, as $n \to \infty$.

(c) Show that

$$|T_\epsilon^{(n)}(X)| \doteq 2^{n(H(X)\pm\delta(\epsilon)}. \tag{4}$$

(d) Let $x^n \in T_\epsilon^{(n)}(X)$, and let $T_\epsilon^{(n)}(Y|x^n)$ be the set of $y^n$ sequences such that $(x^n, y^n) \in T_\epsilon^{(n)}(X,Y)$. Show that

$$|T_\epsilon^{(n)}(Y|x^n)| \doteq 2^{n(H(Y|X)\pm\delta(\epsilon)}. \tag{5}$$

(e) Let $x^n \in T_\epsilon^{(n)}(X)$, and $Y^n$ be drawn independently of $x^n$ i.i.d. $\sim P(y)$. Show that

$$\Pr(x^n, Y^n \in T_\epsilon^{(n)}(X,Y)| \doteq 2^{n(I(X,Y)\pm\delta(\epsilon)}. \tag{6}$$

(Note that in (d) and (e) the bounds do not depend on $x^n$.)

3. **(10 points bonus to the final grade for the first two students who solve the following question)**

   (5 points.) **MAC with causal state information at the encoder** Find the capacity region of a MAC $P_{Y|X_1,X_2,S}$, where the state information $S$ is known *causally* to both encoders but not to the decoder, as depicted in Fig. 1

   ( 5 points.) **MAC with non causal state information at the encoder** Find the capacity region of a MAC $P_{Y|X_1,X_2,S}$, where the state information $S$ is known *noncausally* to both encoders but not to both decoder, as depicted in Fig. 1

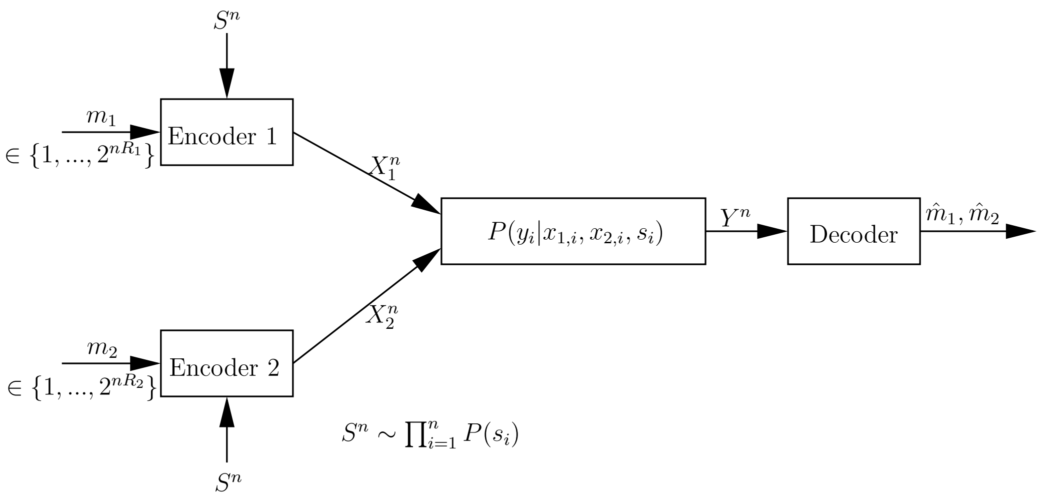


Figure 1: Mac with state information. Find the capacity region of the MAC with state information at the encoder. The state is known causally or non-causally to the encoders.