Homework Set #3
Methods of types, MAC, and Csiszar Sum

1. **Large deviations.**
   Let $X_1, X_2, \ldots$ be i.i.d. random variables drawn according to the Bernoulli distribution
   $$\Pr\{X_i = 1\} = \Pr\{X_i = -1\} = \frac{1}{2}.$$
   Let S_n be the random walk defined by
   $$S_n = \sum_{i=1}^{n} X_i.$$
   Find the function $f(\alpha)$ such that, for all $\alpha > 0$,
   $$\lim_{n\to\infty} -\frac{1}{n} \log \Pr\{S_n \geq n\alpha\} = f(\alpha).$$

2. **Counting.**
   Let $\mathcal{X} = \{1, 2, \ldots, m\}$. Show that the number of sequences $x^n \in \mathcal{X}^n$ satisfying $\frac{1}{n}\sum_{i=1}^{n} g(x_i) \geq \alpha$ is approximately equal to 2^{nH^*}, to first order in the exponent, where
   $$H^* = \max_{P:\sum P(i)g(i)\geq\alpha} H(P).$$

3. **The cooperative capacity of a multiple access channel.**

   (a) Suppose X_1 and X_2 have access to *both* indices $W_1 \in \{1, 2^{nR_1}\}, W_2 \in \{1, 2^{nR_2}\}$. Thus the codewords $X_1^n(W_1, W_2), X_2^n(W_1, W_2)$ depend on both indices. Find the capacity region.

   (b) Evaluate this region for the binary erasure multiple access channel $Y = X_1 + X_2$, $X_i \in \{0, 1\}$. Compare to the non-cooperative region.

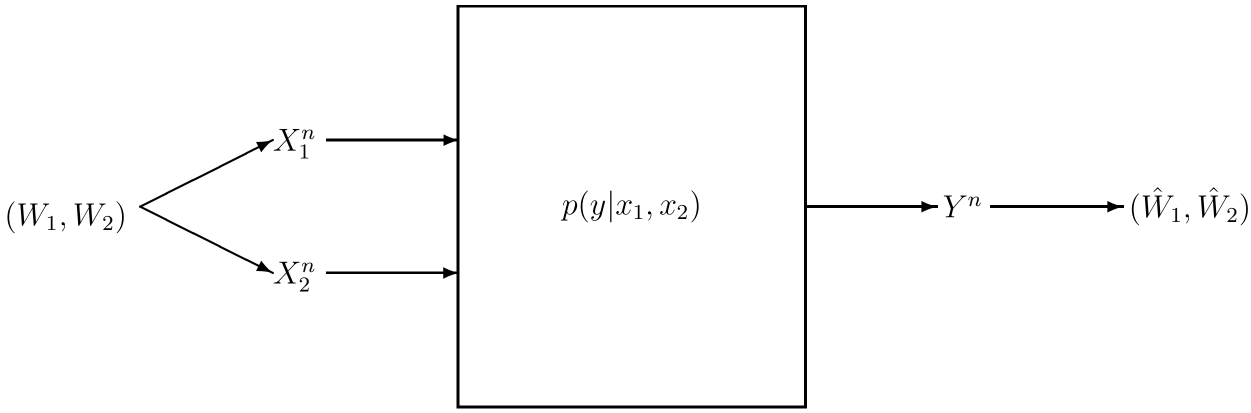


Figure 1: Multiple access channel with cooperating senders.

4. **Capacity of multiple access channels.**
   Find the capacity region for each of the following multiple access channels:

   (a) Additive modulo 2 multiple access access channel. $X_1 \in \{0, 1\}, X_2 \in \{0, 1\}, Y = X_1 \oplus X_2$.

   (b) Multiplicative multiple access channel. $X_1 \in \{-1, 1\}, X_2 \in \{-1, 1\}, Y = X_1 \cdot X_2$.

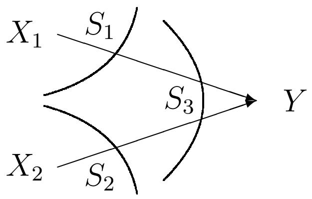
5. **Cut-set interpretation of capacity region of multiple access channel.**
   For the multiple access channel we know that (R_1, R_2) is achievable if

$$R_1 < I(X_1; Y \mid X_2), \quad (1)$$
$$R_2 < I(X_2; Y \mid X_1), \quad (2)$$
$$R_1 + R_2 < I(X_1, X_2; Y), \quad (3)$$

   for X_1, X_2 independent. Show, for X_1, X_2 independent, that

$$I(X_1; Y \mid X_2) = I(X_1; Y, X_2).$$

   Thus R_1 is less than the mutual information between X_1 and everything else.



Interpret the information bounds as bounds on the rate of flow across cutsets S_1, S_2 and S_3.

6. **Multiple-access channel.**

Let the output Y of a multiple-access channel be given by

$$Y = X_1 + \mathrm{sgn}(X_2)$$

where X_1, X_2 are both real and power limited,

$$\begin{aligned} E[X_1^2] &\leq P_1, \\ E[X_2^2] &\leq P_2, \end{aligned}$$

and $\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$.

Note that there is interference but no noise in this channel.

(a) Find the capacity region.

(b) Describe a coding scheme that achieves the capacity region.

7. **Csiszar Sum Equality**. Let X^n and Y^n be two random vectors with arbitrary joint probability distribution. Show that:

$$\sum_{i=1}^{n} I(X_{i+1}^n; Y_i | Y^{i-1}) = \sum_{i=1}^{n} I(Y^{i-1}; X_i | X_{i+1}^n) \tag{4}$$

As we shall see this inequality is useful in proving converses to several multiple user channels. (Hint: You can prove this by induction or by expanding the terms on both sides using the chain rule.)