Homework Set #5
Polar codes

1. **Polarization and the idea of polar codes**: The question is about polarization effect in memoryless channels that can lead to simple coding schemes that achieve the capacity which are called polar codes.

   (a) Consider the channel in Fig. 1 where two parallel binary erasure channels can be used at once (the input is $X = (X_1, X_2)$). The inputs alphabets are binary, so that Y_1 and

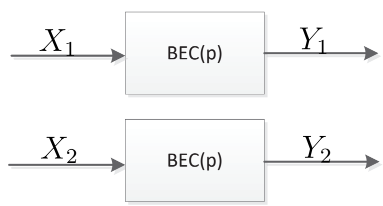


Figure 1: Two parallel binary erasure channels

   Y_2 are the outputs of a BEC(p) with inputs X_1 and X_2, respectively. Compute the capacity of this channel, namely,

$$\max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2). \tag{1}$$

   What is the input distribution $p(x_1, x_2)$ that achieves the capacity?

   (b) Consider the system in Fig. 2, where addition is modulo 2:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

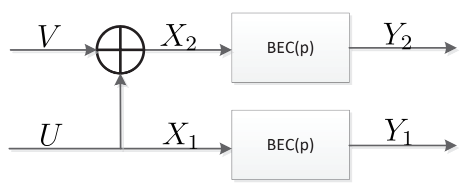


Figure 2: Two parallel binary erasure channels with modified inputs

Compute the capacity of the new channel, i.e. $\max_{p(u,v)} I(U,V;Y_1,Y_2)$.

What is the $p(u,v)$ that achieves the capacity?

Next, the channel is decomposed into two parallel channels as appears in Fig. 3. The input of Channel 1 is U and its output is (Y_1,Y_2,V). The input of Channel 2 is V and its output is (Y_1,Y_2).

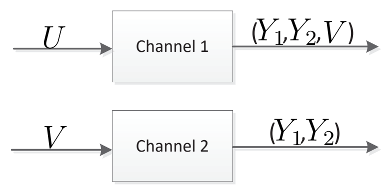


Figure 3: Two new channels

(c) Compute the expressions $I(U;Y_1,Y_2,V)$ and $I(V;Y_1,Y_2)$ with respect to the $p(u,v)$ that achieves the maximum in (b). What is the sum of the expressions you computed?

(d) Compare the mutual information of Channels 1 and 2 with the capacity of a binary erasure channel (that is, write $<$, $>$ or $=$ with simple proof).

*For large n, repeating this decomposition n times, ends up in nc clean channels and in $n(1-c)$ totally noisy channels. This is the main idea of polar codes, which achieves capacity.

2. **Polar codes** Consider a binary erasure channel W with erasure probability p. One step of the polarization process creates a better channel W^+ and a worse channel W^- from two independent copies of W.

(a) The polar code creates 4 effective channels W^{++}, W^{+-}, W^{-+}, W^{--}. Write down the capacities of these 4 channels in terms of information-theoretic quantities.

(b) Compute explicitly the capacities of the 4 channels in terms of the parameter p.

(c) Suppose we would like to send at the rate 3/4 bits per channel use. Which of the U_i's should be frozen, and which should be set as information bits?

(d) We repeat the polarization process n times to create 2^n different channels from 2^n compies of W. Let $\overline{W}$ and $\underline{W}$ be the best and worst channels among these 2^n channels. Compute explicitly the capacities of $\overline{W}$ and $\underline{W}$ in terms of the parameters p and n.

(e) What happens to the capacities of $\overline{W}$ and $\underline{W}$ as $n \to \infty$?

3. **Polar compressor** For a positive integer N, let $n = 2^N$ and consider the invertible matrix $P_n \in \mathbb{F}_2^{n\times n}$ defined by:

$$P_n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes N}.$$

Further, consider $Z^n = (Z_1, \ldots, Z_n) \sim \text{Bern}(p)^n$ where $p \in (0, 0.5)$, and let $W^n = Z^n \cdot P_n$.

   (a) For both channel coding and source coding, we do polarization. Explain briefly the difference in polarization between channel coding and source coding.

   (b) Define the rate for source coding and channel coding, and explain whether you want to maximize or minimize those rates.

   (c) Assume $n = 4$ and consider the entropy terms $H(W_i|W^{i-1})$ for $i \in \{1, \ldots, 4\}$. Determine and explain which one is the highest, and calculate this specific entropy term explicitly in terms of p.

   (d) Define the set S_τ as follows:

$$S_\tau = \{i \in \{1, \ldots, 4\} \mid H(W_i|W^{i-1}) \geq \tau\}.$$

   For $\hat{\tau} = -\mathbb{E}[\log_2(P_{W_1})]$, write explicitly the set $S_{\hat{\tau}}$. Explain your result.

   (e) Consider $z^4 = [1, 0, 1, 1]$ and the set $S_{\hat{\tau}}$ that you found in the previous item. What is the output of the encoder?

   (f) This time let $n = 2$, and assume that $Z_1 \sim \text{Bern}(p_1)$ and $Z_2 \sim \text{Bern}(p_2)$ are sampled conditioned on $Z_1 + Z_2 = a$ (for $a \in \mathbb{F}_2$). Let $b(p_1, p_2, a)$ denote the probability of Z_2 being 1 conditioned on $Z_1 + Z_2 = a$. Find $b(p_1, p_2, a)$ for both $a = 0$ and $a = 1$.

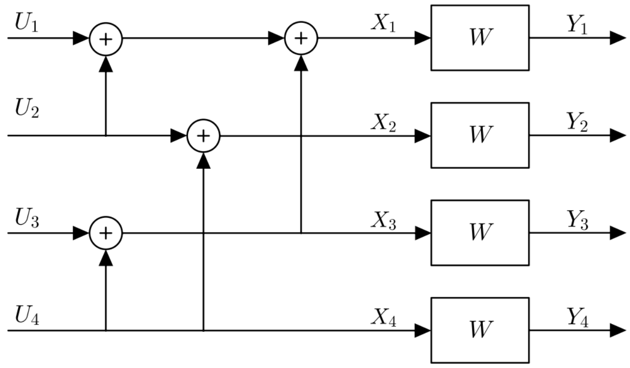
4. **Polar code design:** In this problem, we explore how the selection of frozen bits affects the decoding process in the construction of polar codes. Consider a binary erasure channel (BEC) W, and the encoding scheme of block size $N = 4$ in the figure below.

   For items (a) to (c), assume that the bits U_1 and U_2 are frozen to 0, while U_3 and U_4 are used for message bits.

   (a) Calculate the code rate.

   (b) Given the message bits $(U_3, U_4) = (1, 1)$, perform the encoding to determine the codeword (X_1, X_2, X_3, X_4).

   (c) Apply successive cancellation decoding to the received vector $(Y_1, Y_2, Y_3, Y_4) = (1, ?, ?, 1)$. Determine if decoding is successful, and if so, identify the decoded message.



For items (d) to (f), consider a different scenario where U_2 and U_3 are frozen to 0, while U_1 and U_4 are message bits.

(d) Apply successive cancellation decoding for received vector $(Y_1, Y_2, Y_3, Y_4) = (1, ?, ?, 0)$, and show that the decoding might fail when decoding U_1.

(e) Explain the reason for the decoding failure.

(f) Perform optimal maximum likelihood decoding for the same received vector $(1, ?, ?, 0)$. To do this:

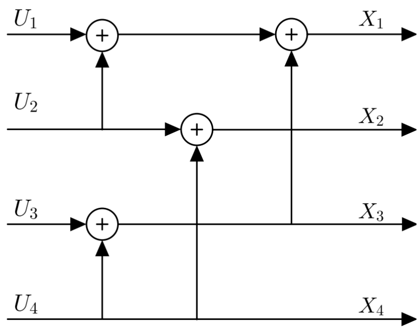
- First, compute the codeword (X_1, X_2, X_3, X_4) for all four possible input messages (U_1, U_4).
- Then, identify the input message(s) (U_1, U_4) that could result in the received vector $(1, ?, ?, 0)$. If more than one message corresponds to the received vector, declare a failure in decoding.

Determine if the decoding is successful, and if so, specify the decoded message.

(g) For a code rate $R = 0.5$, determine which bits should be set as frozen to achieve the best performance in terms of bit error rate when using the successive cancellation decoding algorithm. Justify your selection.

5. **Decoding a Compressed Message:** In this problem, you will learn how to decode a compressed message using polar codes. Consider the encoding scheme with a block size of $N = 4$ shown in the figure below, where $U_i \sim \text{Bern}(0.1)$ for $i = 1, \ldots, 4$.

Items (a) to (c) are related and should be solved in sequence.



(a) Suppose we want to compress the message $(U_1, \ldots, U_4)$ with a rate of $R = 0.75$. Specify which bits $\{X_i\}_{i=1}^4$ should be used to represent the compressed message. Justify your answer.

(b) Given the message $(U_1, U_2, U_3, U_4) = (1, 0, 1, 1)$, and based on your answer to the previous item, write the compressed message.

(c) We would like to decode the original message from the compressed message. Apply the successive cancellation (SC) decoding algorithm, and determine if decoding succeeded or not.

Items (d) to (f) are independent and not related to the previous items.

(d) Can the encoder predict which errors the SC decoder might make? Explain your answer in no more than two lines.

(e) What additional information could the encoder send to the decoder to help correct errors?

(f) Consider that we aim to achieve lossless compression (without errors during decoding) by allowing a variable block length. Using the SC algorithm, propose a method to ensure this while maintaining the compression rate as low as possible.
Remark: Variable block length refers to a compression scheme where the length of the blocks of data being encoded or compressed can vary and depend on the transmitted data.