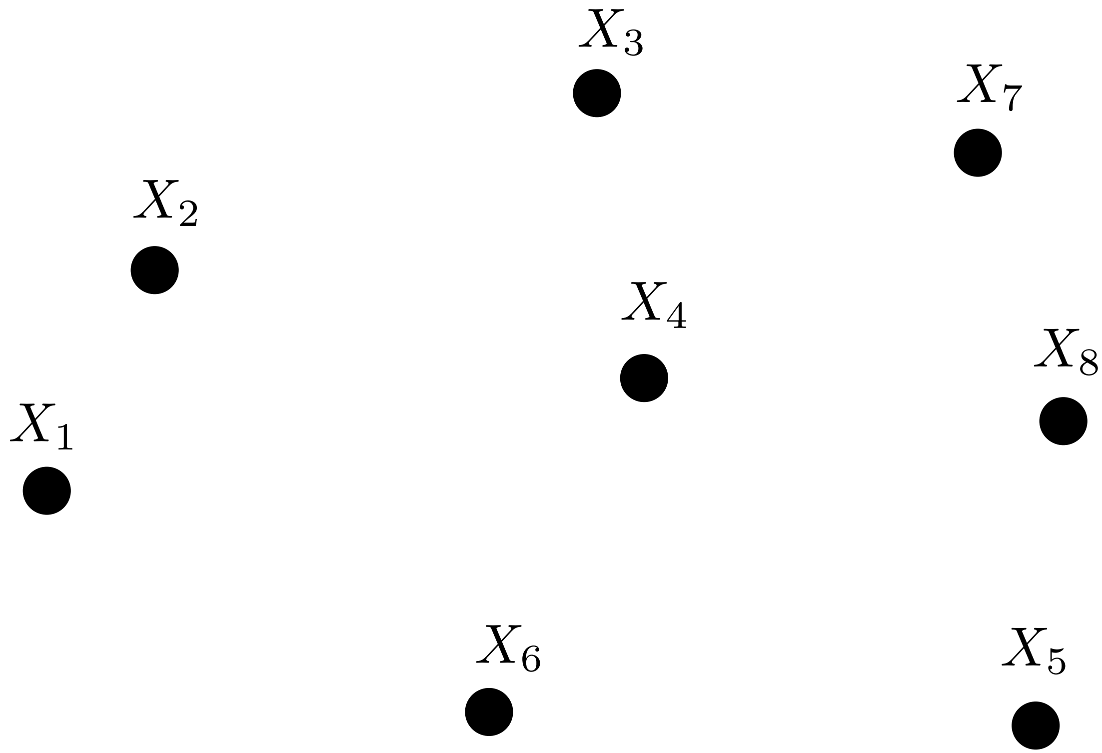
Capacity of Coordinated Actions

Haim Permuter & Thomas Cover

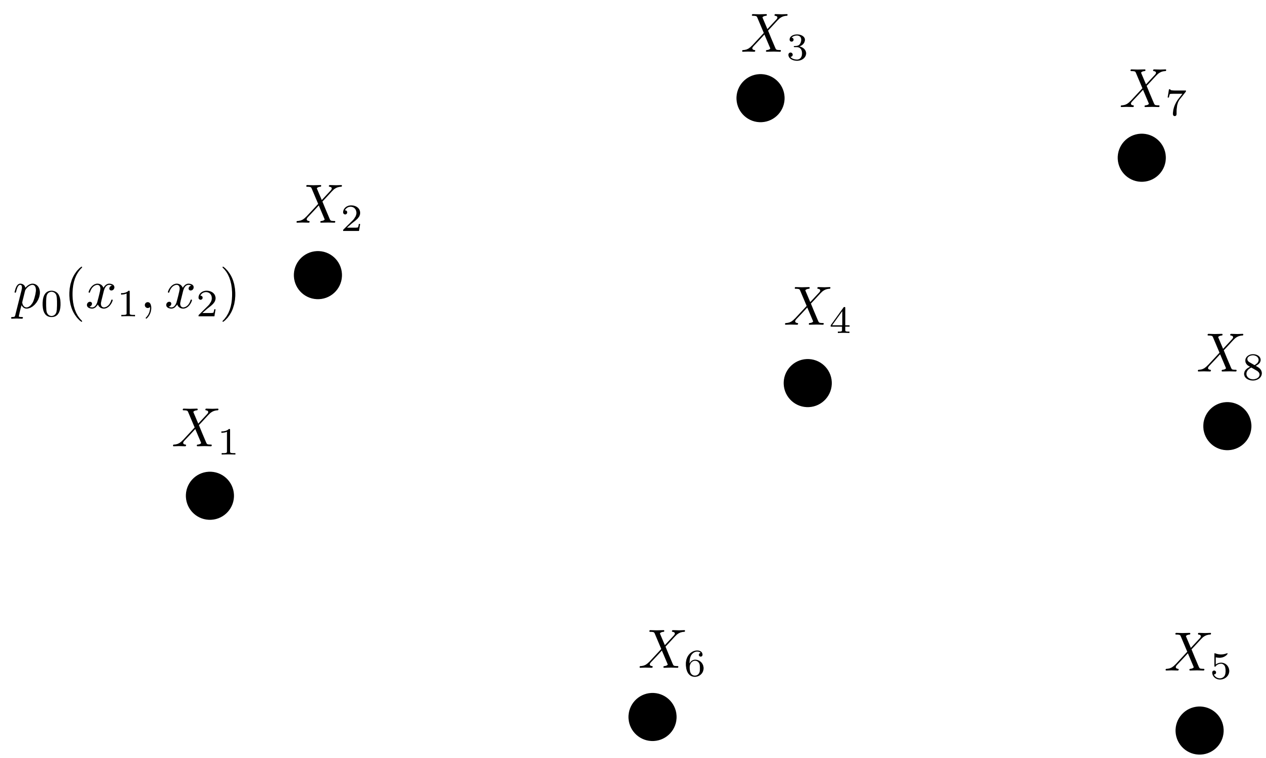
Stanford University

Coordinated Actions



$$p(x_1, x_2, x_3, ..., x_8) = ?$$

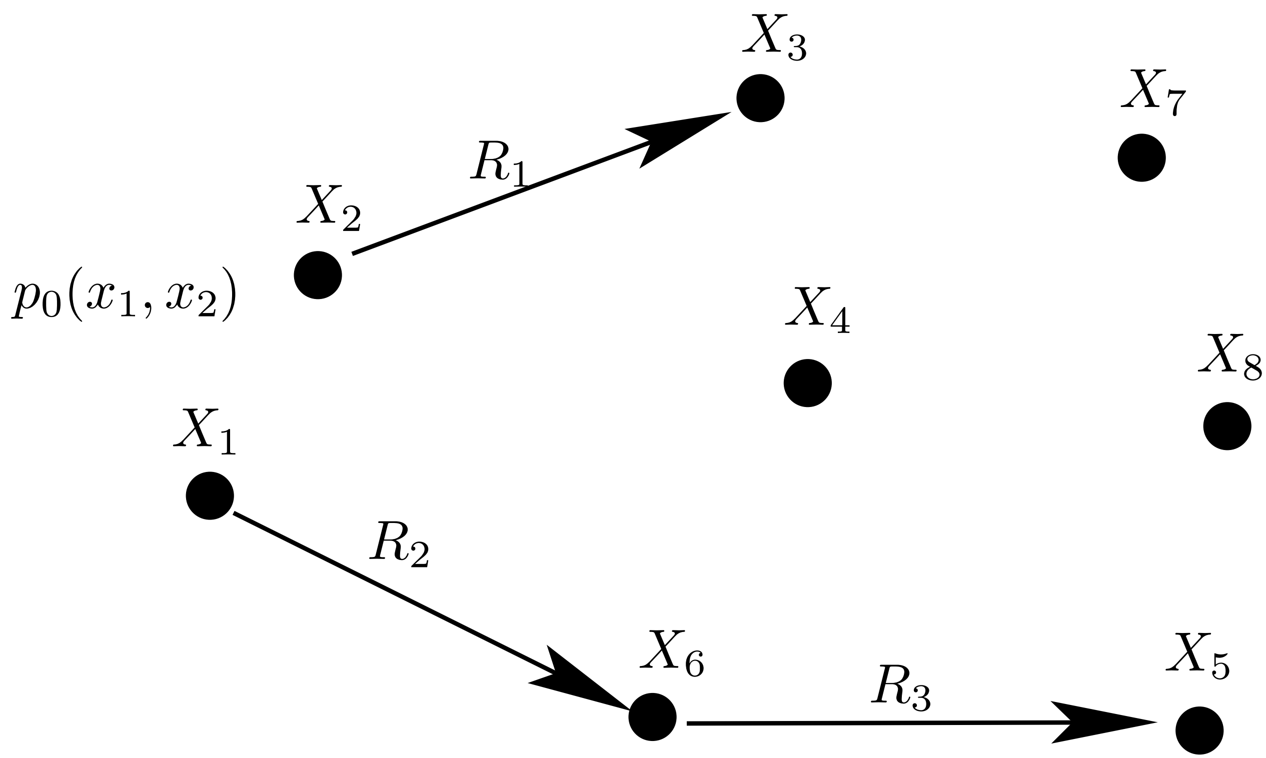
Coordinated Actions



$$p_0(x_1, x_2)p(x_3, ..., x_8|x_1, x_2) =?$$

X_1, X_2 are specified by nature according to $p_0(x_1, x_2)$

Coordinated Actions



$$p_0(x_1, x_2)p(x_3, ..., x_8|x_1, x_2) =?$$

X_1, X_2 are specified by nature according to $p_0(x_1, x_2)$

Question 1

Three–card deck

Bob

1

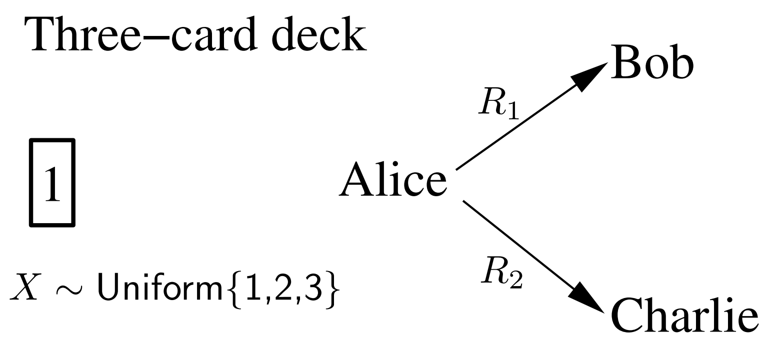
Alice

$X \sim \text{Uniform}\{1,2,3\}$

Charlie

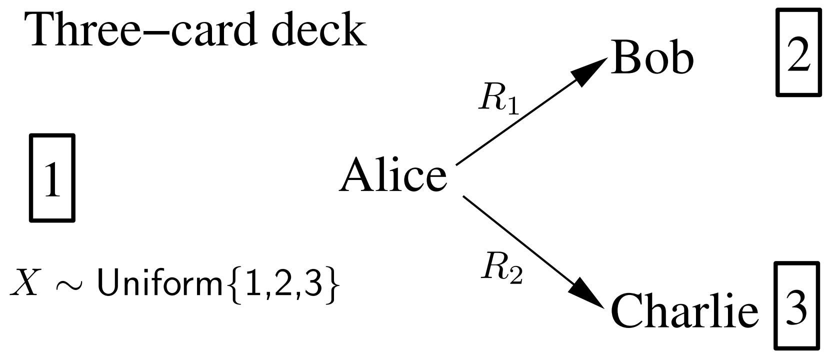
Alices's card X is specified by nature.

Question 1



Alices's card X is specified by nature.

Question 1

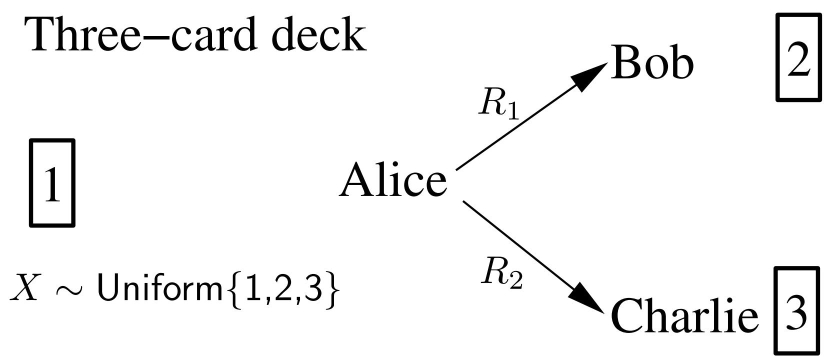


Alices's card X is specified by nature.

<u>The Goal</u>

to achieve uniform distribution over the six permutations of $(1, 2, 3)$

Question 1



Alices's card X is specified by nature.

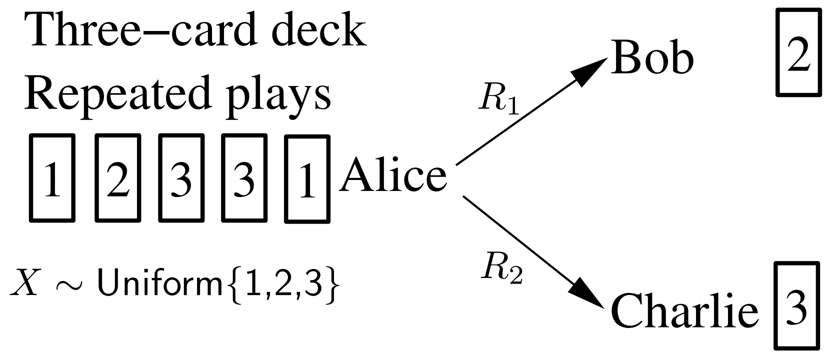
The Goal

to achieve uniform distribution over the six permutations of $(1, 2, 3)$

Question

How much information must Alice send to Bob and to Charlie to achieve the goal?

Question 1



Alices's card X is specified by nature.

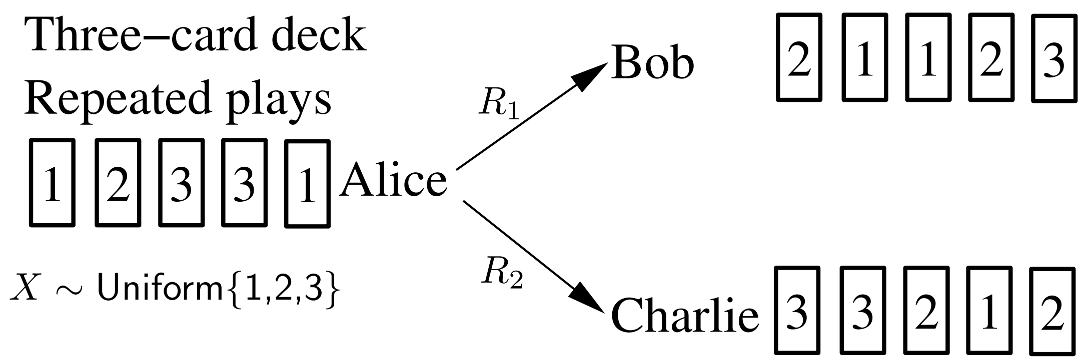
The Goal

to achieve uniform distribution over the six permutations of $(1, 2, 3)$

Question

How much information must Alice send to Bob and to Charlie to achieve the goal?

Question 1



Alices's card X is specified by nature.

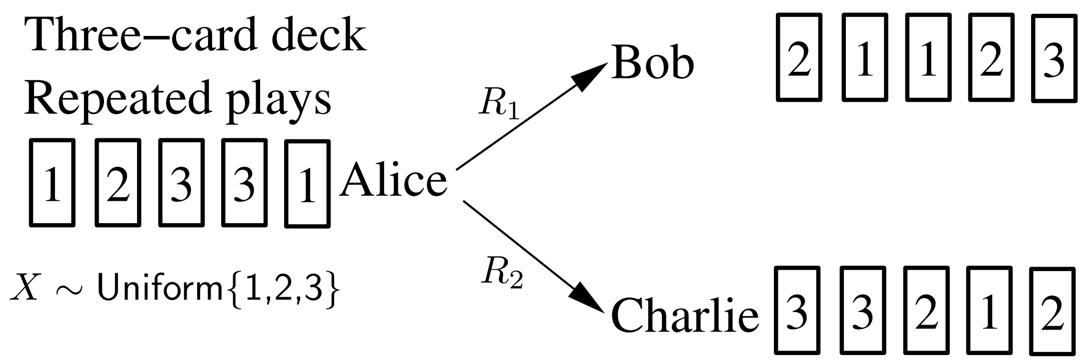
<u>The Goal</u>

to achieve uniform distribution over the six permutations of $(1,2,3)$

<u>Question</u>

How much information must Alice send to Bob and to Charlie to achieve the goal?

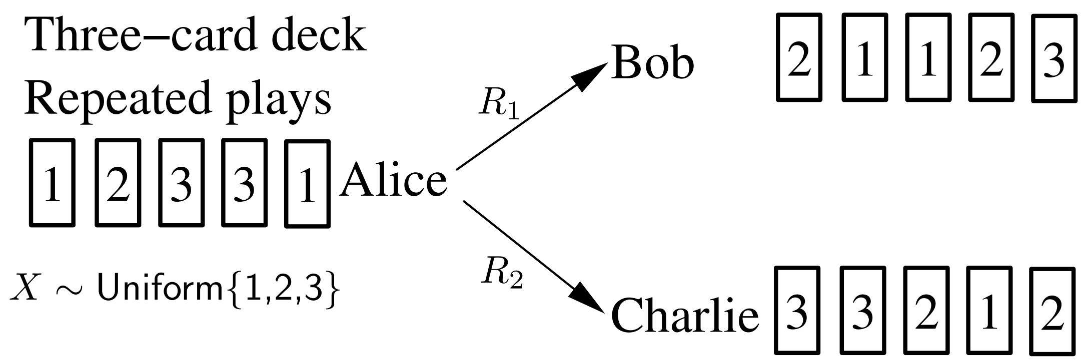
Question 1



A brute-force solution

Alice transmits to Bob and Charlie her card number. This requires
$$R_1 = R_2 = \log 3.$$

Question 1



A brute-force solution

Alice transmits to Bob and Charlie her card number. This requires
$R_1 = R_2 = \log 3$.

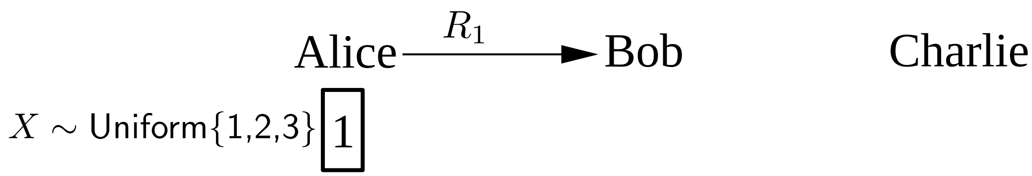
Is it optimal?

Question 2 (Cascade)

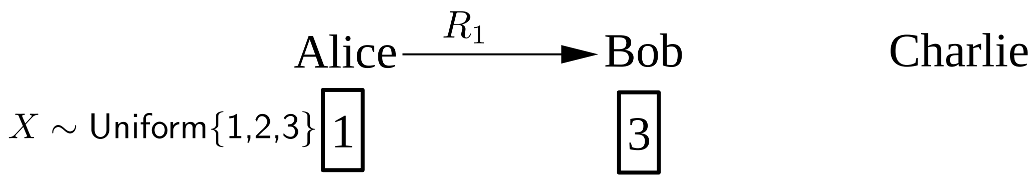
Alice Bob Charlie

$X \sim \text{Uniform}\{1,2,3\}$ $\boxed{1}$

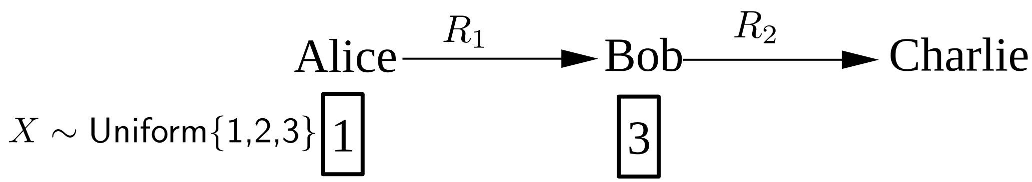
Question 2 (Cascade)



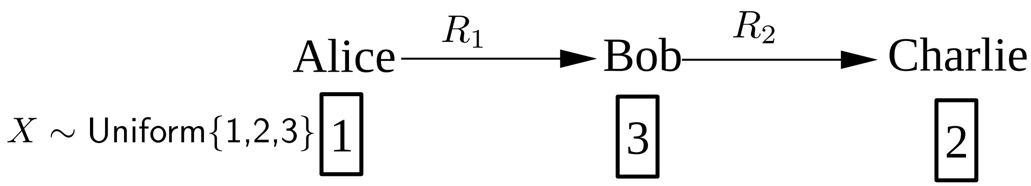
Question 2 (Cascade)



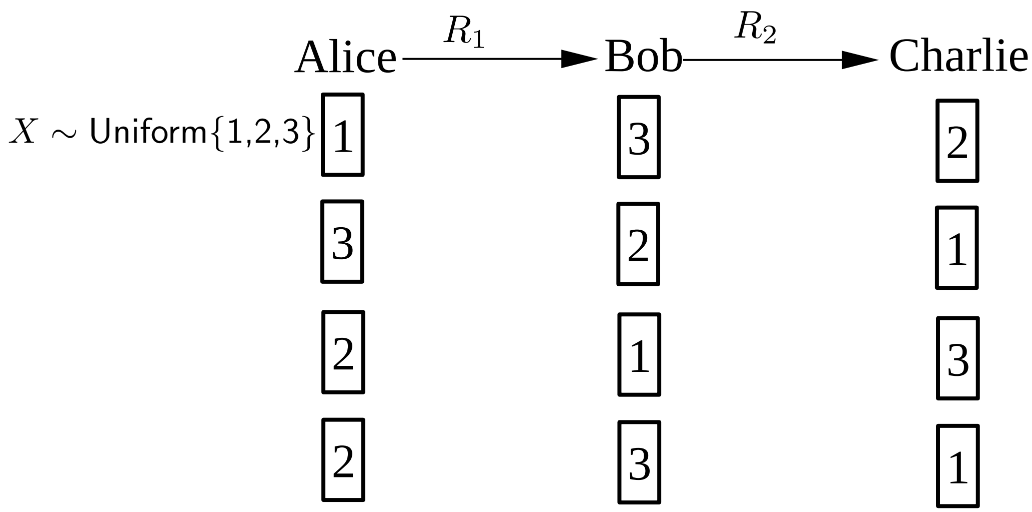
Question 2 (Cascade)



Question 2 (Cascade)



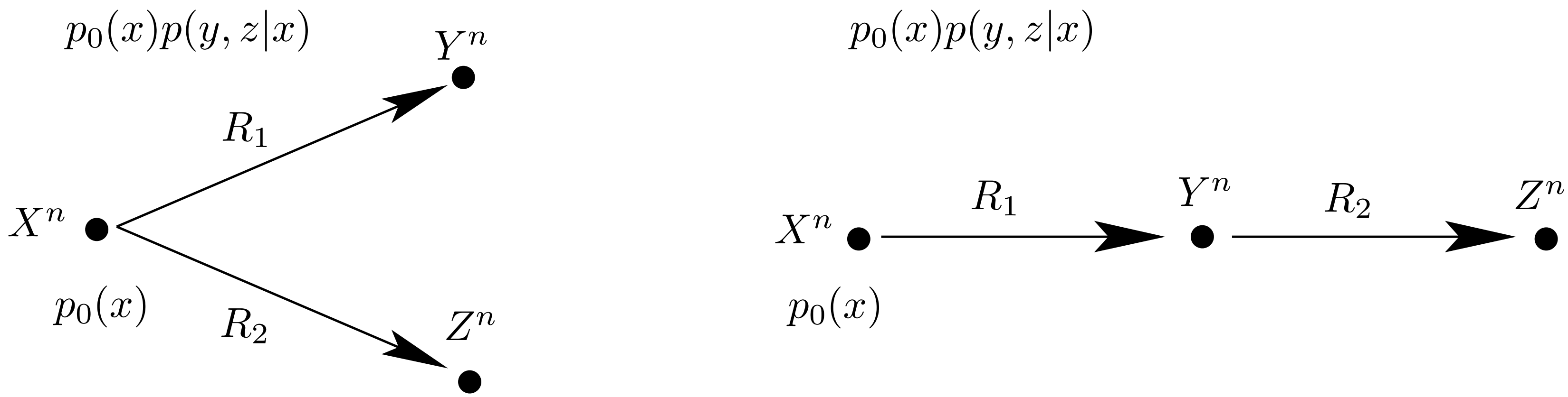
Question 2 (Cascade)



Question

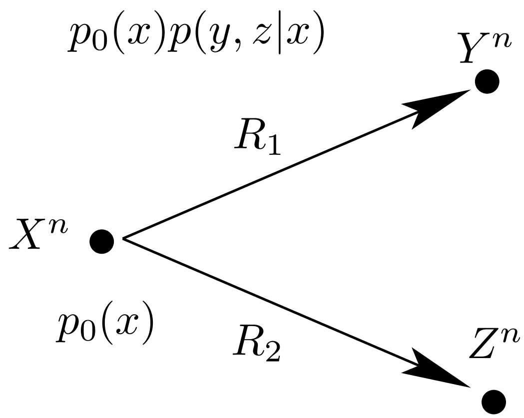
How much information must Alice send to Bob and Bob to Charlie to have uniform distribution over the six permutations of $(1, 2, 3)$?

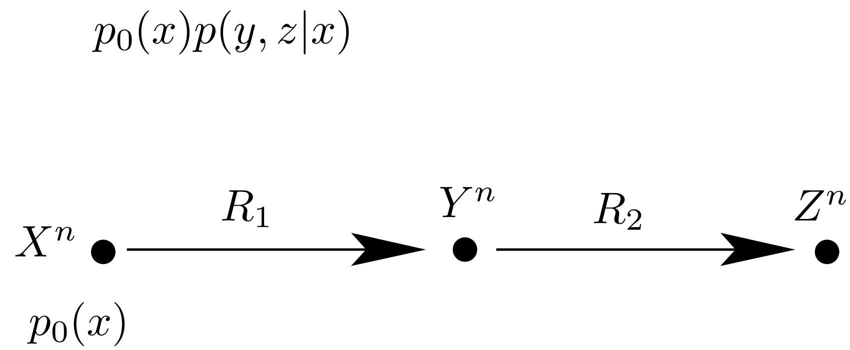
Coordinated Actions: The Setting



- the actions at node X are specified by nature: $p(x^n) = \prod_{i=1}^{n} p_0(x_i)$
- the actions at nodes Y, Z are chosen according to the information received at the nodes

Coordinated Actions: The Goal



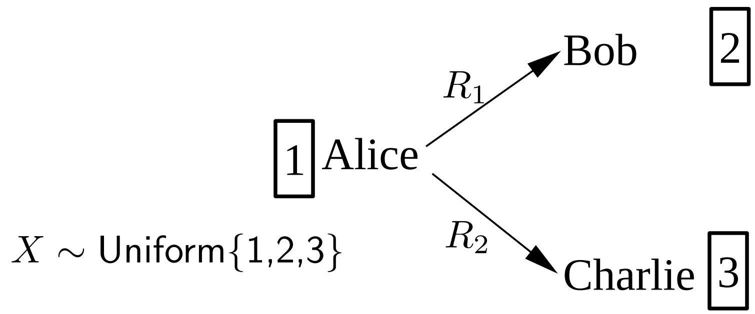


- to find the rates (R_1, R_2) that can achieve the joint distribution $p_0(x)p(y,z|x)$

- to find the set of all distributions $p_0(x)p(y,z|x)$ that can be achieved with communication rate (R_1, R_2)

Applications

- Task Assignments - agents must perform different jobs



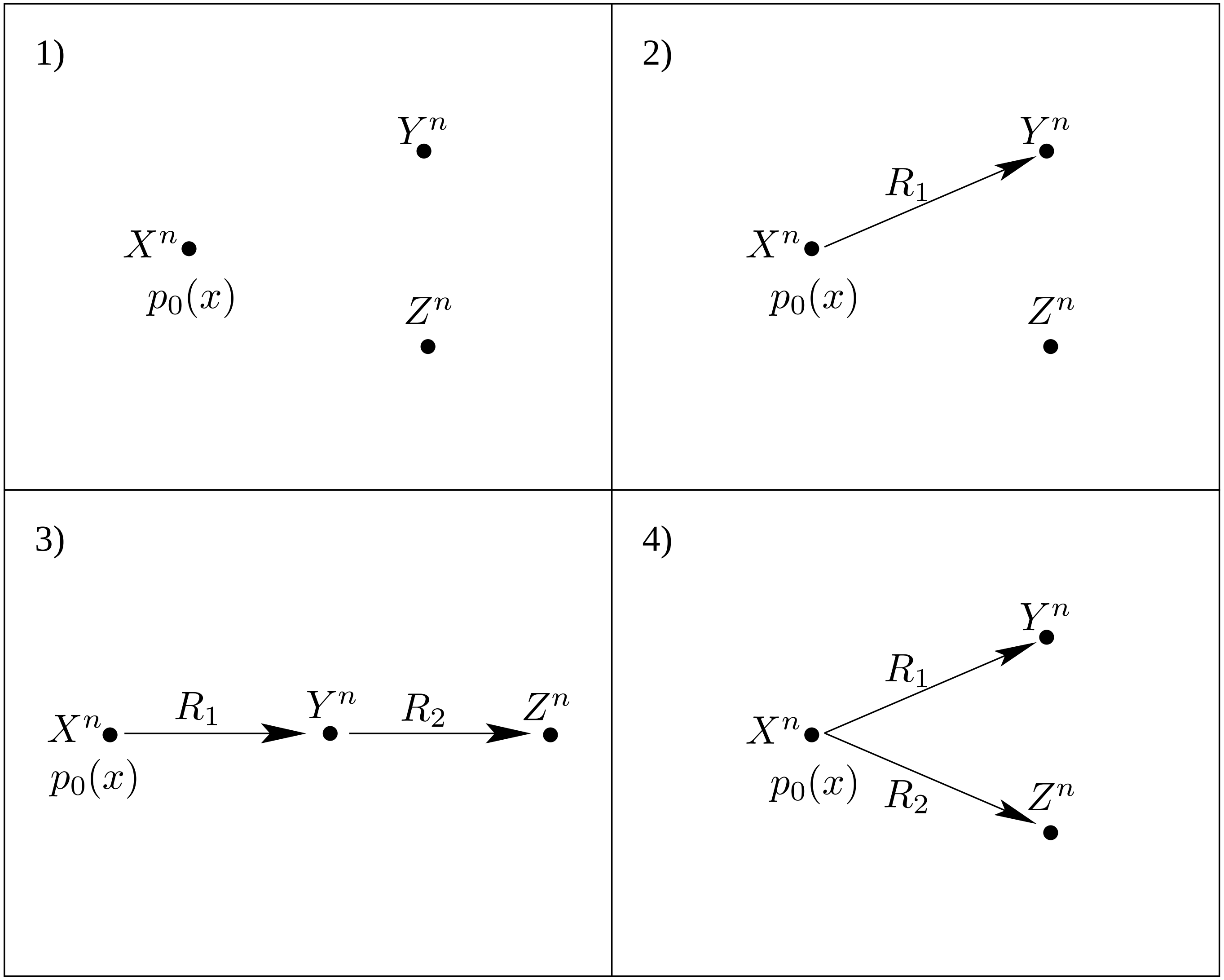
- Computation: parallelization and recombination

- Game theory - players must take actions according to an optimal distribution [Anantharam/Borkar05].

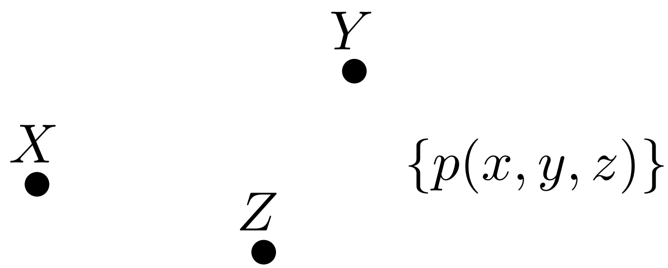
- Quantum information - quantum coding of mixed states

  [Barbum/Caves/Fuchs/Schumacher01], [Kramer/Savari07]

Building Blocks of Distributed Action Problems



Three Nodes and No Communication



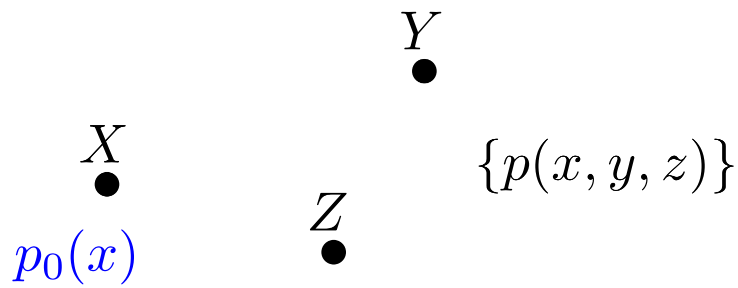
Any joint distribution $p(x, y, z)$ can be achieved.

- $(\Omega, \mathcal{B}, P), X(\omega), Y(\omega), Z(\omega)$

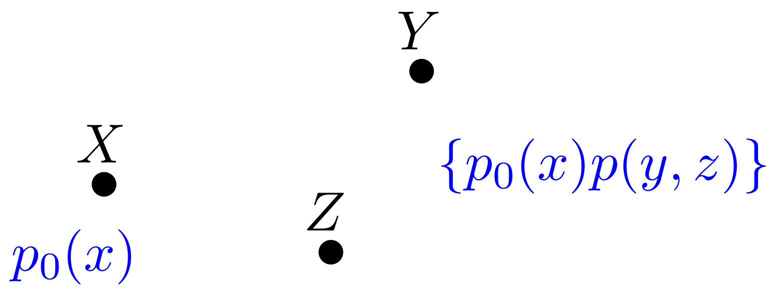
- Time sharing

- Using a codebook that is generated by $p(x, y, z)$.

Three Nodes and No Communication

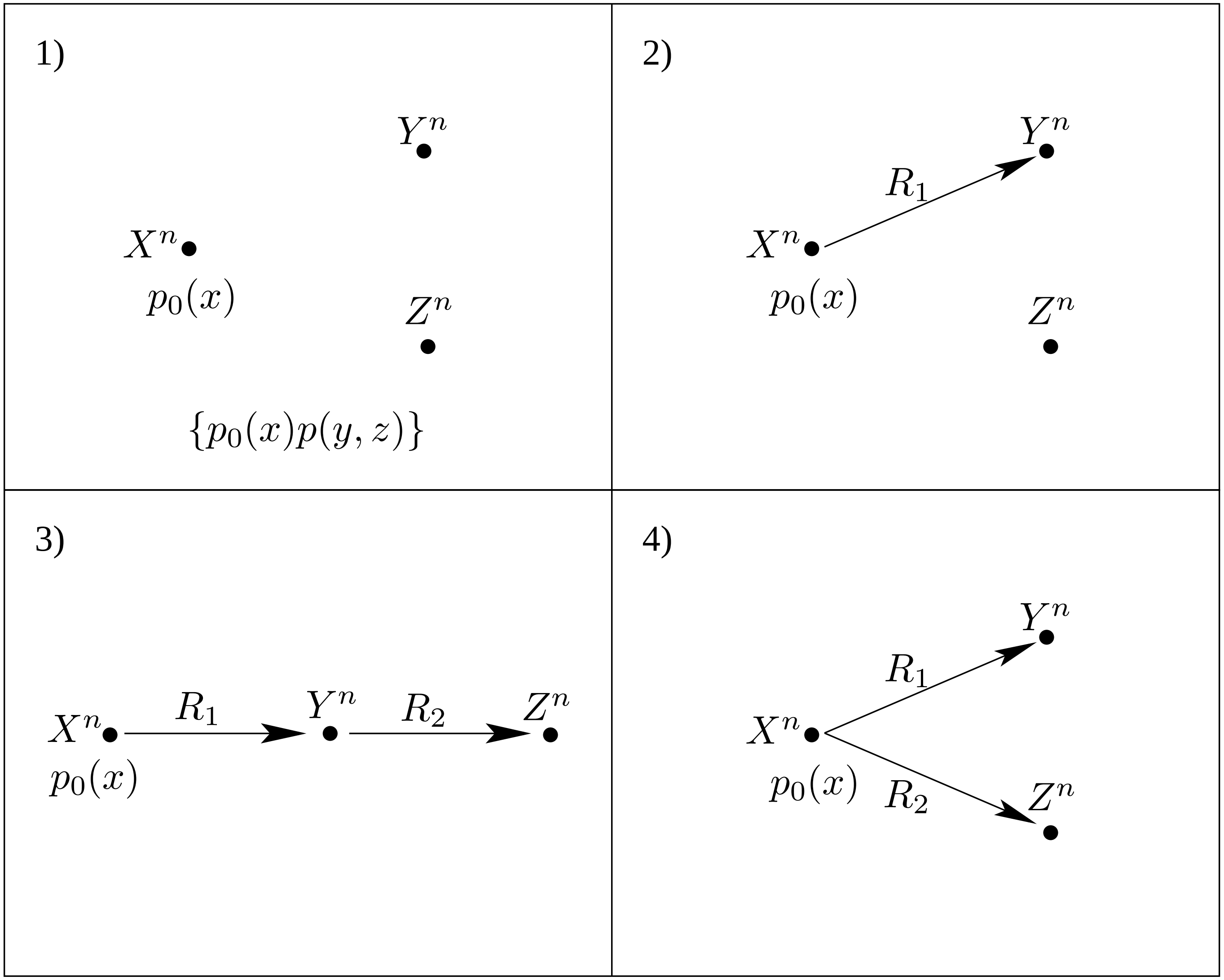


Three Nodes and No Communication

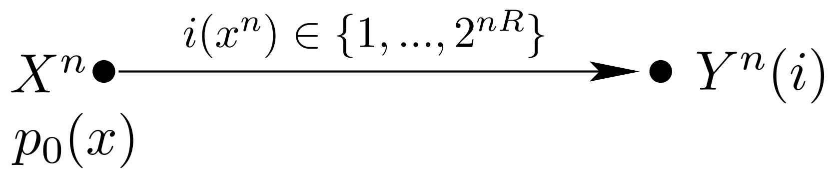


If X is specified to take a certain value distributed according to $p_0(x)$, then only $p(x, y, z) = p_0(x)p(y, z)$ can be achieved.

Building Blocks of Distributed Action Problems



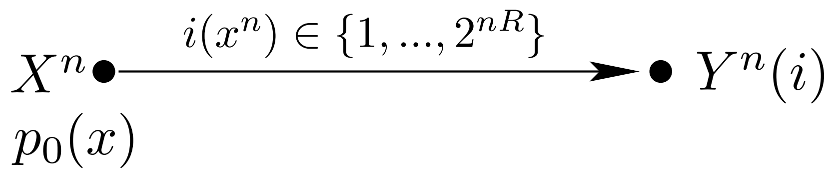
One Link



Definition. *The pair $(R, p(x,y))$ is $achievable$, if there exists a sequence of rate R codes such that $P_{X^n,Y^n}(x,y) \to p(x,y)$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$.*

$P_{X^n,Y^n}(x,y)$ is the joint type.

One Link



Theorem. *A desired distribution* $p(x, y) = p_0(x)p(y|x)$ *is achievable if*

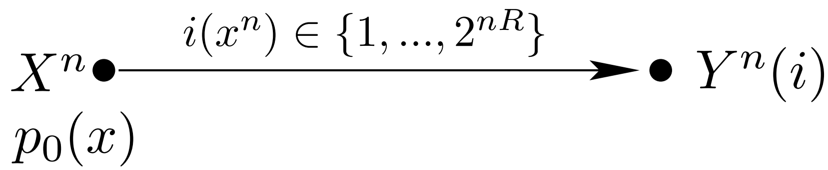
$$R > I(X; Y),$$

and is not achievable if

$$R < I(X; Y).$$

Like rate distortion but without the distortion

Outline of the Proof



Achievability proof is similar to rate distortion.

Converse is based on the following lemma:

Lemma. *If a sequence of R code* $(2^{nR}, n)$ *satisfies*

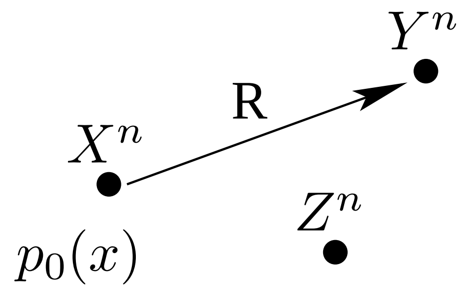
$$P_{X^n,Y^n}(x,y) \xrightarrow{\text{prob}} p(x,y), \ \ \forall x,y \in \mathcal{X} \times \mathcal{Y}$$

then

$$\frac{1}{n}\sum_{k=1}^{n} \Pr(X_k = x, Y_k(i) = y) \to p(x,y), \ \ \forall x,y \in \mathcal{X} \times \mathcal{Y}.$$

$$nR \overset{(a)}{\geq} I(X^n;Y^n) \overset{(b)}{\geq} \sum_{k=1}^{n} I(X_k;Y_k) \overset{(c)}{\geq} n(I(X;Y) - \epsilon_n)$$

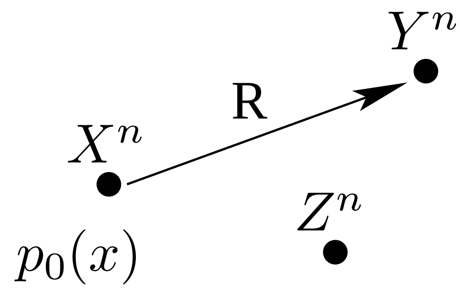
One Link



What is the achievable region?

Is it the set of all distributions $\{p_0(x)p(y|x)p(z) : I(X;Y) \leq R\}$?

One Link

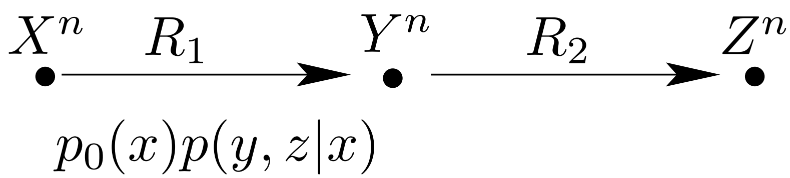


What is the achievable region?

Is it the set of all distributions $\{p_0(x)p(y|x)p(z) : I(X;Y) \leq R\}$?

No. Any joint distribution $p_0(x)p(y,z)$ can be achieved without communication.

Chain of two agents



Theorem. *The achievable region for a distribution* $p_0(x)p(y,z|x)$ *is*

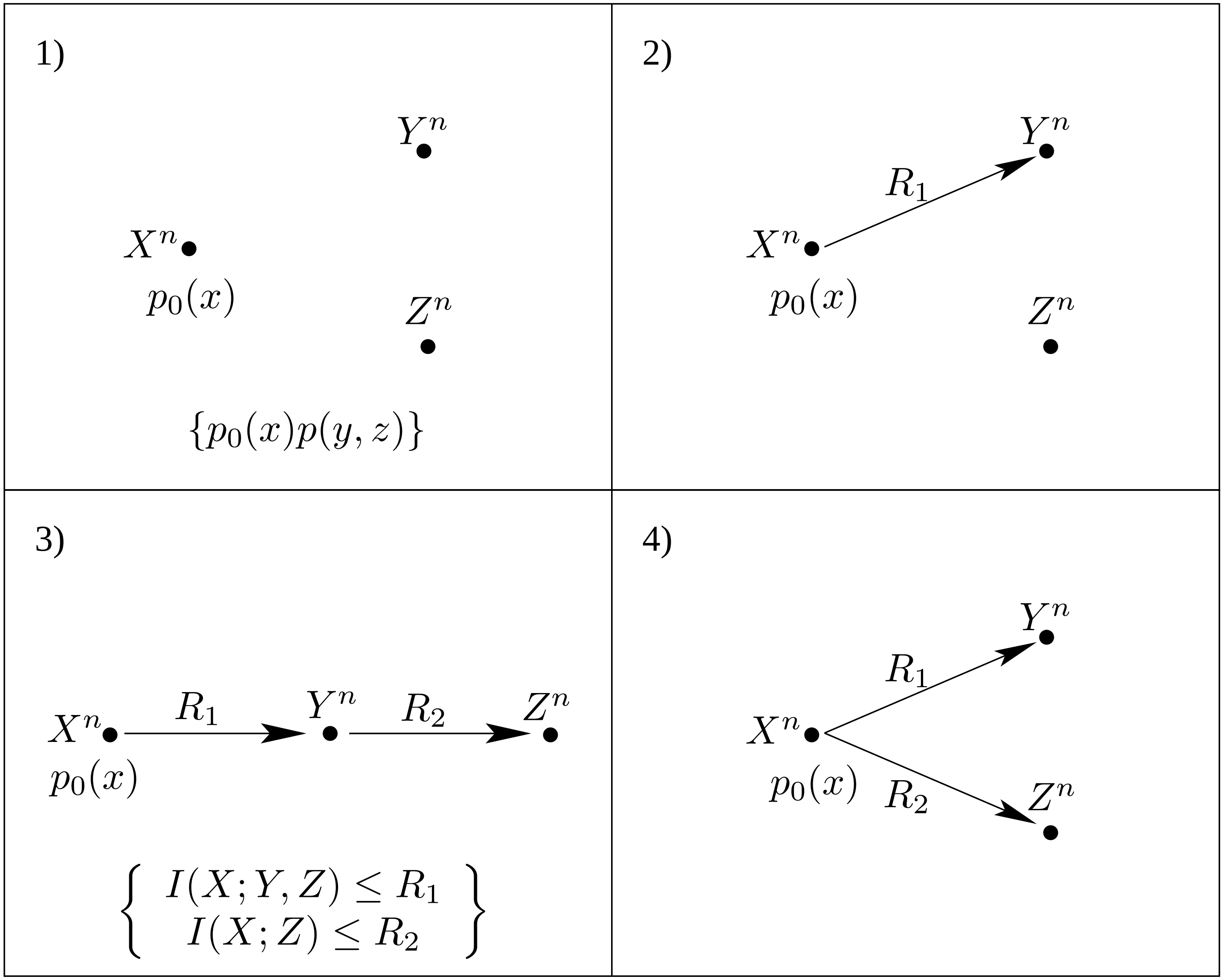
$$\begin{aligned} R_1 &\geq I(X;Y,Z), \\ R_2 &\geq I(X;Z). \end{aligned}$$

Converse: Follows from the two node case.

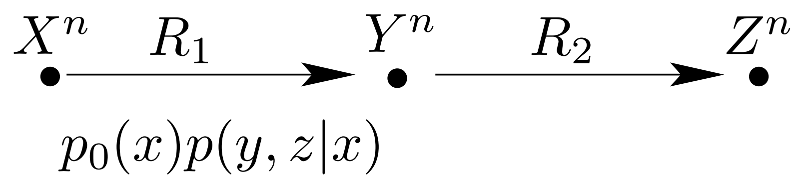
Achievability: First transmit information from X to Z (i.e., $R_2 = I(X;Z)$).

Then use Z as side information for X and Y (i.e., $R_1 = R_2 + I(X;Y|Z)$).

Building Blocks of Distributed Action Problems



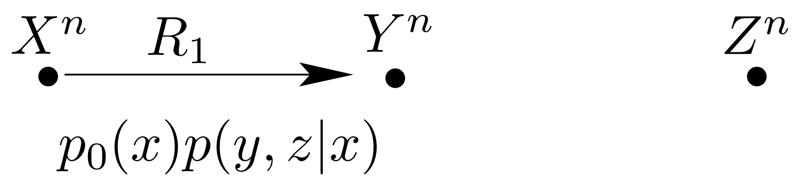
Three nodes one rate



Theorem. *The achievable region for a distribution* $p_0(x)p(y,z|x)$ *is*

$$\begin{aligned} R_1 &\geq I(X;Y,Z), \\ R_2 &\geq I(X;Z). \end{aligned}$$

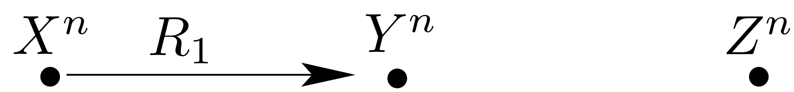
Three nodes one rate



Theorem. *The achievable region for a distribution* $p_0(x)p(y,z|x)$ *is*

$$
\begin{aligned}
R_1 &\geq I(X;Y,Z), \\
0 &= I(X;Z).
\end{aligned}
$$

Three nodes one rate

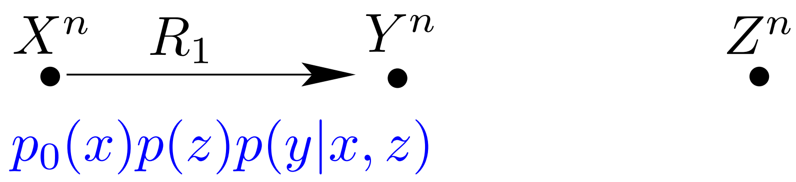


$p_0(x)p(z)p(y|x,z)$

Theorem. *The achievable region for a distribution* $p_0(x)p(z)p(y|x,z)$ *is*

$$R_1 \quad \geq \quad I(X;Y,Z).$$

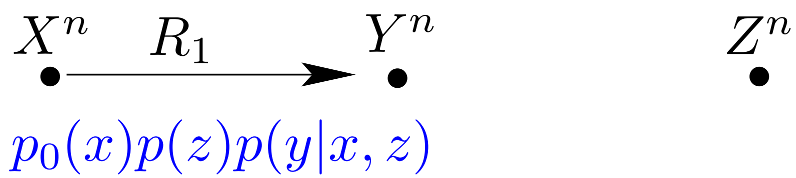
Three nodes one rate



Theorem. *The achievable region for a distribution* $p_0(x)p(z)p(y|x,z)$ *is*

$$R_1 \geq I(X;Y|Z).$$

Three nodes one rate

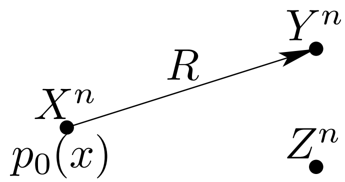


Theorem. *The achievable region for a distribution* $p_0(x)p(z)p(y|x,z)$ *is*

$$R_1 \geq I(X;Y|Z).$$

There is a tension between the dependence of X and Y, and between the dependence of Y and Z.

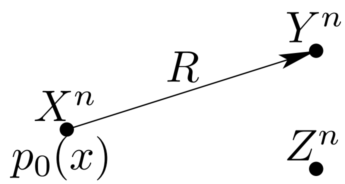
Three nodes one rate: The Gaussian Case



$p_0(x)$

$$(X,Y,Z) \sim N\left(0, \begin{bmatrix} 1 & \rho_{xy} & 0 \\ \rho_{xy} & 1 & \rho_{yz} \\ 0 & \rho_{yz} & 1 \end{bmatrix}\right)$$

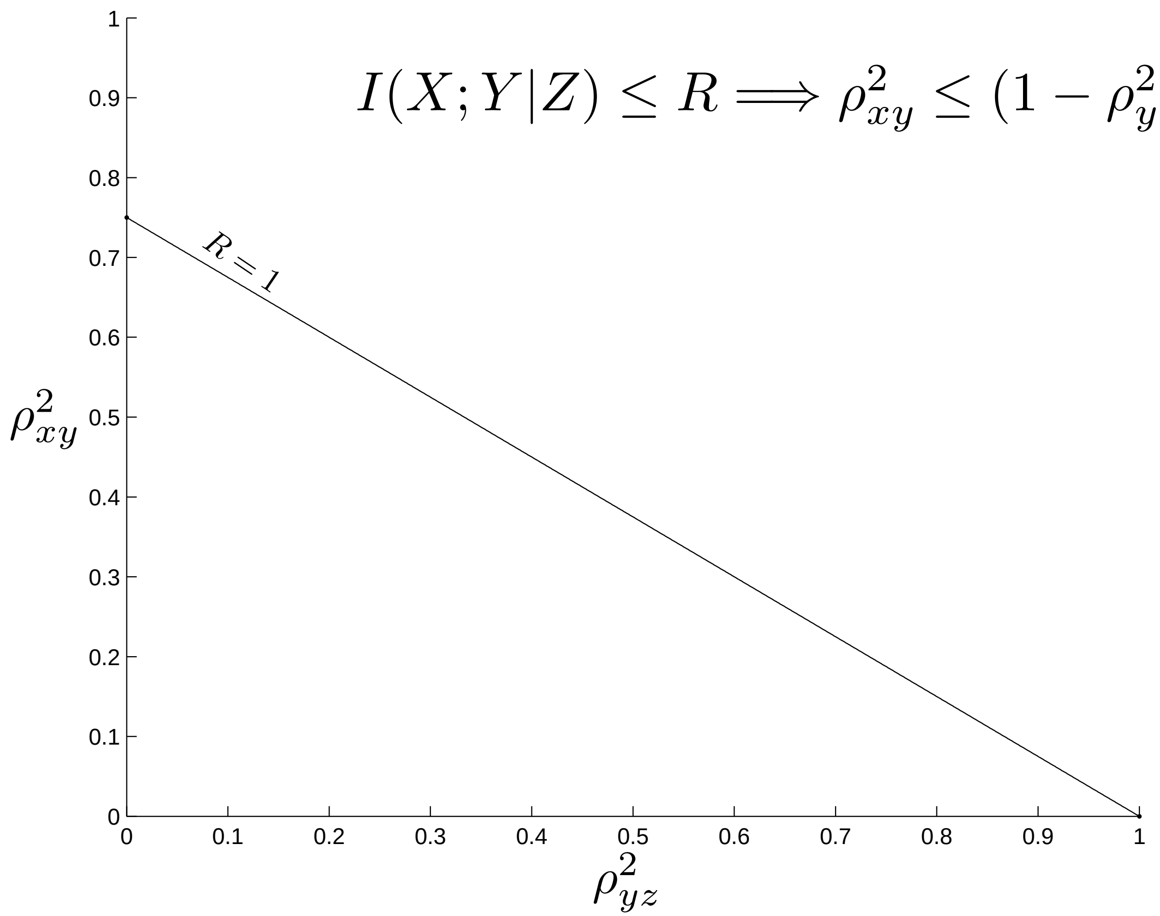
Three nodes one rate: The Gaussian Case



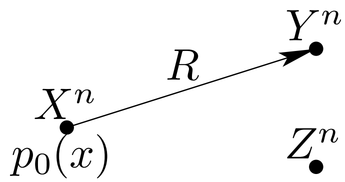
$p_0(x)$

$$(X,Y,Z) \sim N\left(0, \begin{bmatrix} 1 & \rho_{xy} & 0 \\ \rho_{xy} & 1 & \rho_{yz} \\ 0 & \rho_{yz} & 1 \end{bmatrix}\right)$$

$$I(X;Y|Z) \leq R \Longrightarrow \rho_{xy}^2 \leq (1-\rho_{yz}^2)(1-2^{-2R})$$

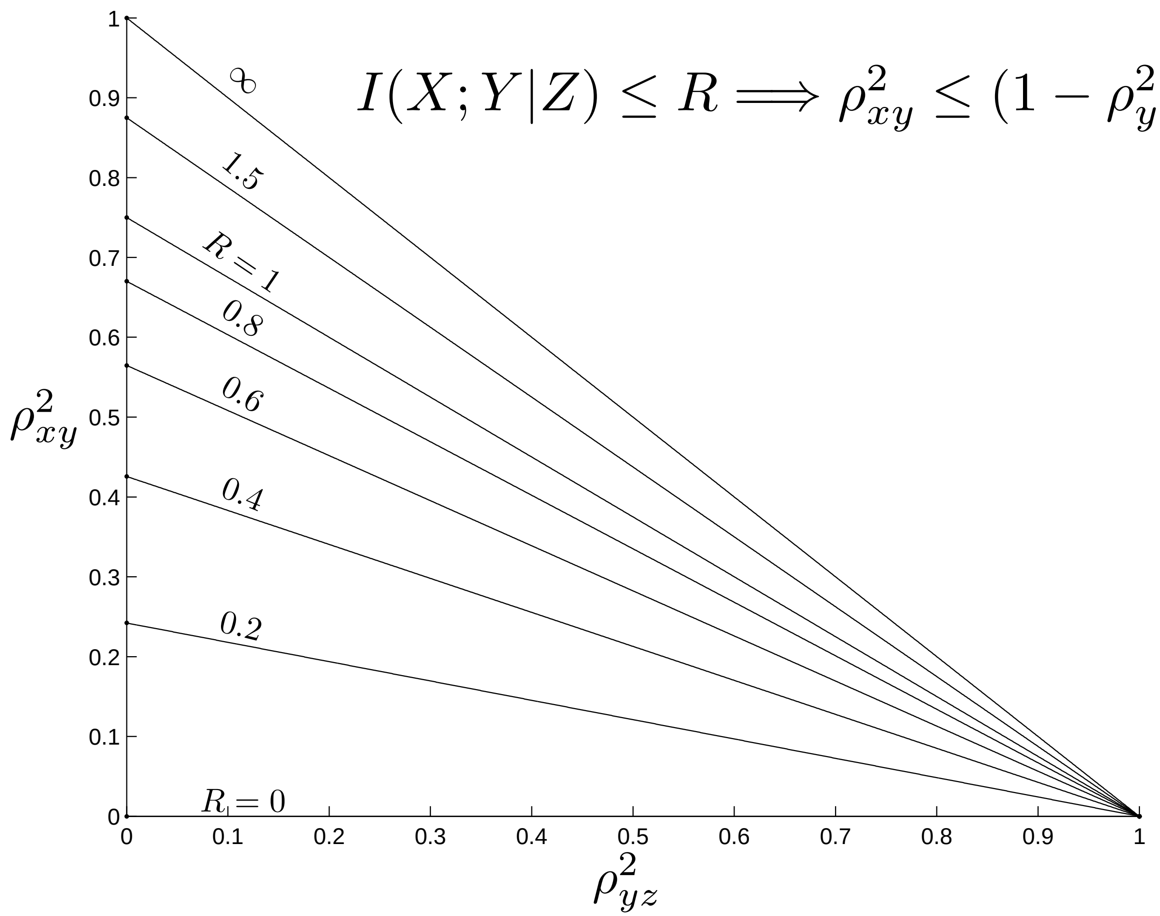


Three nodes one rate: The Gaussian Case

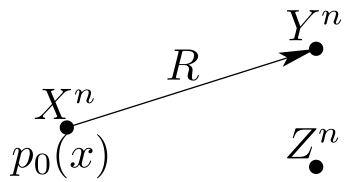


$$(X,Y,Z) \sim N\left(0, \begin{bmatrix} 1 & \rho_{xy} & 0 \\ \rho_{xy} & 1 & \rho_{yz} \\ 0 & \rho_{yz} & 1 \end{bmatrix}\right)$$

$$I(X;Y|Z) \leq R \Longrightarrow \rho_{xy}^2 \leq (1-\rho_{yz}^2)(1-2^{-2R})$$

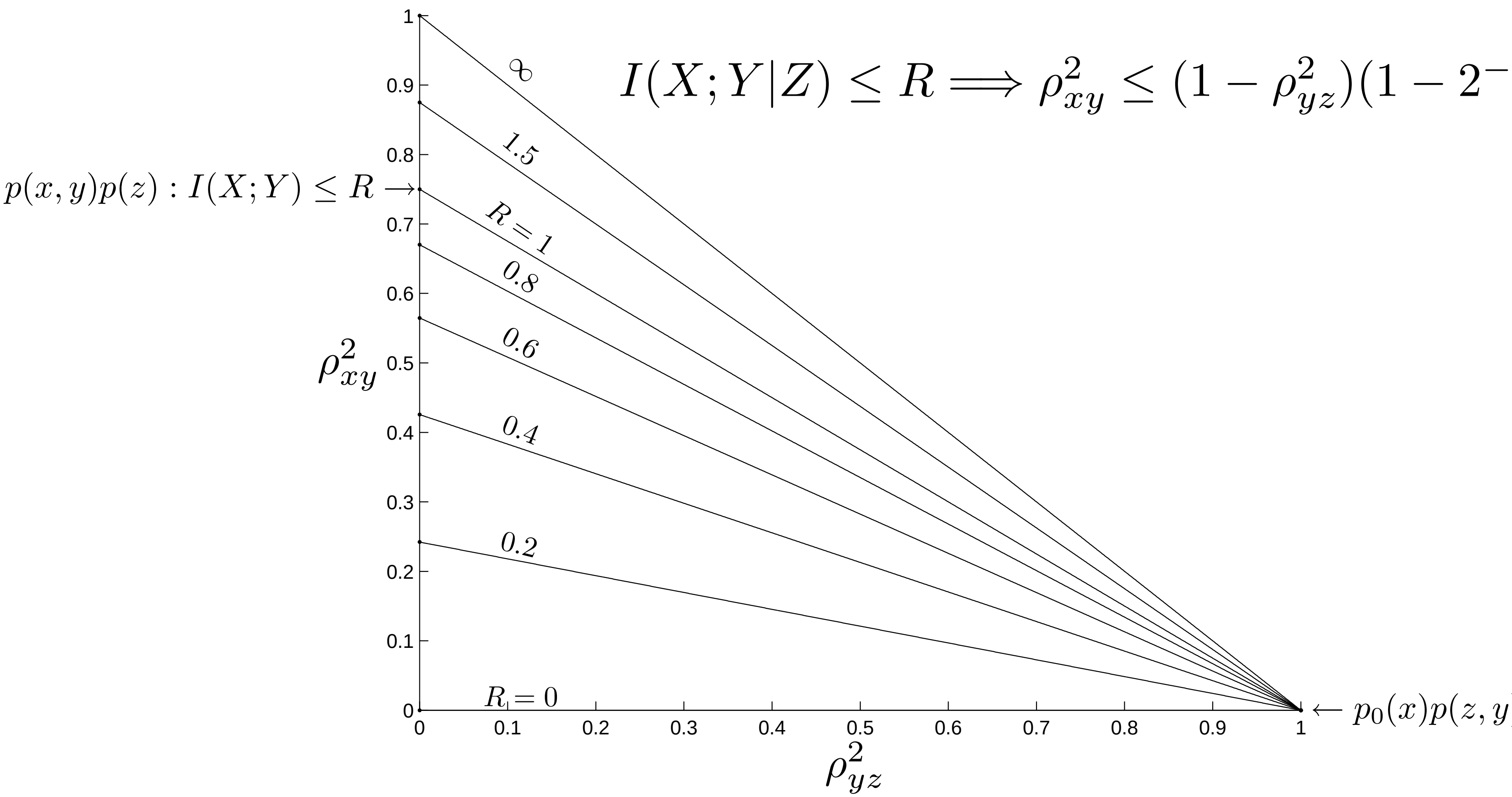


Three nodes one rate: The Gaussian Case

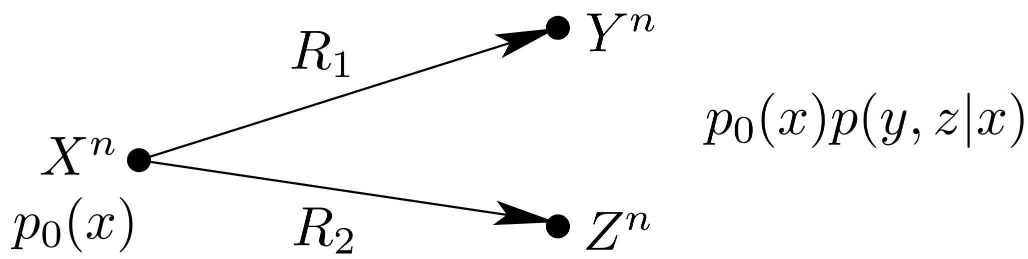


$$(X,Y,Z) \sim N\left(0, \begin{bmatrix} 1 & \rho_{xy} & 0 \\ \rho_{xy} & 1 & \rho_{yz} \\ 0 & \rho_{yz} & 1 \end{bmatrix}\right)$$

$$I(X;Y|Z) \le R \Longrightarrow \rho_{xy}^2 \le (1-\rho_{yz}^2)(1-2^{-2R})$$



Broadcast Distributed Action



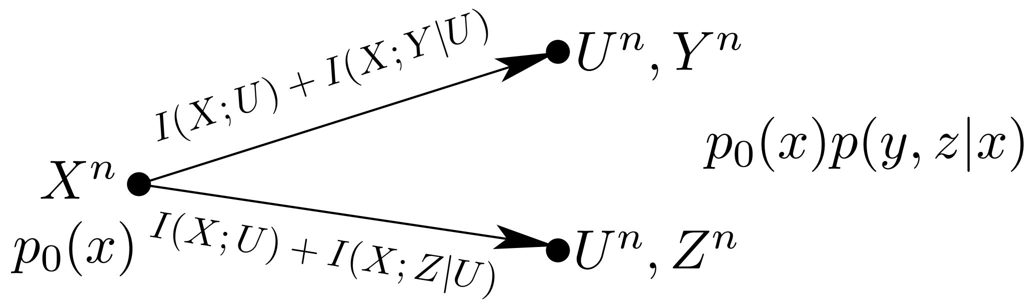
An achievable region

Theorem. *$p_0(x)p(y,z|x)$ is achievable if the rates R_1, R_2 satisfies*

$$\begin{aligned} R_1 &> I(X;U,Y), \\ R_2 &> I(X;U,Z), \end{aligned}$$

for some $Y - (X,U) - Z$.

Broadcast Distributed Action



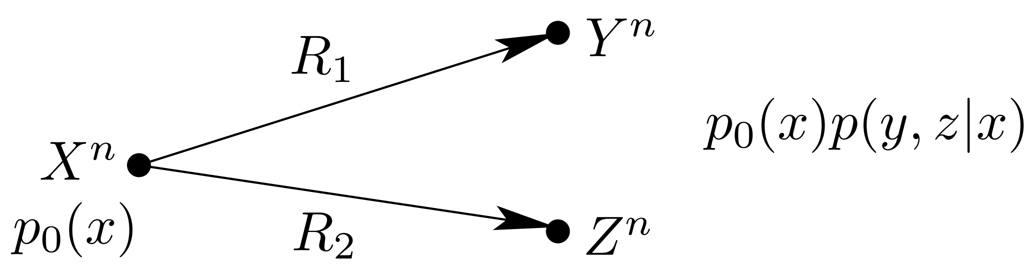
An achievable region

Theorem. $p_0(x)p(y,z|x)$ *is achievable if the rates* R_1, R_2 *satisfies*

$$
\begin{aligned}
R_1 &> I(X;U,Y),\\
R_2 &> I(X;U,Z),
\end{aligned}
$$

for some $Y - (X,U) - Z$.

Broadcast Distributed Action



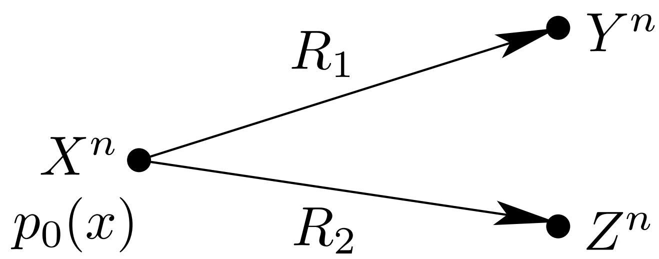
An achievable region

Theorem. $p_0(x)p(y,z|x)$ *is achievable if the rates* R_1, R_2 *satisfies*

$$\begin{aligned} R_1 &> I(X;V) + \min\{I(X;U,Y|V), I(X,U;Y|V)\}, \\ R_2 &> I(X;V) + \min\{I(X;U,Z|V), I(X,U;Z|V)\}, \end{aligned}$$

for some $Y - (V, X, U) - Z$.

Broadcast Distributed Action



$p_0(x)p(y,z|x)$

<u>An achievable region</u>

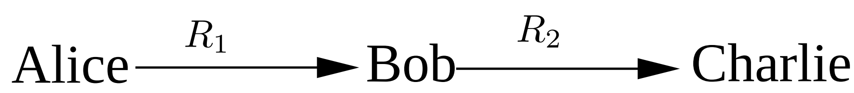
Theorem. $p_0(x)p(y,z|x)$ *is achievable if the rates* R_1, R_2 *satisfies*

$$\begin{aligned} R_1 &> I(X;V) + \min\{I(X;U,Y|V), I(X,U;Y|V)\}, \\ R_2 &> I(X;V) + \min\{I(X;U,Z|V), I(X,U;Z|V)\}, \end{aligned}$$

for some $Y - (V, X, U) - Z$.

If $X - Y - Z$ or $Y - X - Z$, then the region is optimum.

Solving the Cascade Question

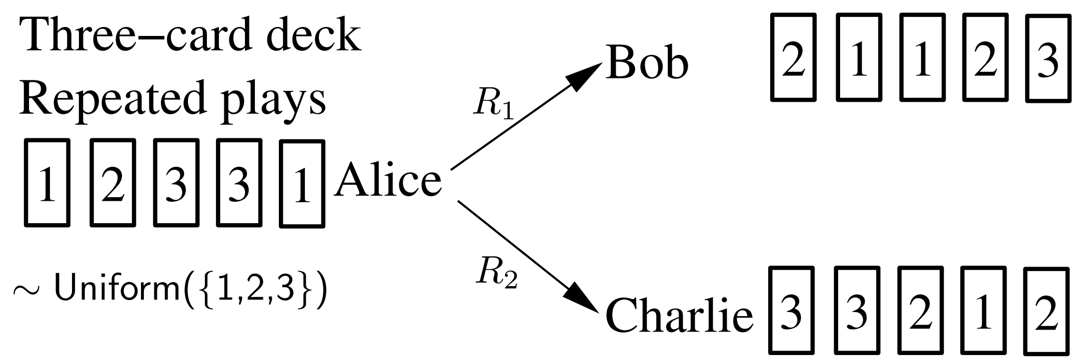


$X \sim$ Uniform{1,2,3}

Alice (X), Bob (Y) and Charlie's (Z) actions need to be distributed uniformly over the six possible permutations of $\{1, 2, 3\}$.

$$
\begin{aligned}
R_1 &\geq I(X; Y, Z) = \log 3, \\
R_2 &\geq I(X; Z) = \log \frac{3}{2}.
\end{aligned}
$$

An Achievable Scheme for Question 1



We restrict $Y = \{1, 2\}$ and $Z = \{2, 3\}$. Y will choose 1 and Z will choose 3 as default, unless X tells to one of them to choose 2.

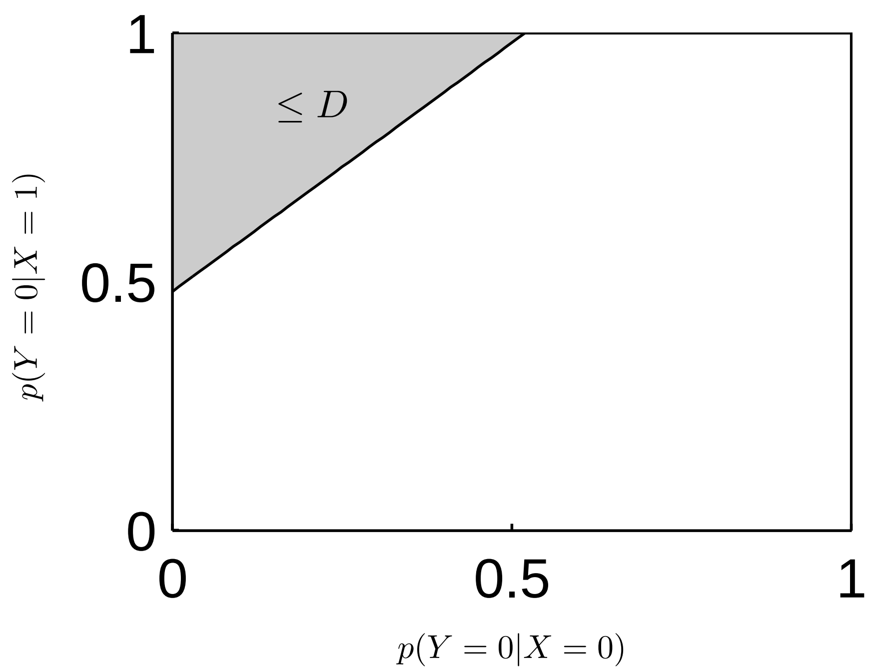
$$R_1 \quad \geq \quad I(X; U, Y) = H(\frac{1}{3}) - 0 = 0.918$$

Paul Cuff showed that $R_1 = R_2 = 0.890$ is achievable.

Coordinated Action and Rate Distortion

In rate distortion problems the distortion is a **linear function** of the distribution. Consider a source $X \sim$ Bernoulli($\frac{1}{2}$) and a reconstruction Y.

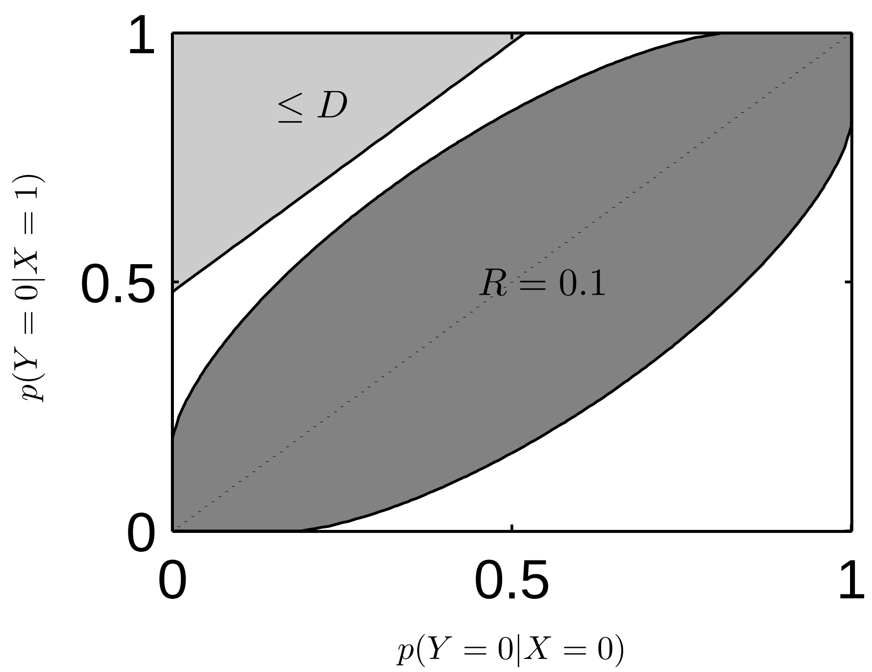
$$D = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x,y) p_0(x) p(y|x) = \frac{1}{2}(1 - p(Y=0|X=0) + p(Y=0|X=1)) \quad (1)$$



Coordinated Action and Rate Distortion

In rate distortion problems the distortion is a **linear function** of the distribution. Consider a source $X \sim$ Bernoulli($\frac{1}{2}$) and a reconstruction Y.

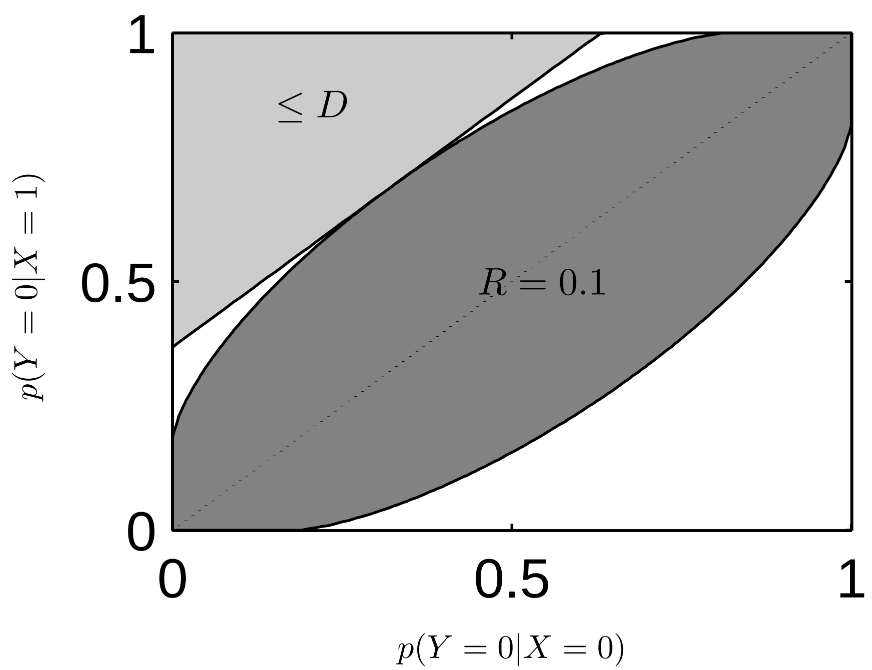
$$D = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x,y) p_0(x) p(y|x) = \frac{1}{2}(1 - p(Y=0|X=0) + p(Y=0|X=1)) \tag{2}$$



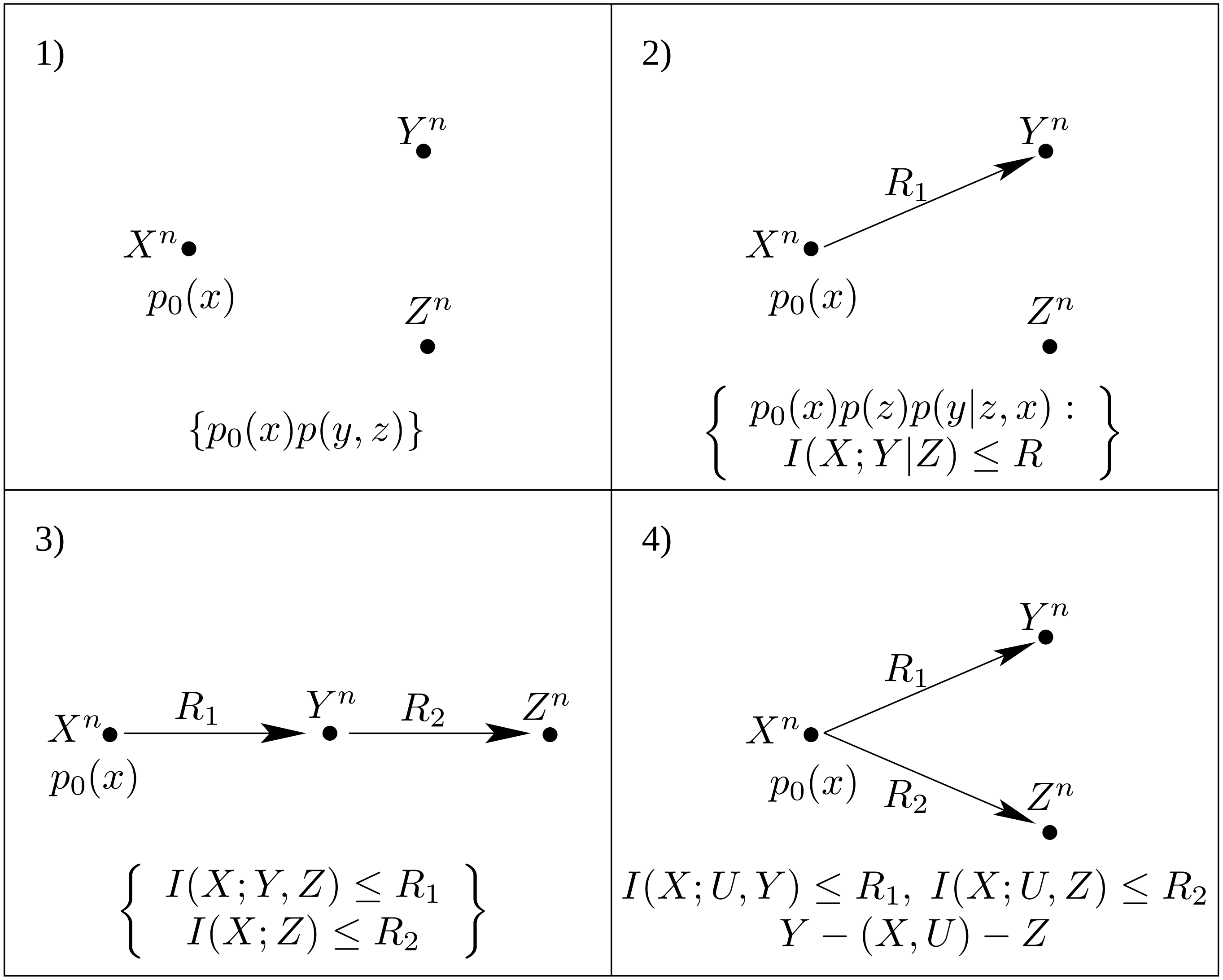
Coordinated Action and Rate Distortion

In rate distortion problems the distortion is a **linear function** of the distribution. Consider a source $X \sim$ Bernoulli($\frac{1}{2}$) and a reconstruction Y.

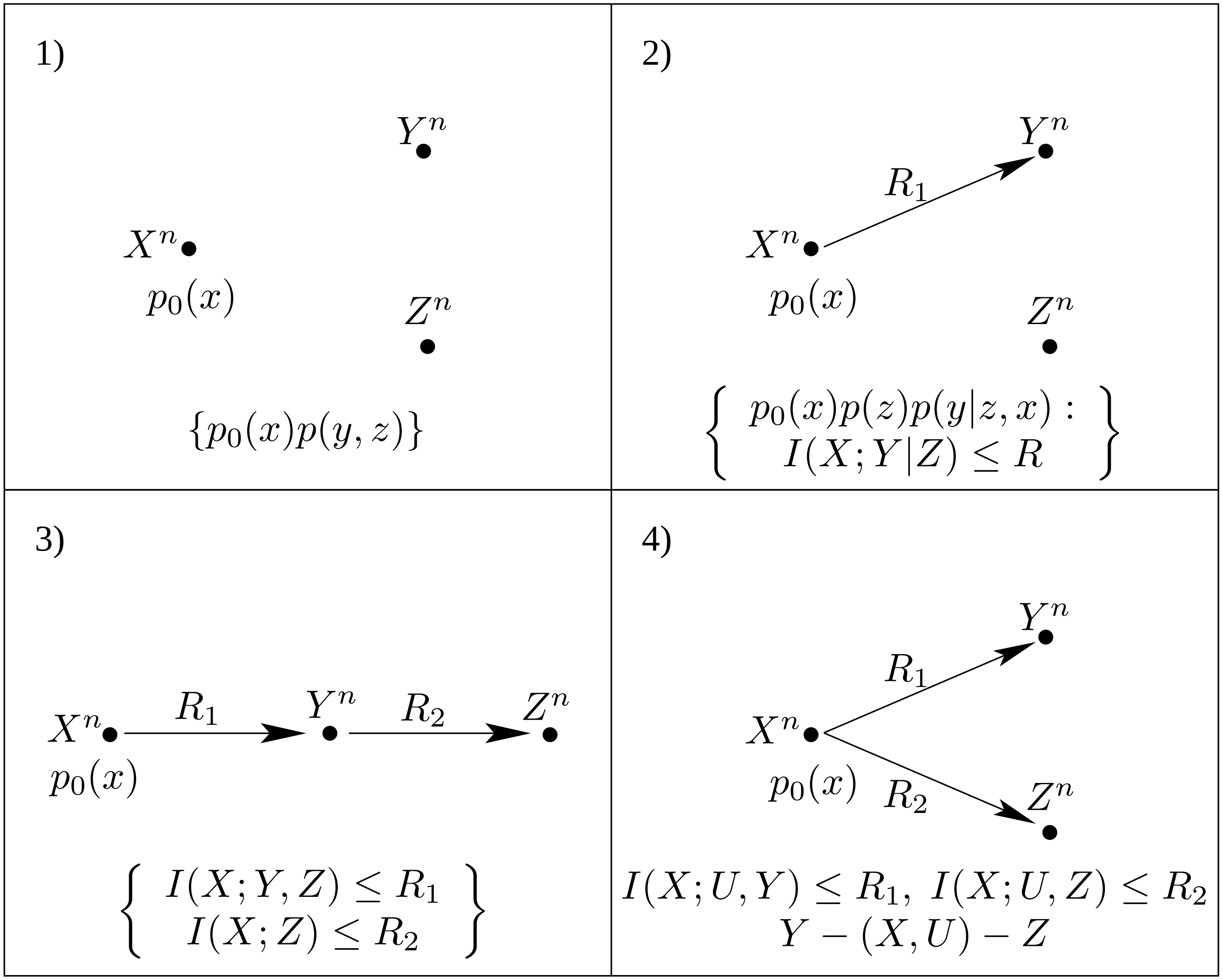
$$D = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x,y) p_0(x) p(y|x) = \frac{1}{2}(1 - p(Y=0|X=0) + p(Y=0|X=1)) \tag{3}$$



Summary

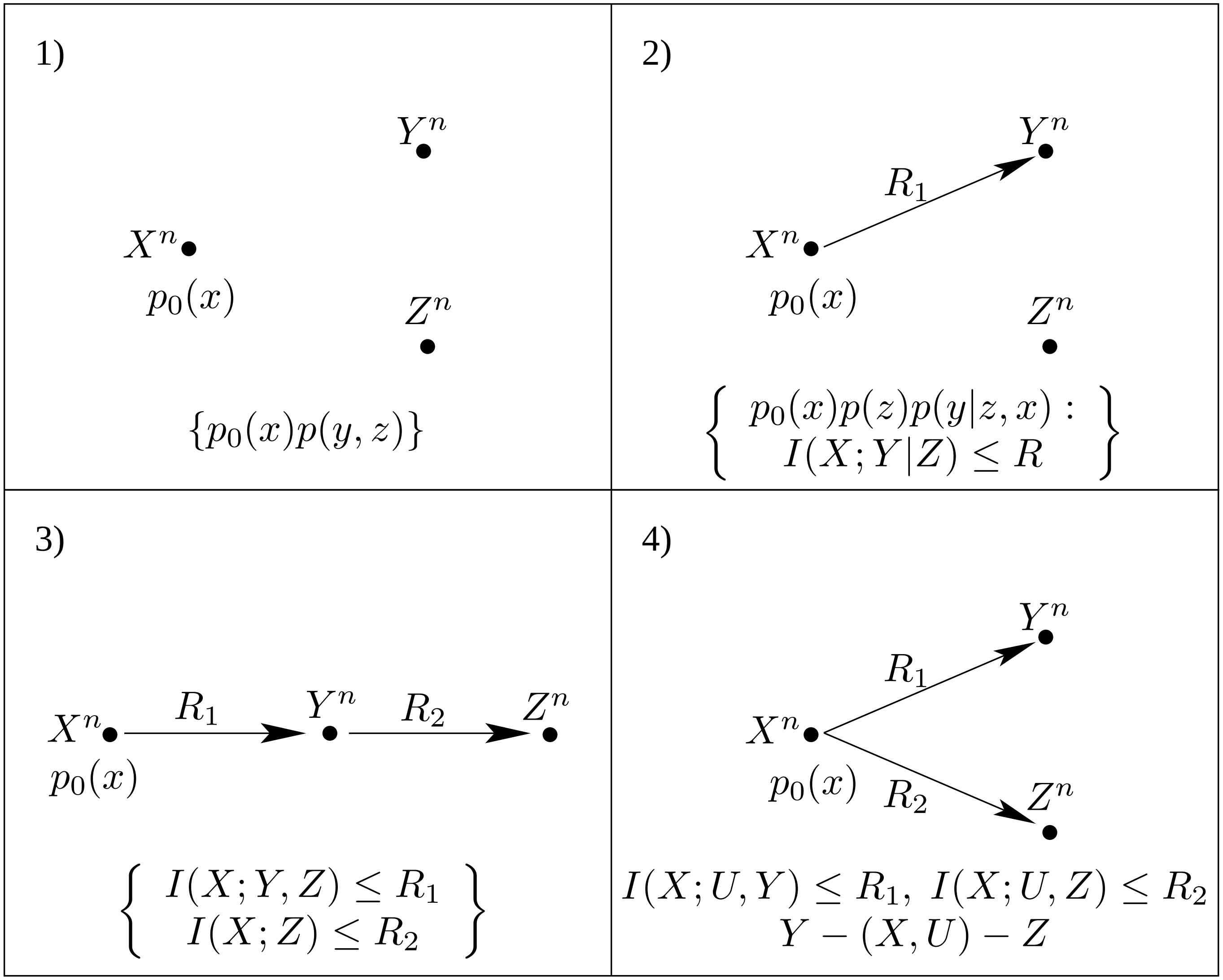

1)
Y^n
X^n
$p_0(x)$
Z^n
$\{p_0(x)p(y,z)\}$
2)
Y^n
R_1
X^n
$p_0(x)$
Z^n
$\left\{ \begin{array}{c} p_0(x)p(z)p(y|z,x) : \\ I(X;Y|Z) \leq R \end{array} \right\}$
3)
X^n
R_1
Y^n
R_2
Z^n
$p_0(x)$
$\left\{ \begin{array}{c} I(X;Y,Z) \leq R_1 \\ I(X;Z) \leq R_2 \end{array} \right\}$
4)
Y^n
R_1
X^n
$p_0(x)$
R_2
Z^n
$I(X;U,Y) \leq R_1,\ I(X;U,Z) \leq R_2$
$Y - (X,U) - Z$

Summary



Acknowledgments: Paul Cuff, Young-Han Kim, Taesup Moon, Tsachy Weissman

Summary



Acknowledgments: Paul Cuff, Young-Hun Kim, Taesup Moon, Tsachy Weissman

$Thank\ You$!