Cascade and Triangular Source Coding with Side Information at the First Two Nodes

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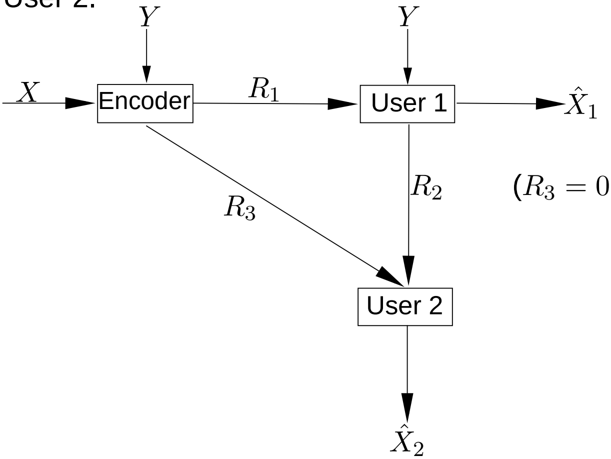
Triangular setting

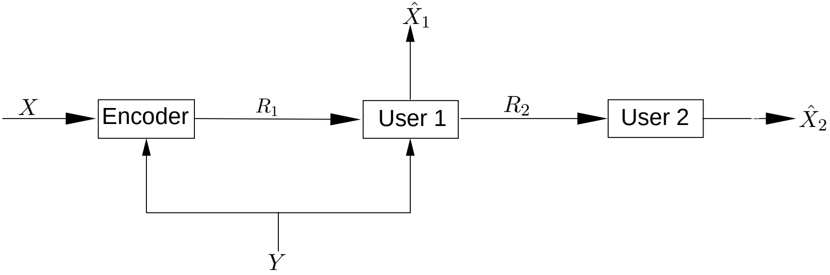
Side information Y is known to the encoder and User 1, but not to User 2.



($R_3 = 0$, cascade case)

The goal: $\mathbb{E}\left[d(X^n, \hat{X}_1^n)\right] \leq D_1, \mathbb{E}\left[d(X^n, \hat{X}_2^n)\right] \leq D_2.$

Cascade setting



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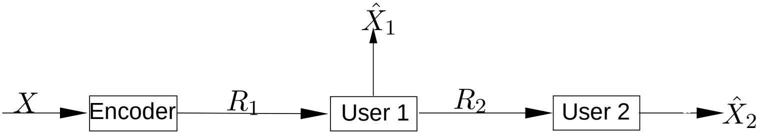
Previous work: Cascade case, no side information



Theorem

$$\begin{aligned} R_1 &\geq I(X;\hat{X}_1,\hat{X}_2) \\ R_2 &\geq I(X;\hat{X}_2) \end{aligned}$$

for some joint distribution $P_{X,\hat{X}_1,\hat{X}_2}$ *s.t.*
$\mathbb{E}d_i(X,\hat{X}_i) \leq D_i, \quad i=1,2.$ *Yamamoto81*

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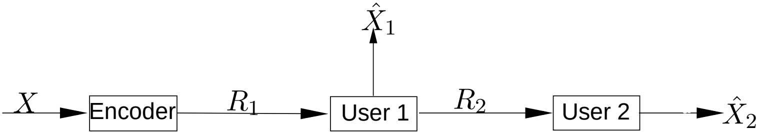

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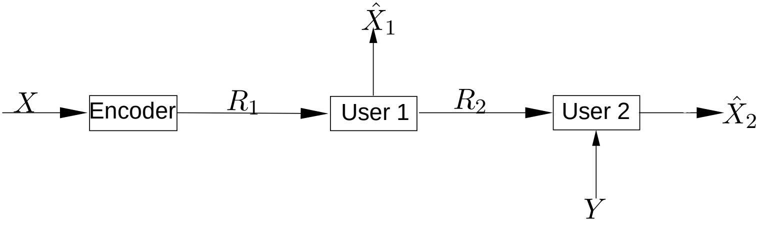
Converse: Cut set bound.
Achievability: Use $\hat{X}_2$ as side information for User 1. Note:
$R_1 \geq I(X;\hat{X}_1|\hat{X}_2) + I(X;\hat{X}_2)$

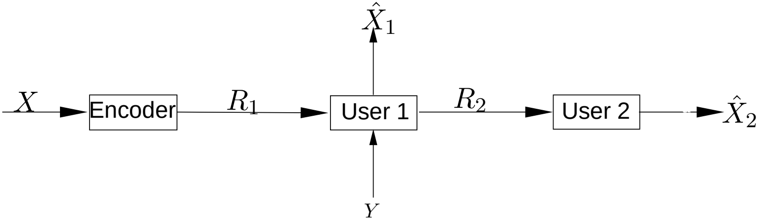
Previous work: Cascade case, with side information at User 2



Vasudevan/Tian/Diggavi06

- provided inner and outer bound
- solved for the Gaussian case
- still open for the general case

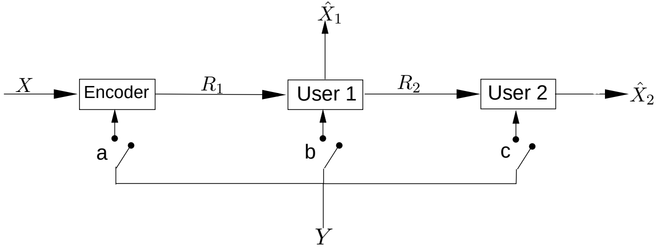
Previous work: Cascade case, with side information at User 1



Su/Cuff/El Gamal09

- provided inner and outer bound
- both relaying and recompressing information are needed
- still open for the Gaussian and the general case

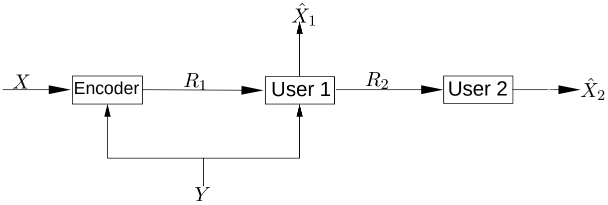
Literature summary



Switch a	Switch b	Switch c	Gaussian case	General case
open	open	open	Solved [1]	Solved [1]
open	open	closed	Solved [2]	bounds [2]
open	closed	open	bounds [3]	bounds [3]
open	closed	closed	Solved [2]	bounds [2]
closed	open	closed	Solved [2]	bounds [2]
closed	closed	open	In this talk	In this talk

[1] Yamamoto81, [2] Vasudevan/Tian/Diggavi06,
[3]Su/Cuff/El-Gamal09

Cascade with side information

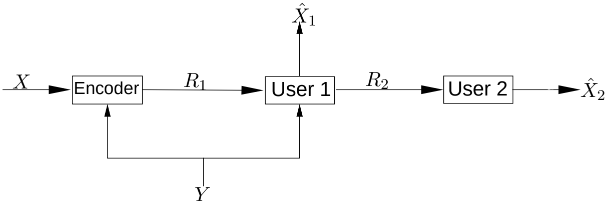


Theorem

$$\begin{aligned} R_1 &\geq I(X;\hat{X}_1,\hat{X}_2|Y), \\ R_2 &\geq I(Y,X;\hat{X}_2), \end{aligned}$$

for $P_{X,Y,\hat{X}_1,\hat{X}_2}$ *s.t.* $\mathbb{E}d_i(X,\hat{X}_i) \leq D_i, \quad i=1,2.$

Cascade with side information



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$$\begin{aligned} R_1 &\geq I(X; \hat{X}_1, \hat{X}_2 | Y), \\ R_2 &\geq I(Y, X; \hat{X}_2), \end{aligned}$$

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Achievability:

$$R_1 \geq \underbrace{I(X, Y; \hat{X}_2) - I(Y; \hat{X}_2)}_{\text{Achieved by binning}} + \underbrace{I(X; \hat{X}_1 | \hat{X}_2, Y)}_{\text{Achieved by Side information}}$$

Converse: Cut set bound.

Cascade Gaussian case

Lemma

It suffices to consider only Gaussian distributions

Cascade Gaussian case

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Proof.

- Given a distribution $P_{X,Y,\hat{X}_1,\hat{X}_2}$ with a covariance K we generate a new distribution $\tilde{P}_{X,Y,\hat{X}_1,\hat{X}_2} \sim \text{Norm}(0, K)$.
- we show

$$\begin{aligned} R_1 &\geq I_P(X;\hat{X}_1,\hat{X}_2|Y), \\ &\geq I_{\tilde{P}}(X;\hat{X}_1,\hat{X}_2|Y), \end{aligned}$$

$$\begin{aligned} R_2 &\geq I_P(Y,X;\hat{X}_2) \\ &\geq I_{\tilde{P}}(Y,X;\hat{X}_2|Y), \end{aligned}$$

- Clearly, the quadratic distortion depends only on the covariance matrix K.

Finding an explicit solution for the Gaussian case

- assume Gaussian distribution and obtain the following optimization problem:

$$\begin{aligned} \text{maximize (over } \alpha) \quad & W \\ \text{subject to} \quad & W \geq \alpha^2 + b\alpha + c \\ & W \leq a'\alpha^2 + b\alpha + c' \end{aligned}$$

  a', b, b', c, c' are constants.
- divide into four cases according to the coefficients of the quadratic forms. Each case can be solved analytically.

The explicit rate Gaussian rate-region

W.l.o.g assume $Y = X + Z$

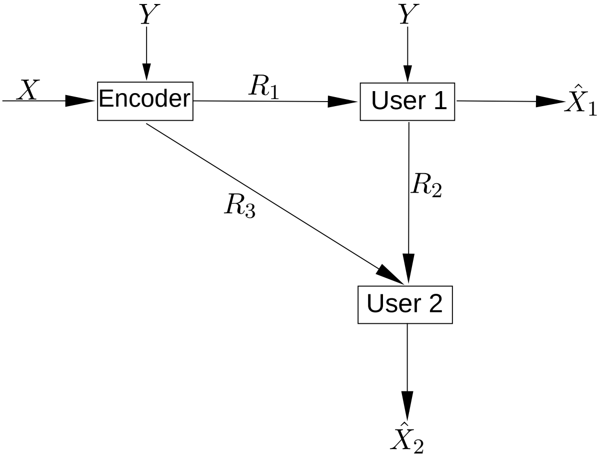
$$R_1(D_1, D_2, R_2) = \frac{1}{2}\max\left(\log\frac{\sigma^2_{X|Y}}{\sigma^2_{X|W,Y}}, \log\frac{\sigma^2_{X|Y}}{D_1}, 0\right),$$

where $\sigma^2_{X|W,Y}$ is given by the following four cases

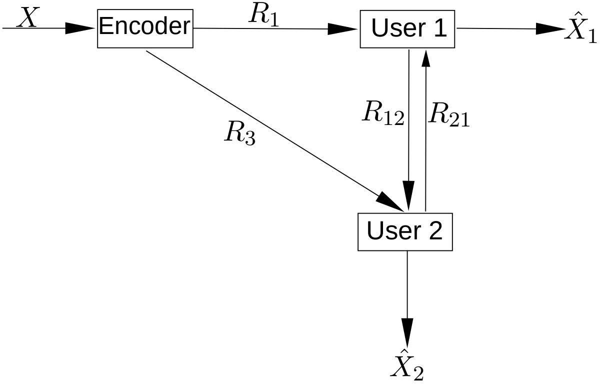
$$\begin{cases} \left(\frac{2^{2R_2}D_2-\sigma_X^2}{\sigma_Z^2\sigma_X^2\alpha^2}+\sigma_{X|Y}^{-2}\right)^{-1}, & \text{if } D_2 \le \sigma^2_{X|Y} \text{ and } \frac{\sigma_X^2}{D_2} \le 2^{2R_2} \le \frac{\sigma_Z^2(\sigma_X^2-D_2)}{\sigma_Z^2\sigma_X^2-D_2\sigma_Z^2-D_2\sigma_X^2}\frac{\sigma_X^2}{D_2} \\ D_2, & \text{if } D_2 \le \sigma^2_{X|Y} \text{ and } 2^{2R_2} \ge \frac{\sigma_Z^2(\sigma_X^2-D_2)}{\sigma_Z^2\sigma_X^2-D_2\sigma_Z^2-D_2\sigma_X^2}\frac{\sigma_X^2}{D_2} \\ \left(\frac{2^{2R_2}D_2-\sigma_X^2}{\sigma_Z^2\sigma_X^2\alpha^2}+\sigma_{X|Y}^{-2}\right)^{-1}, & \text{if } D_2 \ge \sigma^2_{X|Y} \text{ and } \frac{\sigma_X^2}{D_2} \le 2^{2R_2} \le \frac{\sigma_X^4}{\sigma_X^2D_2+\sigma_Z^2D_2-\sigma_X^2\sigma_Z^2} \\ \sigma^2_{X|Y}, & \text{if } D_2 \ge \sigma^2_{X|Y}, \text{ and } 2^{2R_2} \ge \frac{\sigma_X^4}{\sigma_X^2D_2+\sigma_Z^2D_2-\sigma_X^2\sigma_Z^2} \end{cases}$$

and $\alpha = \left(\frac{\sigma_Z}{\sigma_X}\sqrt{\frac{\sigma_X^2-D_2}{D_2-\sigma_X^2 2^{-2R_2}}}-1\right)^{-1}$.

Triangular source coding with side information



Triangular source coding **without** side information



Solved by Yamamoto96

Rate-region of triangular source coding with side information

Theorem

$$
\begin{aligned}
R_1 &\geq I(X;\hat{X}_1,U|Y), &(1)\\
R_2 &\geq I(Y,X;U), &(2)\\
R_3 &\geq I(X;\hat{X}_2|U), &(3)
\end{aligned}
$$

for some joint distribution $P(x,y)P(\hat{x}_1,\hat{x}_2,u|x,y)$ *for which*

$$
\mathbb{E}d_i(X,\hat{X}_i) \leq D_i, \quad i=1,2. \qquad (4)
$$

Rate-region of triangular source coding with side information

Theorem

$$R_1 \geq I(X; \hat{X}_1, U | Y), \quad (1)$$
$$R_2 \geq I(Y, X; U), \quad (2)$$
$$R_3 \geq I(X; \hat{X}_2 | U), \quad (3)$$

for some joint distribution $P(x,y)P(\hat{x}_1, \hat{x}_2, u | x, y)$ *for which*

$$\mathbb{E} d_i(X, \hat{X}_i) \leq D_i, \quad i = 1, 2. \quad (4)$$

Achievability:

- show that it does not decrease when $\hat{X}_2 - (X, U) - (\hat{X}_1, Y)$.
- Use cascade achievability and then use U as side information $R_3 \geq I(X; \hat{X}_2 | U)$

Gaussian triangular case

Can we claim that it suffices to consider Gaussian distribution (like in the Cascade case)?

- Problem: Let K be a covariance matrix induced by a distribution P. Let $\tilde{P}$ be Norm$(0, K)$. Then,

$$R_3 \geq I_P(X; \hat{X}_2|U) \ngeq I_{\tilde{P}}(X; \hat{X}_2|U).$$

Gaussian triangular case

Can we claim that it suffices to consider Gaussian distribution (like in the Cascade case)?

- Problem: Let K be a covariance matrix induced by a distribution P. Let $\tilde{P}$ be Norm$(0, K)$. Then,

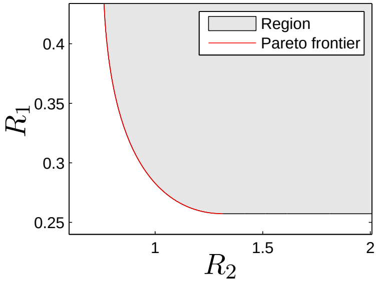
$$R_3 \quad \geq \quad I_P(X;\hat{X}_2|U) \not\geq I_{\tilde{P}}(X;\hat{X}_2|U).$$

- Solution
  - claim that it is enough to consider equalities instead on inequalities using Pareto frontier definition.
  - change the equalities
  - show that Gaussian is optimal for the new set of equations

Pareto frontier

Definition

Pareto frontier of a region is the set of all points, for which there is no strictly better point in the region.



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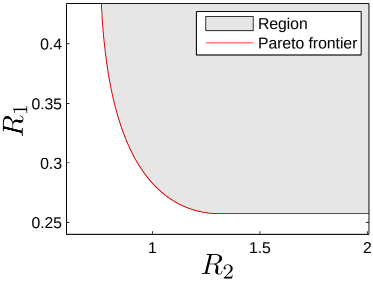


Lemma

If two rate-regions, $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same Pareto frontier, then they are identical.

Optimality of Gaussian distribution for the Gaussian triangular case

The Pareto frontier is

$$R_1 = I(X;\hat{X}_1,U|Y), \quad (5)$$
$$R_2 = I(Y,X;U), \quad (6)$$
$$R_3 = I(Y,X;\hat{X}_2|U). \quad (7)$$

Equivalently, (7) may be replaced by

$$R_3 + R_2 = I(Y,X;\hat{X}_2,U).$$

Now, we can show that

$$\begin{aligned} R_1 &= I_P(X;\hat{X}_1,U|Y) \geq I_{\tilde{P}}(X;\hat{X}_1,U|Y), \\ R_2 &= I_P(Y,X;U) \geq I_{\tilde{P}}(Y,X;U), \\ R_3 + R_2 &= I_P(Y,X;\hat{X}_2,U) \geq I_{\tilde{P}}(Y,X;\hat{X}_2,U). \end{aligned}$$

Hence, $\tilde{P} = \text{Norm}(0,K)$ suffices.

Explicit solution for the Gaussian triangular case

- Assume a joint Gaussian distribution $P_{X,Y,U,\hat{X}_1,\hat{X}_2}$
- We note that the third inequality

$$R_3 \geq I(X;\hat{X}_2|U)$$

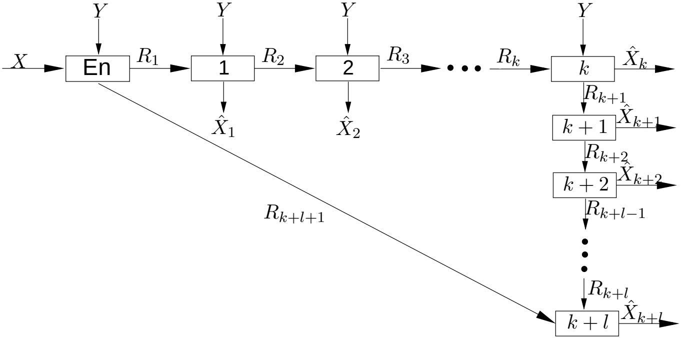
can be always assumed to be equality. Therefore

$$\sigma^2_{X|\hat{X}_2,U} = \sigma^2_{X|U} 2^{-2R_3}. \tag{8}$$

- Using (8), we obtain an optimization problem identical to the cascade case but D_2 is replaced by $D_2 2^{2R_3}$

$$\mathcal{R}_{triangular}(R_1, R_2, R_3, D_1, D_2) = \mathcal{R}_{cascade}(R_1, R_2, D_1, D_2 2^{2R_3})$$

Extensions to $k + l$ users



Rate-distortion region:

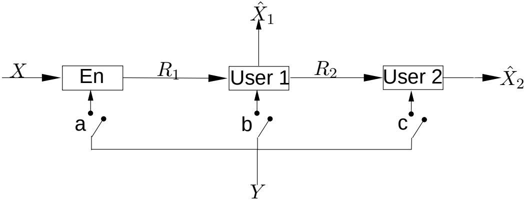
$$\begin{aligned} R_i &\geq I(X;\hat{X}_i,\hat{X}_{i+1},...,\hat{X}_{k+l-1},U|Y),\ 1\leq i\leq k \\ R_j &\geq I(X;\hat{X}_j,...,\hat{X}_{k+l-1},U), \qquad k+1\leq j\leq k+l \\ R_{k+l+1} &\geq I(X;\hat{X}_{k+l}|U), \end{aligned}$$

for some distribution

$P_{X,Y,\hat{X}_1,\hat{X}_2,...,\hat{X}_k,U|X,Y}$ s.t. $\mathbb{E}d_i(X,\hat{X}_i)\leq D_i,\ \ 1\leq i\leq k+l.$

Summary

- Most of the cascade and triangular source coding with side information are still open problems



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- Some proof ideas:
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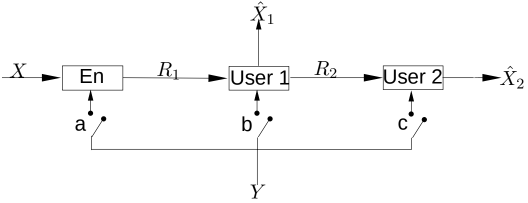

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Thank you very much!