Multiple-Access Channel with Delayed State Information Via Directed Information

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Outline

- Motivation
- Channel model
- Main results
- Proof via directed information
- Capacity region for a finite state additive Gaussian MAC
- Capacity region for a finite state multiple-access fading channel
- Summary

Motivation

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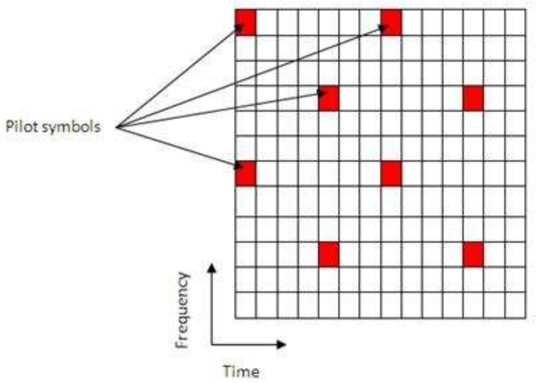
- Instantaneous channel state information (CSI) at the transmitters is often an unrealistic assumption in wireless communications
- The CSI can be transmitted to the transmitters through feedback
- But the CSI feedback is not instantaneous
- Motivated by this we studied the problem of Finite State Markov (FSM) Multiple-Access Channel (MAC) with delayed state information

Motivation-LTE



Channel Model

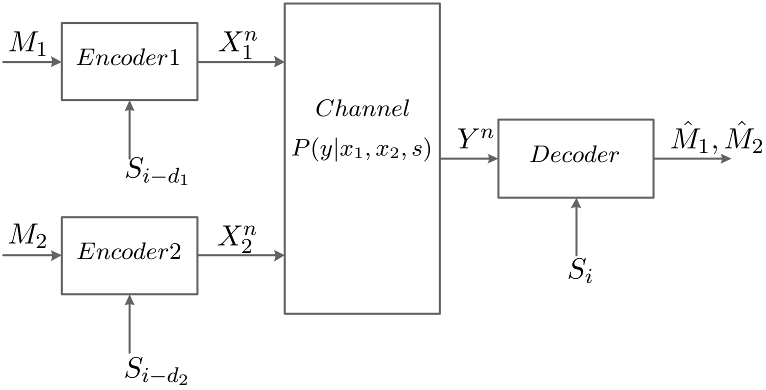


Figure: FSM-MAC with CSI at the decoder and delayed CSI at the encoders. We consider the above problem setting in the following cases: $d_1 > d_2$, $d_1 = d_2$, and $d_2 < d_1 = \infty$.

Strictly causal CSI [Steinberg/Lapidoth10]
Point-to-point case [Viswanathan99]

Channel Model and Notation

- Finite number of states $\mathcal{S} < \infty$.
- Channel state is a stationary Markov process independent of the messages.
- The random variables S_i ,S_{i-d} denote the channel state at time i, and $i-d$, respectively.
- The (S_i, S_{i-d}) joint distribution is stationary and is given by

$$P(S_i = s_l, S_{i-d} = s_j) = \pi(s_j)K^d(s_l, s_j).$$

- The channel transition probability at time $i$is given by $P(y_i|x_{1,i}, x_{2,i}, s_i)$.

Code Description

For each encoder an encoding function,

$$X_{1,i} = \begin{cases} f_{1,i}(M_1), & 1 \leq i \leq d_1 \\ f_{1,i}(M_1, S^{i-d_1}), & d_1 + 1 \leq i \leq n \end{cases} \quad (1)$$

$$X_{2,i} = \begin{cases} f_{2,i}(M_2), & 1 \leq i \leq d_2 \\ f_{2,i}(M_2, S^{i-d_2}), & d_2 + 1 \leq i \leq n \end{cases} \quad (2)$$

Main Results- asymmetrical delayed CSI ($d_1 \geq d_2$)

Theorem

The capacity region of FSM-MAC with CSI at the decoder and asymmetrical delayed CSI at the encoders with delays d_1 and d_2 ($d_1 \geq d_2$), is given by:

$$\mathcal{R} = \bigcup_{P(u|\tilde{s}_1)P(x_1|\tilde{s}_1,u)P(x_2|\tilde{s}_1,\tilde{s}_2,u)} \left(\begin{array}{r} R_1 < I(X_1;Y|X_2,S,\tilde{S}_1,\tilde{S}_2,U) \\ R_2 < I(X_2;Y|X_1,S,\tilde{S}_1,\tilde{S}_2,U) \\ R_1 + R_2 < I(X_1,X_2;Y|S,\tilde{S}_1,\tilde{S}_2,U), \end{array} \right)$$

where U is an auxiliary random variable with cardinality $|\mathcal{U}| \leq 3$.

The joint distribution $(S, \tilde{S}_1, \tilde{S}_2)$ is the same joint distribution as $(S_i, S_{i-d_1}, S_{i-d_2})$.

Main Results- symmetrical delayed CSI ($d = d_1 = d_2$)

Theorem

The capacity region of FSM-MAC with CSI at the decoder and symmetrical delayed CSI at the encoders with delay d ($d = d_1 = d_2$), is given by:

$$\mathcal{R} = \bigcup_{P(u|\tilde{s})P(x_1|\tilde{s},u)P(x_2|\tilde{s},u)} \left(\begin{array}{r} R_1 < I(X_1;Y|X_2,S,\tilde{S},U) \\ R_2 < I(X_2;Y|X_1,S,\tilde{S},U) \\ R_1 + R_2 < I(X_1,X_2;Y|S,\tilde{S},U), \end{array} \right)$$

where U is an auxiliary random variable with cardinality $|\mathcal{U}| \leq 3$.

Main Results- delayed CSI only to encoder 2 ($d_1 = \infty$)

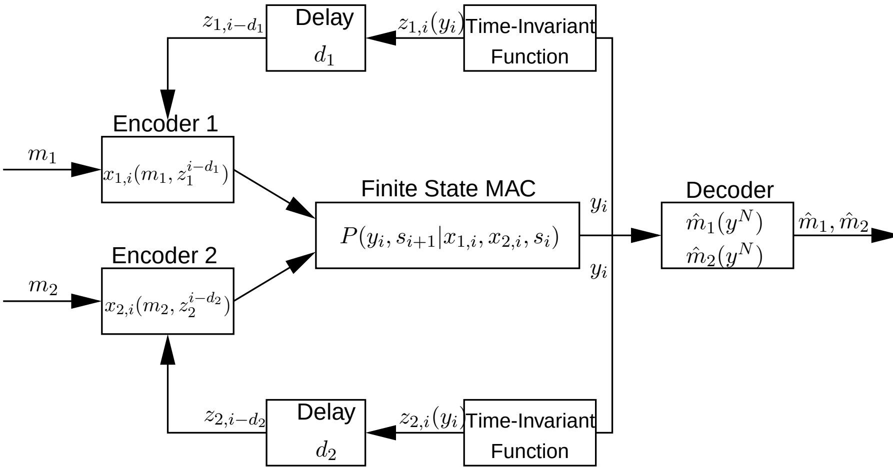
Theorem

The capacity region of FSM-MAC with CSI at the decoder and delayed CSI only to one encoder is given by :

$$\mathcal{R} = \bigcup_{P(q)P(x_1|q)P(x_2|\tilde{s},q)} \left(\begin{array}{r} R_1 < I(X_1;Y|X_2,S,\tilde{S},Q) \\ R_2 < I(X_2;Y|X_1,S,\tilde{S},Q) \\ R_1 + R_2 < I(X_1,X_2;Y|S,\tilde{S},Q), \end{array} \right)$$

where Q is an auxiliary random variable with cardinality $|\mathcal{Q}| \leq 3$.

MAC with time-invariant feedback



[P/Weissman/Chen09]

Definitions

$$I(X^n;Y^n) = \sum_{i=1}^{n} I(X^n;Y_i|Y^{i-1})$$

$$P(x^n|z^n) = \prod_{i=1}^{n} P(x_i|z^n,x^{i-1})$$

Definitions

Directed Information [Massey90]

$$I(X^n \to Y^n) \triangleq \sum_{i=1}^{n} I(X^{i}; Y_i | Y^{i-1})$$

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$$I(X^n; Y^n) = \sum_{i=1}^{n} I(X^n; Y_i | Y^{i-1})$$

Causal conditioning [Kramer98]

$$P(x^n || z^{n-d}) \triangleq \prod_{i=1}^{n} P(x_i | z^{i-d}, x^{i-1})$$

$$P(x^n | z^n) = \prod_{i=1}^{n} P(x_i | z^n, x^{i-1})$$

Finite State MAC

Let

$$\underline{\mathcal{R}}_n = \bigcup \begin{cases} R_1 \leq \frac{1}{n} I(X_1^n \to Y^n || X_2^n, s_0), \\ R_2 \leq \frac{1}{n} I(X_2^n \to Y^n || X_1^n, s_0), \\ R_1 + R_2 \leq \frac{1}{n} I((X_1, X_2)^n \to Y^n | s_0), \end{cases}$$

the union is over input distribution $P(x_1^n || z_1^{n-1}) P(x_2^n || z_2^{n-1})$.

Theorem

For Finite state MAC with time invariant feedback and Markovian state,the capacity region is

$$\mathcal{R} = \lim_{n \to \infty} \mathcal{R}_n,$$

where the limit exists P.&Weissman&Chen07

Proof via Directed information

- Adapt the feedback model:
  - State information at the decoder as a part of the channel's output
  - Choosing the deterministic function of the output:

$$z_{1,i}(y_i, s_i) = z_{2,i}(y_i, s_i) = s_i$$

- Multi-letter expression $\rightarrow$ Single-letter expression.

$$\begin{aligned} R_1 &\leq \frac{1}{n} I(X_1^n \rightarrow Y^n, S^n || X_2^n) \\ R_1 &\leq \frac{1}{n} I(X_2^n \rightarrow Y^n, S^n || X_1^n) \\ R_1 + R_2 &\leq \frac{1}{n} I((X_1, X_2)^n \rightarrow Y^n, S^n) \end{aligned}$$

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$$z_{1,i}(y_i, s_i) = z_{2,i}(y_i, s_i) = s_i$$

- Multi-letter expression $\rightarrow$ Single-letter expression.

$$\begin{aligned}
R_1 &\leq \tfrac{1}{n} I(X_1^n \rightarrow Y^n, S^n || X_2^n) = I(X_1; Y | X_2, S, \tilde{S}_1, \tilde{S}_2, U), \\
R_1 &\leq \tfrac{1}{n} I(X_2^n \rightarrow Y^n, S^n || X_1^n) = I(X_2; Y | X_1, S, \tilde{S}_1, \tilde{S}_2, U), \\
R_1 + R_2 &\leq \tfrac{1}{n} I((X_1, X_2)^n \rightarrow Y^n, S^n) = I(X_1, X_2; Y | S, \tilde{S}_1, \tilde{S}_2, U).
\end{aligned}$$

Capacity Region for a Finite State Additive Gaussian MAC

FS additive Gaussian noise (AGN) MAC,

$$Y_i = X_{1,i} + X_{2,i} + N_{S_i}, \tag{3}$$

we apply the sum rate formula on the finite state Markov AGN MAC with transmitters power constrains $\mathcal{P}_1$ and $\mathcal{P}_2$.

$$R_1 + R_2 < \max_{p(u|\tilde{s}_1)p(x_1|\tilde{s}_1,u)p(x_2|\tilde{s}_1,\tilde{s}_2,u)} I(X_1, X_2; Y|S, \tilde{S}_1, \tilde{S}_2, U),$$

subject to the power constraints,

$$\sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{u} P(u|\tilde{s}_1) E[X_1^2|\tilde{s}_1, u] \leq \mathcal{P}_1,$$

$$\sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} P(\tilde{s}_2|\tilde{s}_1) \sum_{u} P(u|\tilde{s}_1) E[X_2^2|\tilde{s}_1, \tilde{s}_2, u] \leq \mathcal{P}_2.$$

Capacity Region for a Finite State Additive Gaussian MAC

We bound the sum rate,

$$I(X_1, X_2; Y|S, \tilde{S}_1, \tilde{S}_2, U) \leq \frac{1}{2}\sum_{\tilde{s}_1}\pi(\tilde{s}_1)\sum_{\tilde{s}_2}P(\tilde{s}_2|\tilde{s}_1)\sum_{s}P(s|\tilde{s}_2) \times \log\left(1 + \frac{\mathcal{P}_1(\tilde{s}_1) + \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2)}{\sigma_s^2}\right). \quad (4)$$

We can achieve (4) if we choose:

- $X_1(\tilde{s}_1, u) \sim \mathcal{N}(0, \mathcal{P}_1(\tilde{s}_1))$.
- $X_2(\tilde{s}_1, \tilde{s}_1, u) \sim \mathcal{N}(0, \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2))$
- $X_1(\tilde{s}_1, u) \perp X_2(\tilde{s}_1, \tilde{s}_1, u) \perp N_s$

Capacity Region for a Finite State Additive Gaussian MAC

We get the following optimization problem,

$$R_1 + R_2 = \max_{\mathcal{P}_1(\tilde{s}_1), \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2)} \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1 - d_2}(\tilde{s}_2, \tilde{s}_1) \sum_{s} K^{d_2}(s, \tilde{s}_2) \times \log\left(1 + \frac{\mathcal{P}_1(\tilde{s}_1) + \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2)}{\sigma_s^2}\right), \quad (5)$$

subject to the power constraints,

$$\sum_{\tilde{s}_1} \pi(\tilde{s}_1) \mathcal{P}_1(\tilde{s}_1) \leq \mathcal{P}_1, \quad (6)$$

$$\sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} P(\tilde{s}_2 | \tilde{s}_1) \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2) \leq \mathcal{P}_2. \quad (7)$$

- At any given time i the channel is in one of two possible states $Good$ or Bad.

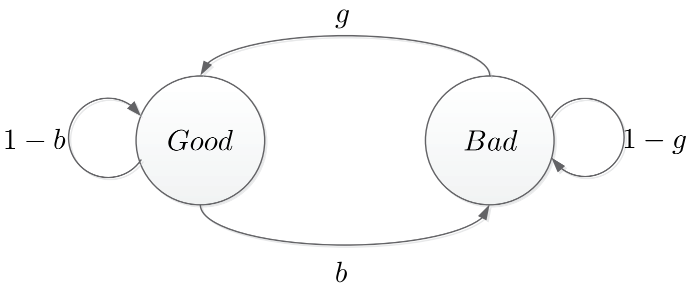


Figure: Two-state AGN channel.

- At any given time i the channel is in one of two possible states $Good$ or Bad.
- $\sigma_B^2 > \sigma_G^2$.

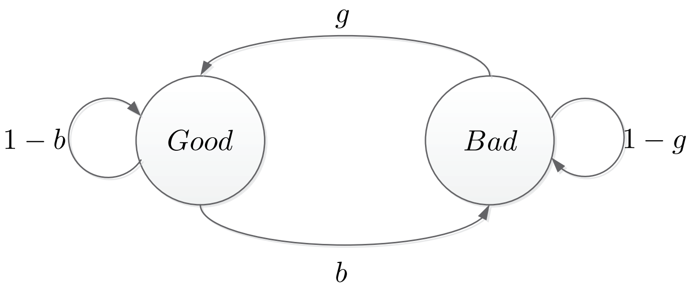
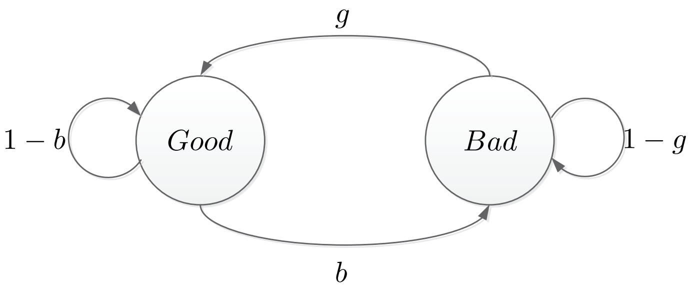


Figure: Two-state AGN channel.

Gilbert-Elliot Gaussian MAC

- At any given time i the channel is in one of two possible states $Good$ or Bad.
- $\sigma_B^2 > \sigma_G^2$.
- $\mathcal{P}_1 = 10, \mathcal{P}_2 = 10, \sigma_G^2 = 1, \sigma_B^2 = 100, g = 0.1, b = 0.1$.



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Two State AGN MAC- Sum Rate

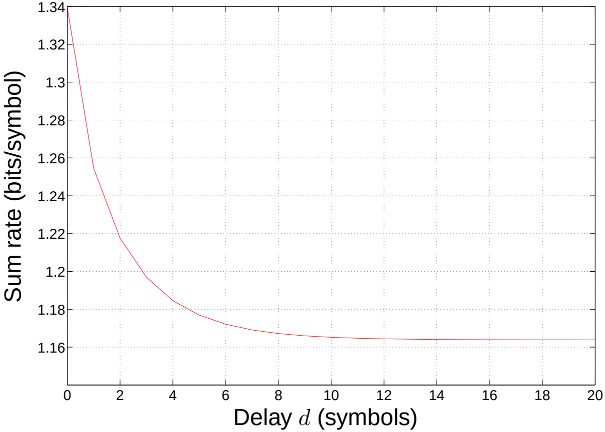


Figure: Sum rate versus delay d (symmetrical delay $d_1 = d_2 = d$).

Two State AGN MAC- Power Control Policy

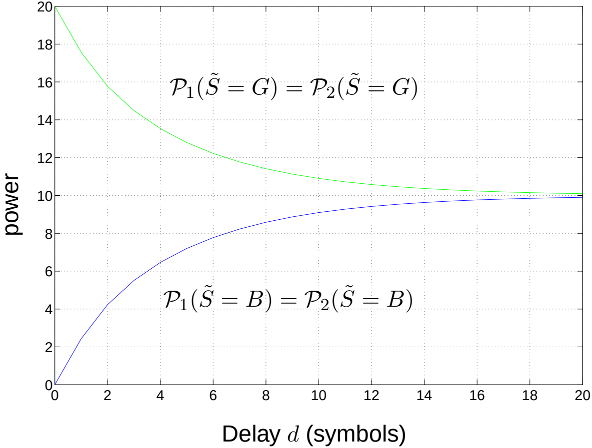


Figure: Power control policy versus delay d (symmetrical delay $d_1 = d_2 = d$).

Two State AGN MAC- Capacity Region

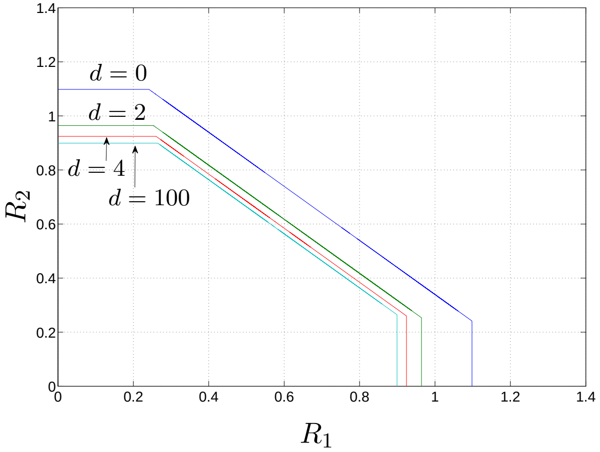


Figure: Capacity rate region for the two states AGN-MAC- symmetrical case $d = d_1 = d_2$.

Two State AGN MAC- Capacity Region

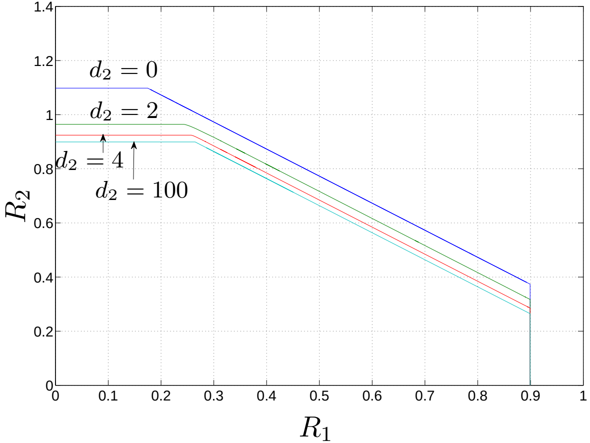
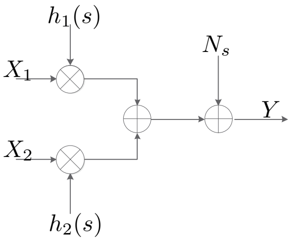


Figure: Capacity rate region for the two states AGN-MAC-Transmitter 1 doesn't have the CSI $d_2 \leq d_1 = \infty$.

Power Constrained FS Multiple-Access Fading Channel

FS multiple-access fading channel,

$$Y_i = h_1(s_i)X_{1,i} + h_2(s_i)X_{2,i} + N_{S_i}, \tag{8}$$



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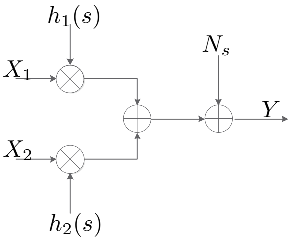


Figure: The fading channel.

- The terms $X_{1,i}, X_{2,i}$ are the transmitted waveform.
- The terms $h_1(s_i), h_2(s_i)$ are the fading process of the users, and are deterministic functions of s_i.

Power Constrained FS Multiple-Access Fading Channel

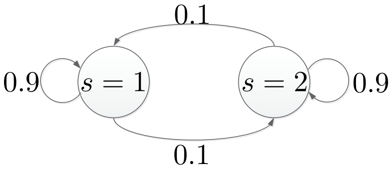
We derive the following optimization problem,

$$\begin{aligned} R_1 + R_2 \quad &= \quad \max_{\mathcal{P}_1(\tilde{s}_1), \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2)} \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1 - d_2}(\tilde{s}_2, \tilde{s}_1) \\ &\quad \times \sum_{s} K^{d_2}(s, \tilde{s}_2) \log\left(1 + \frac{h_1(s)^2 \mathcal{P}_1(\tilde{s}_1) + h_2(s)^2 \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2)}{\sigma_s^2}\right), \end{aligned}$$

subject to the power constraints,

$$\begin{aligned} &\sum_{\tilde{s}_1} \pi(\tilde{s}_1) \mathcal{P}_1(\tilde{s}_1) \leq \mathcal{P}_1, \\ &\sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} P(\tilde{s}_2 | \tilde{s}_1) \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2) \leq \mathcal{P}_2. \end{aligned}$$

- The state process,



- The fading process,
$$h_1 = \left\{ \begin{array}{cc} 1 & s = 1 \\ 0.5 & s = 2 \end{array} \right\}, \quad h_2 = \left\{ \begin{array}{cc} 0.5 & s = 1 \\ 1 & s = 2 \end{array} \right\}.$$
- The noise,
$N \sim \mathcal{N}(0, 1)$.
- The power constraints,
$\mathcal{P}_1 = 2$, $\mathcal{P}_2 = 2$.

Power Constrained FS Multiple-Access Fading Channel

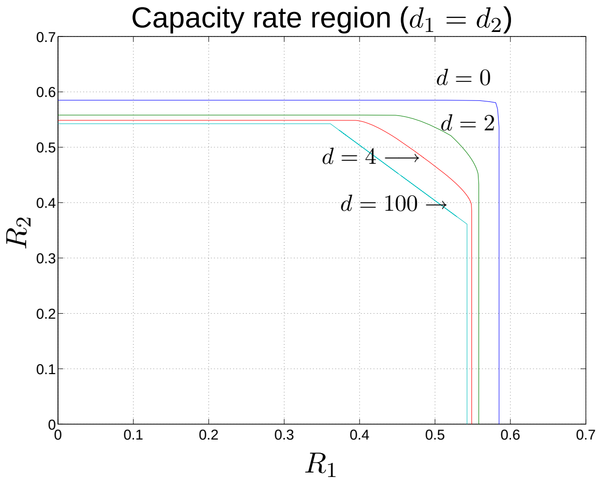


Figure: Capacity rate region for the two states fading channel-symmetrical case $d = d_1 = d_2$.

Power Constrained FS Multiple-Access Fading Channel

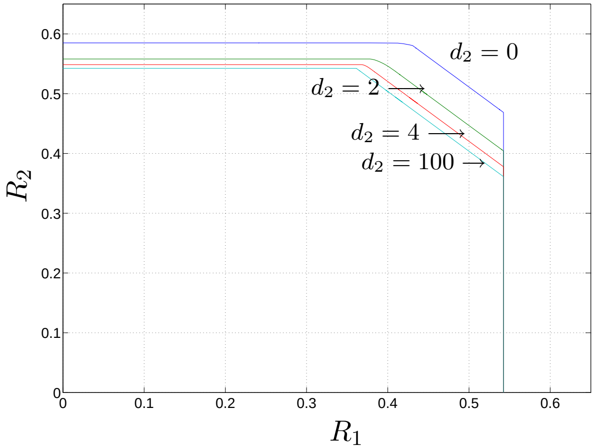


Figure: Capacity rate region for the two states fading channel- Transmitter 1 doesn't have the CSI $d_2 \leq d_1 = \infty$.

Summary

- A single-letter characterization is provided via directed information for the capacity region of FSM-MAC, when the transmitters have access to delayed CSI, and CSI is available at the receiver.

$$\bigcup_{P(u|\tilde{s}_1)P(x_1|\tilde{s}_1,u)P(x_2|\tilde{s}_1,\tilde{s}_2,u)} \left(\begin{array}{r} R_1 < I(X_1;Y|X_2,S,\tilde{S}_1,\tilde{S}_2,U) \\ R_2 < I(X_2;Y|X_1,S,\tilde{S}_1,\tilde{S}_2,U) \\ R_1+R_2 < I(X_1,X_2;Y|S,\tilde{S}_1,\tilde{S}_2,U), \end{array} \right)$$

- the whole statistics $(S,\tilde{S}_1,\tilde{S}_2)$ is important
- Computable result on FS AGN MAC, and on FS multiple-access fading channel.

Summary

- A single-letter characterization is provided via directed information for the capacity region of FSM-MAC, when the transmitters have access to delayed CSI, and CSI is available at the receiver.

$$\bigcup_{P(u|\tilde{s}_1)P(x_1|\tilde{s}_1,u)P(x_2|\tilde{s}_1,\tilde{s}_2,u)} \begin{pmatrix} R_1 < I(X_1;Y|X_2,S,\tilde{S}_1,\tilde{S}_2,U) \\ R_2 < I(X_2;Y|X_1,S,\tilde{S}_1,\tilde{S}_2,U) \\ R_1+R_2 < I(X_1,X_2;Y|S,\tilde{S}_1,\tilde{S}_2,U), \end{pmatrix}$$

- the whole statistics $(S,\tilde{S}_1,\tilde{S}_2)$ is important
- Computable result on FS AGN MAC, and on FS multiple-access fading channel.

Thank you to Benjamin Zaidel and thank you for attending the talk!!

Achievability Proof via Multiplexing Coding

Encoder 1:

- Construct k codebooks $\mathcal{C}_{\tilde{s}_1}$ for all $\tilde{S}_1 \in \mathcal{S}$, when in each codebook $\mathcal{C}_{\tilde{s}_1}$ there are $2^{n_1(\tilde{s}_1)R_1(\tilde{s}_1)}$ codewords.
- Every codeword $\mathcal{C}_{\tilde{s}_1}(i)$ when $i \in \{1, 2, ..., 2^{n_1(\tilde{s}_1)R_1(\tilde{s}_1)}\}$ has a length of $n_1(\tilde{s}_1)$ symbols.
- Each codeword from the $\mathcal{C}_{\tilde{s}_1}$ codebook is built $X_{\tilde{s}_1} \sim$ i.i.d. $P(x_{\tilde{s}_1}|\widetilde{S}_1 = \tilde{s}_1)$.
- Every time that the delayed CSI is $\tilde{S}_1 = \tilde{s}_1$, encoder 1 sends the next symbol from $\mathcal{C}_{\tilde{s}_1}$ codebook.
- Therefore we can send total of $2^{nR_1} = 2^{\sum_{\tilde{s}_1 \in \mathcal{S}} n_1(\tilde{s}_1)R_1(\tilde{s}_1)}$ messages.

Two State AGN MAC- Capacity Region

We present the capacity rate region by solving numerically the following optimization problem for different values of α,

$$\max_{R_1, R_2} \alpha R_1 + R_2, \tag{9}$$

subject to the constraints,

$$\begin{aligned} R_1 &\leq \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1 - d_2}(\tilde{s}_2, \tilde{s}_1) \\ &\quad \times \sum_{s} K^{d_2}(s, \tilde{s}_2) \log \left(1 + \frac{\mathcal{P}_1(\tilde{s}_1)}{\sigma_s^2} \right), \end{aligned} \tag{10}$$

$$\begin{aligned} R_2 &\leq \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1 - d_2}(\tilde{s}_2, \tilde{s}_1) \\ &\quad \times \sum_{s} K^{d_2}(s, \tilde{s}_2) \log \left(1 + \frac{\mathcal{P}_2(\tilde{s}_1, \tilde{s}_2)}{\sigma_s^2} \right), \end{aligned} \tag{11}$$

$$\begin{aligned} R_1 + R_2 &\leq \frac{1}{2} \sum_{\tilde{s}_1} \pi(\tilde{s}_1) \sum_{\tilde{s}_2} K^{d_1 - d_2}(\tilde{s}_2, \tilde{s}_1) \sum_{s} K^{d_2}(s, \tilde{s}_2) \\ &\quad \times \log \left(1 + \frac{\mathcal{P}_1(\tilde{s}_1) + \mathcal{P}_2(\tilde{s}_1, \tilde{s}_2)}{\sigma_s^2} \right). \end{aligned} \tag{12}$$

$$R_1 + R_2 = \max_{\mathcal{P}_1(\tilde{s}),\mathcal{P}_2(\tilde{s})} \frac{1}{2} \sum_{\tilde{s}} \pi(\tilde{s}) \sum_{s} K^d(s,\tilde{s}) \log\left(1 + \frac{\mathcal{P}_1(\tilde{s}) + \mathcal{P}_2(\tilde{s})}{\sigma_s^2}\right), \quad (13)$$

subject to the power constraints,

$$\sum_{\tilde{s}} \pi(\tilde{s})\mathcal{P}_1(\tilde{s}) \leq \mathcal{P}_1, \quad (14)$$

$$\sum_{\tilde{s}} \pi(\tilde{s})\mathcal{P}_2(\tilde{s}) \leq \mathcal{P}_2, \quad (15)$$

$$\mathcal{P}_1(\tilde{s}) \geq 0 \ \ \forall \tilde{s}, \quad (16)$$

$$\mathcal{P}_2(\tilde{s}) \geq 0 \ \ \forall \tilde{s}. \quad (17)$$

The solution can be obtained by the Lagrange multiplier method. Since $\log$ is concave function, and that $\pi(\tilde{s}), K^d(s,\tilde{s}) \geq 0$. We get that objective function is concave in both variables $\mathcal{P}_1(\tilde{s})$, and $\mathcal{P}_2(\tilde{s})$. Also the constraints function are affine. So we can use the Kuhn-Tucker conditions as a sufficient conditions to solve the optimization problem.