Model Based Phase Unwrapping of 2-D Signals

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Abstract

A parametric model and a corresponding parameter estimation algorithm for unwrapping two-dimensional phase functions, are presented. The proposed algorithm performs global analysis of the observed signal. Since this analysis is based on parametric model fitting, the proposed phase unwrapping algorithm has low sensitivity to phase aliasing due to low sampling rates and noise, as well as to local errors. In its first step the algorithm fits a 2-D polynomial model to the observed phase. The estimated phase is then used as a reference information which directs the actual phase unwrapping process: The phase of each sample of the observed field is unwrapped by increasing (decreasing) it by the multiple of 2π which is the nearest to the difference between the principle value of the phase and the estimated phase value at this coordinate. In practical applications the entire phase function cannot be approximated by a single 2-D polynomial model. Hence the observed field is segmented, and each segment is fit with its own model. Once the phase model of the observed field has been estimated we can repeat the model-based unwrapping procedure described earlier for the case of a single segment and a single model field.

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1 Introduction

An important problem in many two-dimensional signal processing applications is the unwrapping of two-dimensional (2-D) phase functions in order to remove existing phase ambiguities. For example, in Interferometric Synthetic Aperture Radar (INSAR) applications, the phase of the observed 2-D signal is a function of the scatterer coordinates. This function measures the elevation of the scattering point on the ground. Hence, ground elevations and terrain maps can be produced from the INSAR data, [3] – [6]. A critical consideration in producing the 3-D terrain maps is the need to perform 2-D phase unwrapping of the observed signal phase to enable a meaningful interpretation of the data. Ideally, in the absence of noise and phase aliasing, one could unwrap the phase information by following an integration path and adding multiples of 2π to the phase whenever a sudden drop from π to $-\pi$ occurs. To ensure that no phase-aliasing occurs, the original scene must be properly sampled, so that phase differences between two adjacent samples are smaller than π radians. This requirement cannot be generally satisfied, and hence in the presence of noise and phase aliasing, this simple phase unwrapping method is inadequate.

Many 2-D phase unwrapping techniques involve *local* analysis of the phase image by means of sequential processing of the differences between adjacent pixels, [7], or by employing edge detection techniques, [4] [5]. The edges, also called fringe lines, represent phase jumps of 2π radians. The idea behind edge detection based techniques is to find the location of the fringe lines, and then to unwrap the phase by adding a multiple of 2π each time the integration path crosses a fringe line. Since in general the phase image is noisy, it has to be first filtered to reduce the noise level. Next, the edges (obtained by some edge detection algorithm) have to be linked by an edge linking algorithm, or by using some interpolation method, to provide an estimate of the fringe line [5]. However, due to the presence of the noise, and since fringe lines can become dense in some areas, it may become very difficult to track the fringe lines. If phase aliasing occurs, the algorithm must identify and avoid crossing "ghost lines" in the integration path [7]. (The "ghost lines" are those segments of the phase discontinuity contour lines in the principal phase image, which are made invisible due to phase-aliasing). If the integration path of the phase differences crosses a ghost line, a phase error of 2π is propagated from the crossing point ahead, along the integration path. Hence, the goodness of the estimate of the ghost line positions becomes critical in the 2-D phase unwrapping procedure. To the best of our knowledge, this problem does not have a satisfactory solution. Thus, the major problem with this family of algorithms is that they are all based on local phase properties, and on finding integration paths in order to perform the phase unwrapping. Since in such schemes local errors result in global errors, their usefulness in the presence of noise is limited.

An alternative method for 2-D phase unwrapping, which is not based on any integration path following, is to obtain a least squares estimate of the true phase by minimizing the differences between the first-order discrete partial derivatives of the wrapped phase function, and those of the unknown unwrapped solution function [8], [9]. It has been shown that this least squares solution is equivalent to the solution of Poisson's equation on a rectangular grid with Neumann boundary conditions [9]. The algorithms are implemented by using the fast cosine transform, or the fast Fourier transform [10]. Recently, [11], has proposed a two stage phase unwrapping procedure such that in the first stage the gradient of the wrapped phase function is estimated from the observed data. In the second stage the estimated phase gradients are employed to estimate the unwrapped phase through minimization of the mean squared error between the gradient of the unwrapped phase and the gradient estimated from the observed data. Similarly to the foregoing algorithms, the proposed solution is equivalent to solving a partial differential equation with Neumann boundary conditions. Hence, any error in estimating the phase gradient at the boundaries (for example, due to noise), would influence the results of the entire phase unwrapping procedure.

In this paper we propose to use a parametric model as an alternative to the foregoing methods. Since continuous functions can be approximated by polynomials, a natural choice for modeling any *continuous* 2-D phase function is a 2-D polynomial of the coordinates. Since the assumption of phase smoothness is implicit to our model, no *explicit* phase unwrapping is required in estimating the observed phase. Moreover, the proposed algorithm performs *global* analysis of the observed signal and hence no integration path following is needed. Since the global analysis of the signal is based on parametric model fitting, the proposed phase unwrapping algorithm has low sensitivity to phase aliasing due to low sampling rates and noise, as well as to local errors. Hence, the proposed algorithm has the potential of alleviating the problems of existing algorithms.

More specifically, in the absence of observation noise, the phase of the 2-D signal is assumed to be a 2-D polynomial function of the coordinates. In the special case of a first order polynomial, this reduces to a homogeneous model – a 2-D sinusoid. When the polynomial order is higher, the model is no longer homogeneous: the spatial frequencies are now a function of location.

The proposed 2-D phase unwrapping algorithm is based on the Phase Differencing (PD) Algorithm [1], [2]. Given the observed 2-D signal, the PD algorithm provides estimates of all the phase parameters. Since the PD algorithm was found to be quite robust in the presence of observation noise, the initial step of the phase unwrapping algorithm is to fit a 2-D polynomial model to the observed phase.

We note that the algorithm attempts to fit a 2-D polynomial phase model to the data itself, and is not at all concerned with the wrapped phase image as some of the phase unwrapping techniques described earlier. Since the model inherently assumes the phase to be a smooth function of the coordinates, it is not concerned with the 2π ambiguities of the phase function. In this method the phase parameters can be estimated even for relatively low SNR scenarios, in which the local edge detection based algorithms are ineffective.

The estimated phase function is smooth due to the assumed model. This property enables the derivation of a relatively simple unwrapping step which operates on each sample of the observed field. The estimated phase is used as a reference information which directs the actual phase unwrapping process: The phase of each sample of the observed field is unwrapped by increasing (decreasing) it by the multiple of 2π which is the nearest to the difference between the principle value of the phase and the estimated phase value at this coordinate.

The paper is organized as follows. In section 2 we describe the parametric model of the observed signal, the 2-D phase difference operator, and the parameter estimation algorithm which is based on the 2-D phase difference operator. In section 3 we present the basic phase unwrapping algorithm assuming that the entire phase function obeys a single 2-D polynomial model. This assumption does not generally hold in practice. Hence, in section 4 we extend the basic algorithm so that it can be applied to any continuous phase function. This is accomplished by segmenting the observed field, and fitting a model to each segment. The segment size must be chosen so that the phase function is sufficiently smooth, and hence can be fit with a 2-D polynomial. Once the phase model of the observed field has been estimated, we repeat the model-based unwrapping procedure we have developed in section 3. In section 5 we summarize our results.

2 The Phase Differencing Algorithm

In this section we define the phase difference operator and summarize the main results of [1] and [2]. These results led to the development of the Phase Differencing Algorithm which is also described later in this section. We start with a description of the type of signal for which the operator was designed.

2.1 The Signal Model

Let $\{y(n,m)\}$ be a discrete complex valued 2-D random field consisting of the sum of a random amplitude deterministic phase signal, and an additive white Gaussian noise. We further assume that the signal phase is a smooth enough function of the field coordinates so that it can be approximated by a 2-D polynomial function of these coordinates. The amplitude function is a sum of a real valued mean and a real valued homogeneous random field. More specifically

$$y(n,m) = v(n,m) + u(n,m) \ , \quad n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1 \ , \tag{1}$$

where

$$v(n,m) = A\Big(1 + z(n,m)\Big) \exp\{j\phi_{S+1}(n,m)\} \ , \tag{2}$$

$$\phi_{S+1}(n,m) = \sum_{\{0 \le k,\ell : 0 \le k+\ell \le S+1\}} c(k,\ell) n^k m^\ell \ . \tag{3}$$

We shall call $\phi_{S+1}(n,m)$ 2-D polynomial of *total-degree* $S+1$. For example, using this terminology we say that a constant valued field is a 2-D polynomial phase signal of total-degree 0, and a 2-D exponential signal is a 2-D polynomial phase signal of total-degree 1. In other words, one might think of the phase polynomial $\phi_S(n,m)$, as if it has S 'layers' since increasing S by one adds a 'layer' of additional $S+2$ parameters to the phase model.

The amplitude A is a real valued positive constant. Since in many physical systems the observed signal amplitude is subject to a real valued multiplicative noise, we model it by $A\Big(1 + z(n,m)\Big)$, where $z(n,m)$ is a real valued, zero mean, noise field. The additive observation noise $u(n,m)$, is assumed to be complex valued, zero mean white Gaussian noise. (Note that the scaling coefficients associated with horizontal and vertical sampling, are absorbed into the coefficients $c(k,\ell)$).

To simplify the presentation we start with the case in which there is no observation noise and $A \equiv 1$. Hence, $y(n,m) = v(n,m) = \exp\{j\phi_{S+1}(n,m)\}$.

2.2 The PD_n and PD_m Operators

Next we define the basic phase differencing operators.

Definition 1: Let τ_m and τ_n be some positive constants. Define

$$\text{PD}_{m^{(0)}}[v(n,m)] \triangleq v(n,m) \ , n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1 \ , \tag{4}$$

$$\text{PD}_{m^{(1)}}[v(n,m)] \triangleq v(n,m) v(n, m+\tau_m)^* \ , \tag{5}$$

where the resulting 2-D signal $\text{PD}_{m^{(1)}}[v(n,m)]$ exists for $n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1-\tau_m$. In the following we keep the same type of notation to indicate the indices for which the left hand-side of the equation exists.

In general we have

$$\begin{aligned}\mathrm{PD}_{m^{(q)}}[v(n,m)] &\triangleq \mathrm{PD}_{m^{(q-1)}}[v(n,m)]\Big(\mathrm{PD}_{m^{(q-1)}}[v(n,m+\tau_m)]\Big)^* , \\ & n=0,1,\ldots,N-1\ ,m=0,1,\ldots,M-1-q\tau_m\ . \end{aligned} \quad (6)$$

Similarly

$$\mathrm{PD}_{n^{(0)}}[v(n,m)] \triangleq v(n,m)\ ,n=0,1,\ldots,N-1\ ,m=0,1,\ldots,M-1\ , \quad (7)$$

$$\begin{aligned}\mathrm{PD}_{n^{(1)}}[v(n,m)] &\triangleq v(n,m)v(n+\tau_n,m)^*\ , \\ & n=0,1,\ldots,N-1-\tau_n\ ,m=0,1,\ldots,M-1\ , \end{aligned} \quad (8)$$

and

$$\begin{aligned}\mathrm{PD}_{n^{(p)}}[v(n,m)] &\triangleq \mathrm{PD}_{n^{(p-1)}}[v(n,m)]\Big(\mathrm{PD}_{n^{(p-1)}}[v(n+\tau_n,m)]\Big)^* , \\ & n=0,1,\ldots,N-1-p\tau_n\ ,m=0,1,\ldots,M-1\ . \end{aligned} \quad (9)$$

Note that applying any of the operators $\mathrm{PD}_{m^{(1)}}[\cdot]$, or $\mathrm{PD}_{n^{(1)}}[\cdot]$ to a 2-D polynomial phase signal of total-degree S, results in a 2-D polynomial phase signal of total-degree $S-1$.

Assume we have sequentially applied the phase difference operator $\mathrm{PD}_{n^{(1)}}$ P times, and the phase difference operator $\mathrm{PD}_{m^{(1)}}$ $S-P$ times, to some complex-valued 2-D signal $v(n,m)$. Then, the resulting signal which we denote by $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]$ is given by, [2],

$$\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \prod_{q=0}^{S-P}\left\{\prod_{p=0}^{P}\left[v^{(*^{(p+q)})}(n+p\tau_n,m+q\tau_m)\right]^{\binom{P}{p}}\right\}^{\binom{S-P}{q}} , \quad (10)$$

where we define

$$v^{(*^{(p+q)})}(n+p\tau_n,m+q\tau_m) = \begin{cases} v(n+p\tau_n,m+q\tau_m), & p+q \text{ even} \\ v^*(n+p\tau_n,m+q\tau_m), & p+q \text{ odd} \end{cases} . \quad (11)$$

Theorem 1: Applying the operator $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[\cdot]$ to the 2-D signal

$$v(n,m) = \exp\{j\sum_{k,\ell\in I} c(k,\ell)n^k m^\ell\}\ , \qquad n=0,1,\ldots,N-1\ ,m=0,1,\ldots,M-1\ , \quad (12)$$

where $I = \{0 \leq k, \ell \text{ and } 0 \leq k + \ell \leq S + 1\}$, results in a 2-D exponential which is given by

$$\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \exp\left\{j[\omega_S n + \nu_S m + \gamma_S(\tau_n, \tau_m)]\right\} \;, \qquad n = 0,1,\ldots,N-1-P\tau_n \;, m = 0,1,\ldots,M-1-(S-P)\tau_m \;, \tag{13}$$

and

$$\omega_S = (-1)^S c(P+1, S-P)(P+1)!(S-P)!\tau_n^P \tau_m^{S-P} \;, \tag{14}$$

$$\nu_S = (-1)^S c(P, S+1-P)P!(S+1-P)!\tau_n^P \tau_m^{S-P} \;, \tag{15}$$

and $\gamma_S(\tau_n, \tau_m)$ is not a function of m nor n.

Theorem 1 implies that applying in some arbitrary sequence, P times the operator $\mathrm{PD}_{n^{(1)}}$, and $S-P$ times the operator $\mathrm{PD}_{m^{(1)}}$, to the observed signal (12), the resulting signal is the 2-D exponential $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \exp\left\{j[\omega_S n + \nu_S m + \gamma_S(\tau_n, \tau_m)]\right\}$ where ω_S and ν_S are given by (14) and (15), respectively. We can thus reduce any 2-D non homogeneous, polynomial phase signal, $v(n,m)$, whose phase is of total-degree $S+1$, to a 2-D single tone signal whose frequency is (ω_S, ν_S).

Hence, estimating (ω_S, ν_S) using any standard frequency estimation technique, results in an estimate of $c(P+1, S-P)$, and $c(P, S+1-P)$. In this paper we estimate the frequency of the exponential using a search for the maximum of the absolute value of the signal 2-D Discrete Fourier Transform (2-D DFT). More specifically, having estimated ω_S and ν_S in (14) and (15), we have

$$\hat{c}(P+1, S-P) = \frac{\hat{\omega}_S}{(-1)^S (P+1)!(S-P)!\tau_n^P \tau_m^{S-P}} \;, \tag{16}$$

and

$$\hat{c}(P, S+1-P) = \frac{\hat{\nu}_S}{(-1)^S P!(S+1-P)!\tau_n^P \tau_m^{S-P}} \;, \tag{17}$$

which constitutes an estimate of two of the parameters of the highest order 'layer', $S+1$, of the phase model parameters (i.e., those $c(k,\ell)$'s for which $0 \leq k, \ell$ and $k + \ell = S+1$).

Recall however that the highest order 'layer', $S+1$, of the phase model parameters has $S+2$ parameters, which need to be estimated. This can be achieved by repeating the procedure which was described above assuming some arbitrary P, for *all* P such that $0 \leq P \leq S$. Note that this procedure results in repeated estimation of some of the phase parameters.

Multiplying $v(n,m)$ by $\exp\{-j \sum_{k=0}^{S+1} \hat{c}(k, S+1-k) m^{S+1-k} n^k\}$ results in a new polynomial phase signal whose total-degree is S. By applying to the resulting signal a procedure similar to the one used to estimate the parameters $c(k,\ell)$ for $k + \ell = S+1$, we obtain an estimate of the $S+1$ parameters in the S 'layer'. Multiplying the 2-D polynomial phase signal of total-degree

S, which was obtained in the previous step by $\exp\{-j\sum_{k=0}^{S}\hat{c}(k,S-k)m^{S-k}n^k\}$ we obtain a new polynomial phase signal whose total-degree is $S-1$.

Let $v^{(s+1)}(n,m)$ denote the 2-D signal, where $s+1$ denotes the *current* total-degree of its phase polynomial. By repeating for all $s=S,\ldots,0$, the two basic steps of estimating the $c(k,\ell)$ parameters of 'layer' $s+1$ through finding the maxima of $\left|\mathrm{DFT}\left(\mathrm{PD}_{n^{(P)},m^{(s-P)}}[v^{(s+1)}(n,m)]\right)\right|$, for all $0\leq P\leq s$, followed by multiplying the already reduced order 2-D polynomial phase signal by $\exp\{-j\sum_{k=0}^{s+1}\hat{c}(k,s+1-k)m^{s+1-k}n^k\}$ in the next step, we obtain estimates for all the phase parameters except $c(0,0)$. The resulting signal after this processing, $v^{(0)}(n,m)$, is a constant phase 2-D signal. Taking now the average of the imaginary part of the logarithm of this signal we obtain an estimate for $c(0,0)$. We have thus completed the estimation of all the coefficients of the 2-D phase polynomial of total-degree $S+1$.

Once the phase parameters were estimated, the amplitude of the polynomial phase signal is obtained by multiplying the original signal by $e^{-j\hat{\phi}(n,m)}$, where $\hat{\phi}(n,m)$ is the estimated phase. Ideally, the resulting 2-D signal is constant with amplitude A. The algorithm which is based on the foregoing results is summarized in Table 1. In the following we refer to the algorithm as the *Phase Differencing Algorithm* (PD Algorithm).

So far we described the parameter estimation algorithm for the case in which no observation noise exists. However, in many practical situations the signal is observed in the presence additive and multiplicative noise, i.e., the observed field is $\{y(n,m)\}$ as given by (1), (2), (3). Thus, a straightforward but computationally prohibitive alternative to the PD Algorithm is to develop a maximum likelihood estimator for the polynomial phase parameters. This estimator involves a multi-dimensional search in the parameter space and is not practical except for very low order models.

Let $S+1$ denote the total-degree of the observed signal phase.

$s = S,\ v^{(s+1)}(n,m) = v(n,m),\ n = 0,\dots,N-1,\ m = 0,\dots,M-1.$

While $s \geq 0$ ($s+1$ is the 'layer' index)

for $P = 0,\dots,s$ (find all the parameters of the $s+1$ 'layer')

$x(n,m) = \mathrm{PD}_{n^{(P)},m^{(s-P)}}[v^{(s+1)}(n,m)]$

$(\hat{\omega}_s, \hat{\nu}_s) = \underset{(\omega,\nu)}{\operatorname{argmax}} \left|\mathrm{DFT}\Big(x(n,m)\Big)\right|$

$\hat{c}(P+1, s-P) = \frac{\hat{\omega}_s}{(-1)^s (P+1)!(s-P)! \tau_n^P \tau_m^{s-P}}$

$\hat{c}(P, s+1-P) = \frac{\hat{\nu}_s}{(-1)^s P!(s+1-P)! \tau_n^P \tau_m^{s-P}}$

end

$v^{(s)}(n,m) = v^{(s+1)}(n,m)\exp\{-j\sum_{\{k+\ell=s+1\}} \hat{c}(k,\ell) n^k m^\ell\}$

s=s-1

end

$c(0,0) = \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \mathrm{Im}\{\log(v^{(0)}(n,m))\}$

Table 1: The Phase Differencing Algorithm.

In [2] we have derived the Cramer-Rao lower bound on the error variance in estimating the phase model parameters when the signal is observed in the presence of additive white Gaussian noise. More specifically, let Λ denote the observations log likelihood function and let $\mathrm{SNR} = \frac{\mathrm{A}^2}{\sigma^2}$ denote the signal-to-noise ratio where σ^2 is the observation noise variance. In [2] we conclude that the elements of Fisher Information Matrix (FIM) block which corresponds to the phase parameters are given by

$$-E\left\{\frac{\partial^2 \Lambda}{\partial c(k,\ell)\partial c(p,q)}\right\} = 2\ \mathrm{SNR} \sum_{n=0}^{N-1} n^{k+p} \sum_{m=0}^{M-1} m^{\ell+q}\ , \tag{18}$$

and that the FIM is block diagonal. Hence the CRB's on the estimation of the phase parameters, the amplitude parameter, and the observation noise variance are decoupled. From the decoupling of the bounds and (18) we conclude that the bound on the phase parameters is a function only of the total-degree of the 2-D polynomial phase function, the SNR, and the dimensions of the observed field, but is independent of the phase parameters.

It turns out that although the PD algorithm is suboptimal (relative to the ML algorithm), its performance is close to the CRB for moderate to high signal to noise ratios. In [12] we have further studied the performance of the PD Algorithm in the presence of additive white Gaussian noise. In particular, we have concentrated on deriving expressions for the bias and the mean squared error in estimating the coefficients $c(k,\ell)$ of the $S+1$ layer of a constant amplitude polynomial phase signal with total-degree $S+1$ (i.e. the coefficients $c(k,\ell)$ such that $k+\ell = S+1$) in the presence of additive circular white Gaussian noise. We show that the estimates are unbiased for any SNR, and derive a rule for optimal selection of the algorithm parameters. It is shown that a nearly optimal choice of the algorithm free parameters τ_n and τ_m is to set $\tau_n = \frac{N}{P+1}$ and $\tau_m = \frac{M}{S-P+1}$. This parameter setting essentially guarantees that the mean squared error (MSE) in estimating the phase model parameters is minimized. For example, for the case in which $N >> P$ we obtain

$$E\{[\Delta c(P+1, S-P)]^2\} \approx \frac{6\,\mathcal{C}(P,S,\mathrm{SNR})}{\Big[(P+1)!(S-P)!\Big]^2 \left(\frac{N}{P+1}\right)^{2P+3} \left(\frac{M}{S-P+1}\right)^{2S-2P+1}} \qquad (19)$$

where $\mathcal{C}(P,S,\mathrm{SNR})$ is a function of P, S, and the SNR only.

3 2-D Phase Unwrapping

The proposed 2-D phase unwrapping algorithm is based on the PD Algorithm. Since the PD Algorithm was found to be quite robust in the presence of observation noise and phase aliasing, the initial step of the phase unwrapping algorithm is to fit a 2-D polynomial model to the observed phase. Note that the algorithm fits a 2-D polynomial phase model to the data itself, and is not concerned with the wrapped phase image, as some of the existing phase unwrapping techniques described earlier. Since the model is inherently smooth, the algorithm is not concerned with the 2π ambiguities of the phase function. Furthermore, while existing phase unwrapping algorithms are severely affected by insufficient spatial sampling (with respect to the instantaneous frequency), and noise, e.g., [11], the proposed phase unwrapping algorithm is highly robust in the presence of phase aliasing due to both low sampling rates and noise, as long as the true phase function is a continuous function of the coordinates. This robustness is achieved by the initial fitting of a parametric model to the phase of the observed signal.

In order to further illustrate this point consider the 2-D signal $y(n,m)$, whose true phase function is shown in Figure 1. Note the very low sampling rate of this phase function (the phase-axis of this figure is measured in radians, and the dimensions of the sampling grid are 100×100). The wrapped phase function of the observed signal as well as the wrapped phase image are shown in Figure 2. The signal has an amplitude $A = 1$, the noise field $\{z(n,m)\}$ is a

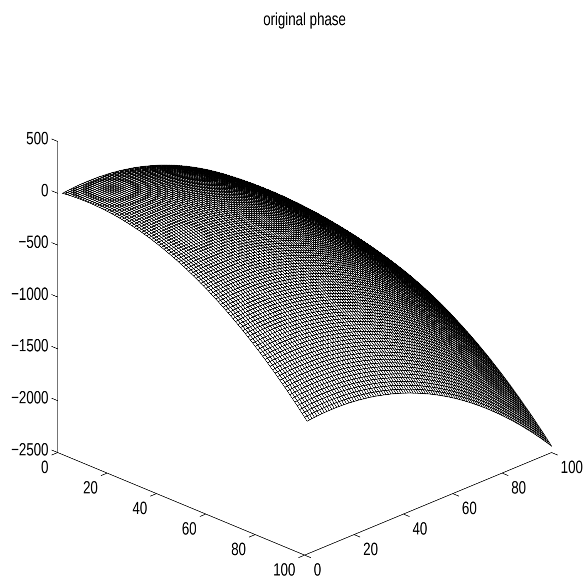

Figure 1: The true phase function of the observed signal.

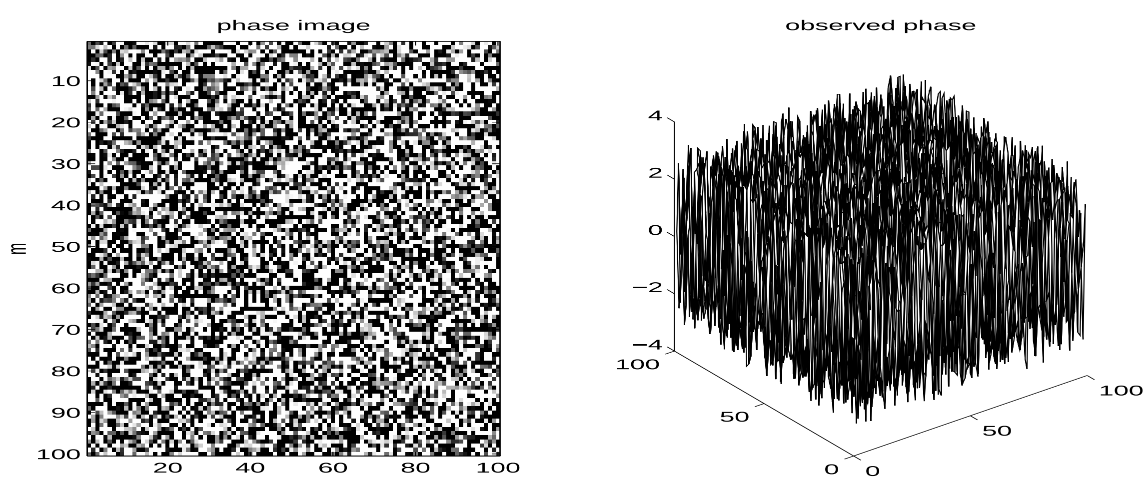

Figure 2: The phase of the observed signal, and the phase image.

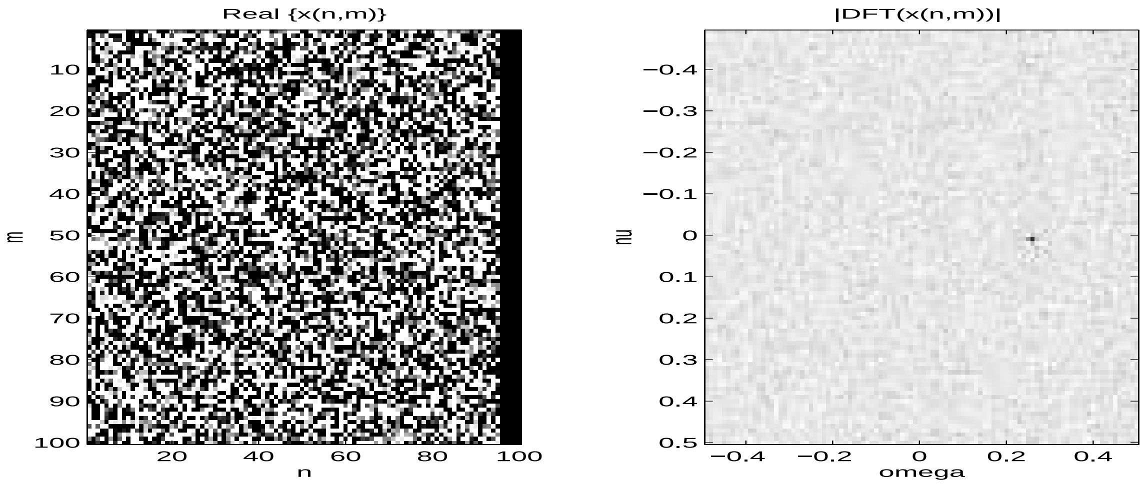

Figure 3: The 2-D polynomial phase signal after applying the operator $\mathrm{PD}_{n^{(0)},m^{(1)}}$ to the observed signal of total-degree 2. (In this iteration, $s = 1$, $p = 0$). Left: Real part of the resulting 2-D signal. Right: Absolute value of the resulting 2-D signal DFT.

zero mean white Gaussian field with variance selected so that the SNR is -5 dB. The variance of the additive noise $\{u(n, m)\}$ is similarly selected so that the SNR is -5 dB. The phase function is a 2-D polynomial of *total-degree* 2. Note that the highly noisy appearance of the phase image suggests that restoration of the phase information using local operators, such as edge detectors and fringe tracking, is impossible.

Since the proposed phase unwrapping algorithm initially fits a 2-D polynomial model to the observed phase, we will first illustrate the first step of the suggested parameter estimation algorithm. Since the polynomial phase total-degree is 2, we start by estimating the parameters of 'layer' 2. In the first step of the PD algorithm we have $s = 1$, and $P = 0$. Hence, applying the operator $\mathrm{PD}_{n^{(0)},m^{(1)}}$ to the observed signal, we obtain the signal $x(n, m)$ which is a 2-D polynomial phase signal of total-degree 1, i.e., a 2-D exponential signal. The real part of this signal is shown in the left image of Figure 3, and the absolute value of its DFT is shown in the right hand side of the same figure. Note that although the noise level is very high, applying the proposed operator to the observed signal results is a clearly observed spectral peak. Estimating the spatial frequency of the spectral peak results in the estimates of $c(1, 1)$, and $c(0, 2)$. Following the remaining steps of the PD Algorithm we obtain estimates of all the phase parameters.

The estimated phase function is smooth, leading to relatively simple unwrapping step. In the unwrapping step the algorithm operates on each sample of the observed field. The estimated phase is now used as a reference information which directs the actual phase unwrapping process:

Recall that the principal value of the observed signal phase is in the interval $[-\pi, \pi]$. On the other hand the true (and the estimated) phase can assume any real value. Hence, the phase of each sample of the observed field is unwrapped by increasing (decreasing) it by the multiple of 2π which is the nearest to the difference between the principle value of the phase and the estimated phase value at this coordinate.

More specifically, let $\phi(n,m)$, $\phi_{PV}(n,m)$, $\psi(n,m)$ denote the phase function of the noiseless signal, the principle value of the observed phase, and the unwrapped phase obtained by the proposed procedure, respectively. Also let $\hat{\phi}(n,m)$ denote the estimated phase obtained using the estimation procedure of section 2. In the absence of noise we have that

$$\phi(n,m) - \phi_{PV}(n,m) = 2\pi k \ , \tag{20}$$

where k is some integer. However, in practice $\phi(n,m)$ is unknown to us, and we only have $\hat{\phi}(n,m)$, which is estimated from the observed noisy measurements. Hence, replacing $\phi(n,m)$ by $\hat{\phi}(n,m)$ we obtain the basic unwrapping formula for the observed signal phase:

$$\psi(n,m) = 2\pi \cdot \mathrm{ROUND}\Big(\frac{\hat{\phi}(n,m) - \phi_{PV}(n,m)}{2\pi}\Big) + \phi_{PV}(n,m) \ . \tag{21}$$

In Figure 4 we illustrate the result of applying the proposed phase unwrapping algorithm to the above example. Note that although the observed signal is critically undersampled, and the SNR is low, the phase unwrapping procedure results in a very good reproduction of the true phase function. Analysis of the error in estimating the phase function shows that the maximal deviation of the unwrapped noisy phase function from the true and noiseless phase is 0.999π. Repeating this example in the absence of observation noise, results in a maximal difference of $4.5 \cdot 10^{-7}$ between the unwrapped and the true phase functions, although the phase is critically undersampled.

Due to the presence of noise, outliers may exist in the unwrapped phase obtained using the foregoing algorithm. By locally smoothing such outliers, an improved result of the phase unwrapping procedure can be obtained. Clearly, such a procedure is effective for medium to high SNRs where outliers can be easily detected, and is less effective for low SNRs.

In [2] it is shown that overestimating the order of the phase polynomial yields estimated coefficients, equal to zero, for the non-existing coefficients. In other words, for all $L > S$ applying L times, in any order, the operators $\mathrm{PD}_{n^{(1)}}[\cdot]$ and $\mathrm{PD}_{m^{(1)}}[\cdot]$ to a 2-D polynomial phase signal $v(n,m)$ of total-degree S yields a constant amplitude signal. This property allows for a relatively simple order estimation in cases where the polynomial total-degree S is unknown. We start with an assumed upper bound on the total-degree S. The decision that $\hat{c}(k,\ell) = 0$ can be based on

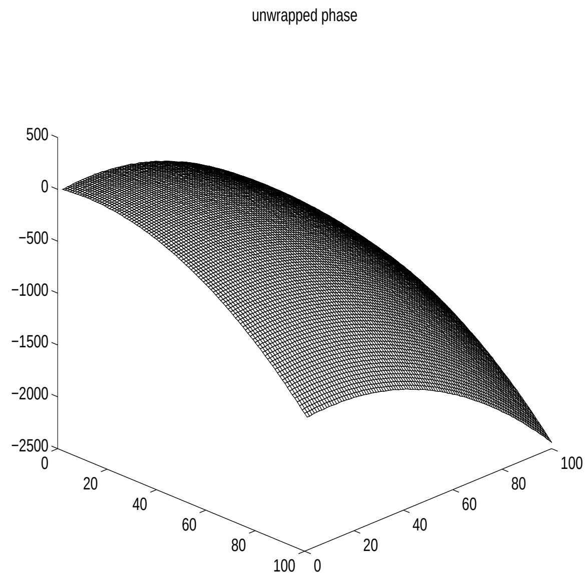

Figure 4: The unwrapped phase.

comparison with the Cramer-Rao bound, [2]. We will decide that $c(k,\ell) = 0$ whenever $|\hat{c}(k,\ell)|$ is not considerably higher than $\{\mathrm{CRB}[c(k,\ell)]\}^{\frac{1}{2}}$. In other words, we expect that overestimating the polynomial phase order will result in estimated coefficients which are approximately zero, for coefficients which belong to 'layers' that are higher than the true total-degree of the polynomial phase signal. Coefficients belonging to 'layers' which are within the true total-degree will, of course, be estimated correctly. As an example we have applied the PD algorithm to the observed field described above, but this time the algorithm assumes that the polynomial phase is of total-degree 3. This assumption led to identical phase estimation and unwrapping results, as in the first case (in which the algorithm assumes a correct total-degree of 2).

4 Phase Unwrapping for Arbitrary Phase Functions

In section 3 we developed the 2-D phase unwrapping algorithm, assuming that the entire phase function obeys a single 2-D polynomial model. However in practical applications this assumption does not generally hold. Hence the observed field has to be segmented, and each segment must be fit with its own model. The segment size must be chosen so that the phase function is smooth enough and hence can be fit with a 2-D polynomial with a total-degree which is small relative to the segment size, so that a meaningful parameter estimate of the phase model can be obtained.

After the observed field has been segmented, we apply to each of the segments the phase estimation algorithm described in section 2. As a result we obtain unwrapped polynomial approximations to the phase function in each segment. However, since the parameter estimation is performed independently for each of the segments, the estimated phase of each segment is known only up to a constant of magnitude $2\pi k$ where k is some unknown integer. Hence, an additional alignment stage has to be incorporated into the phase unwrapping procedure. In this stage a reference segment is arbitrarily chosen, and the phase values of all other segments are sequentially adjusted by the necessary factor of $2\pi k$, so that adjacent segments will form a continuous phase surface.

The proposed alignment procedure is not meant to eliminate phase discontinuities along segments boundaries in cases where the magnitude of the discontinuity is lower than 2π. Such discontinuities may occur due to a mismatch between the polynomial approximation of the phase function at a given segment and the true phase, due to the presence of noise, and since the parameter estimation is performed independently for each of the segments. The effect of such discontinuities can be reduced by an additional alignment stage in which a reference segment is arbitrarily chosen, and the phase values of all other segments are sequentially adjusted, so that the discontinuities of adjacent segments are made smaller, at least in one direction . Note however that if in each segment the true phase function is sufficiently smooth, and the approximating polynomial total degree is sufficiently high, the discontinuities essentially disappear. Moreover, since the true phase function is assumed to be smooth, it is clear that by smoothing the discontinuities near segments boundaries, a further improvement in the phase estimate can be obtained even in cases of high noise or model mismatch.

Once the phase model of the observed field has been estimated we can repeat the model-based unwrapping procedure we have described in the previous section for the case of a single segment and a single model field. Note that for the phase unwrapping purpose, the estimated phase is used only as a reference which indicates which multiple of 2π ought to be added to the observed principal value of the phase in order to unwrap it. Hence in general, small discontinuities in the estimated phase, will not result in errors in the phase unwrapping, even without smoothing the estimated phase.

In order to illustrate the procedure, consider a unit amplitude signal whose true phase function is shown in Figure 5. In this example the observations are subject to noise. The multiplicative noise field $\{z(n,m)\}$ is a zero mean white Gaussian field with variance which was selected such that the SNR is 5 dB. The variance of the additive white Gaussian noise was also selected so that the SNR is 5 dB. A 3-D plot of the wrapped phase function of the observed field as well as the wrapped phase image are shown in Figure 6.

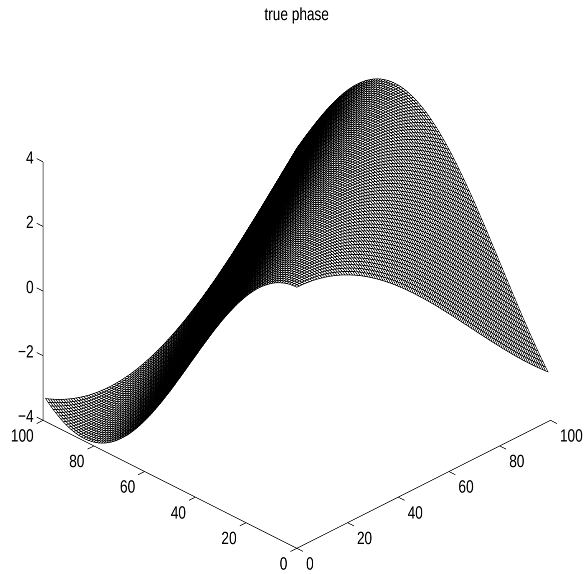

Figure 5: The original phase.

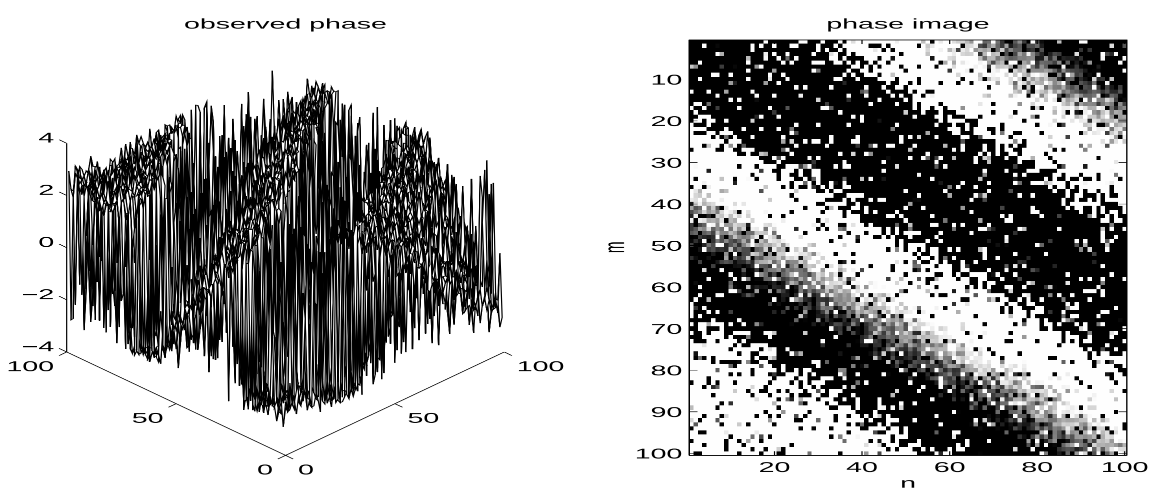

Figure 6: The phase of the observed signal, and the phase image.

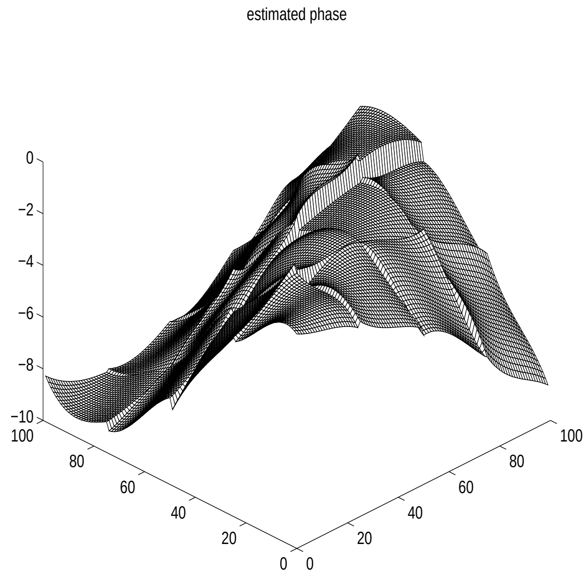

Figure 7: The phase function estimated from noisy measurements.

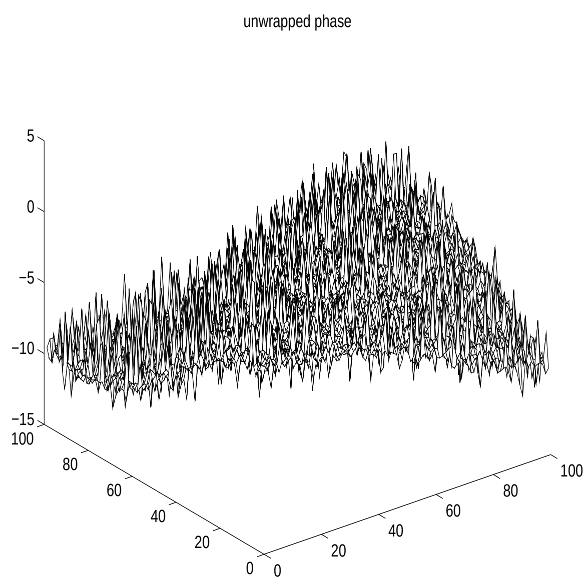

Figure 8: The phase function unwrapped using noisy measurements.

In order to estimate the phase function of the signal, the observed field was divided into 16 non-overlapping blocks of 25×25 samples. A 2-D polynomial of total-degree 3 was fit to each patch. The result of applying the estimation procedure is illustrated in Figure 7. It can be seen that the estimated phase function has some discontinuities along segments boundaries. The discontinuities are the result of the error in estimating the parametric models of the different phase segments, due to the presence of noise. Note however in Figure 8 that these small discontinuities still allow for a proper unwrapping of the observed phase, since the estimated phase is used only as a reference which indicates which integer multiple of 2π should be added to the observed phase.

5 Conclusions

We presented a model based 2-D phase unwrapping algorithm. While many conventional approaches to the 2-D phase unwrapping problem involve *local* analysis of the phase image, the proposed algorithm performs *global* analysis of the observed signal. The algorithm is based on fitting a 2-D polynomial phase model to each segment of the observed signal. Using the estimated phase model of the observed signal, the phase information is restored in a robust and computationally efficient way. Since the proposed phase unwrapping algorithm initially fits a parametric model to the observed data, it is less sensitive to problems of phase aliasing due to noise and undersampling than algorithms that are based on waveform fitting.

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