An Estimation Algorithm for 2-D Polynomial Phase Signals

Benjamin Friedlander* and Joseph M. Francos†

Abstract

We consider nonhomogeneous 2-D signals which can be represented by a constant modulus polynomial-phase model. A novel 2-D phase differencing operator is introduced, and used to develop a computationally efficient estimation algorithm for the parameters of this model. The operation of the algorithm is illustrated using an example.

1 Introduction

A fundamental problem in two-dimensional signal processing and in many image processing applications, is the modeling and analysis of nonhomogeneous two-dimensional (2-D) signals. For example, in almost any image taken by a camera, perspective exists, and hence the acquired 2-D signal is nonhomogeneous, even if the original scene was homogeneous. Conventional approaches to the problems of perspective and camera orientation estimation usually involve *local* analysis of the image, by means of edge detection algorithms, [6]. Recently, a nonparametric method for estimating, and then canceling, the effects of perspective was suggested in [7], using the Chirplet transform. In this method a 1-D cross-section of the image is expanded onto a set of modulated and warped versions of one "mother-waveform", in order to later compute an unwarped representation of the original image.

Parametric models, when used in image processing, generally assume the observed image to be homogeneous, or piece-wise homogeneous. In this paper we consider a parametric model which is *nonhomogeneous*, and attempts to perform global (or at least, less localized) image analysis. More specifically, the proposed model is aimed at modeling images which result from continuous coordinate transformations of homogeneous images. Since 2-D continuous functions can be approximated by 2-D polynomials we will study a model consisting of a sine (or cosine) of a polynomial function of the image coordinates. In the special case of a first order polynomial this reduces to a homogeneous model – a simple 2-D sinusoid. When the polynomial order is higher, the model is no longer homogeneous: the spatial frequencies are now a function of location. This type of model arises, for example, when a homogeneous image consisting of a periodic structure undergoes distortion due to

*Dept. Elec. & Comp. Eng., University of California, Davis, CA 95616.
†Dept. Elec. & Comp. Eng. , Ben-Gurion University, Beer-Sheva 84105, Israel.

perspective. Using the 2-D Wold decomposition, it is shown in [3] that an approximate model for homogeneous textures is a sum of harmonic components in additive noise. Hence, in general, the deterministic component [3], of a homogeneous texture undergoing nonlinear continuous warping can be approximated by a sum of 2-D sinusoids of a polynomial function of the image coordinates.

The proposed model belongs to the general class of AM-FM signals which has been recently investigated both for 1-D and 2-D signals using nonparametric methods, [9], [8], [10], [11]. It is shown that the Teager-Kaiser energy operator can be used to approximately estimate the amplitude envelope of the AM component as well as the instantaneous frequency of the FM component. However, using this method an approximation error exists even when no observation noise is present. The estimation algorithm of 2-D multicomponent AM-FM signals, [11], initially uses multiband bank of Gabor wavelets to isolate the different components, thus avoiding the interference between the various components, and increasing the effective SNR. The estimation of the AM and FM parts of each component follows in the next stage.

For reasons which will become clear later, it is more convenient to work with a complex valued model in which the sinusoidal function is replaced by a complex exponential. In some applications, such as Synthetic Aperture Radar imaging, the 2-D signal is complex valued to begin with. In other applications the 2-D signal is real, but can be converted subject to some restrictive conditions, into complex form through the Hilbert Transform [2].

Throughout this paper we will consider 2-D signals which can be represented by a constant amplitude complex exponential whose phase is a polynomial function of the coordinates. Having defined the model, we will study the problem of estimating its parameters given observations on the 2-D signal. In the presence of additive white Gaussian noise a straightforward but computationally prohibitive approach is to develop a maximum likelihood estimator for the polynomial phase parameters. This estimator involves a multi-dimensional search in the parameter space and is not practical except for very low order models. In this paper we present a suboptimal, but computationally efficient algorithm for estimating the parameters of 2-D constant amplitude polynomial phase signals. This algorithm is an extension of the so-called Polynomial Phase Transform which was introduced in [1]. The algorithm is based on the properties of a 2-D polynomial phase difference operator, which is defined in the next section.

The paper is organized as follows. In section 2 we define the parametric model of the observed signal, define the 2-D polynomial phase difference operator, and present some properties of the operator. In section 3 we present the proposed parameter estimation algorithm which is based on the 2-D polynomial phase difference operator and its properties. We then illustrate the algorithm operation using a numerical example.

2 The Phase Difference Operator

In this section we define the phase difference operator and present some of its basic properties. We start with a description of the type of signal for which the operator was designed.

2.1 The Signal Model

Let $\{v(n,m)\}$ be a discrete 2-D constant amplitude polynomial phase signal. More specifically

$$v(n,m) = A\exp\{j\phi_{S+1}(n,m)\} \ , \quad n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1 \ , \tag{1}$$

where $\phi_{S+1}(n,m) = \sum_{(k,\ell)\in I} c(k,\ell)n^k m^\ell$, and $I = \{0 \le k,\ell \text{ and } 0 \le k+\ell \le S+1\}$. We shall call $\phi_{S+1}(n,m)$ a 2-D polynomial of *total-degree* $S+1$. For example, using this terminology we say that a constant valued field is a 2-D polynomial phase signal of total-degree 0, and a 2-D exponential signal is a 2-D polynomial phase signal of total-degree 1. In other words, one might think of the phase polynomial $\phi_S(n,m)$, as if it has S 'layers' since increasing S by one adds a 'layer' of additional $S+2$ parameters to the phase model. The amplitude A is a real valued positive constant. To simplify the presentation we assume that $A \equiv 1$. Note that the scaling coefficients associated with horizontal and vertical sampling, are absorbed into the coefficients $c(k,\ell)$. The performance of the algorithm in the presence of additive noise is analyzed in [5].

2.2 The PD Operators

In this section we define the two Polynomial Phase Difference Operators which we denote by PD_n and PD_m. First we give a brief heuristic explanation of the idea behind the operator.

Consider the observed signal which is given by (1), and assume for the moment that m and n are continuous variables. Differentiating the observed signal P times along the m axis and $S-P$ times along the n axis, (in any order, as long as the total number of differentiation operations in both axes is S), results in a 2-D complex exponential signal. It can be shown that the spatial frequency (ω,ν) of this complex exponential is a function of two of the coefficients of the highest 'layer', $S+1$, of the phase polynomial parameters, and other known quantities. The exact functional relation of the exponential spatial frequency and the phase parameters is given in the next section. By estimating the frequency of the complex exponential (using standard frequency estimation techniques), we obtain estimates of two of the coefficients of the highest 'layer' of the phase polynomial model. Repeating this procedure for all $0 \le P \le S$, all the coefficients of the highest 'layer', $S+1$, of the phase polynomial model are estimated.

Having completed the estimation of the phase parameters in the highest 'layer', their contribution to the signal phase can be eliminated, thus resulting in a polynomial phase signal of total-degree

S. By repeating this entire process for all the 'layers' in the phase model, all the phase parameters are estimated. The details of how that works will be presented later.

Since in our problem the variables n and m are discrete, phase differentiating will be replaced by phase differencing. In principle, this could be accomplished by computing the phase of the 2-D signal and then performing the differencing operation. However, extraction of the phase function is difficult because of the need to perform phase unwrapping. As we will show next, phase differencing can be accomplished *without* phase unwrapping, by performing a certain nonlinear operation on the 2-D signal, using what we call "the phase differencing (PD) operator". We next define the basic polynomial phase differencing operators.

Definition 1: Let τ_m and τ_n be some positive integers. Define

$$\mathrm{PD}_{m^{(0)}}[v(n,m)] \triangleq v(n,m) \ , n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1 \ , \tag{2}$$

and in general,

$$\mathrm{PD}_{m^{(q)}}[v(n,m)] \triangleq \mathrm{PD}_{m^{(q-1)}}[v(n,m)]\left(\mathrm{PD}_{m^{(q-1)}}[v(n,m+\tau_m)]\right)^* \ , \tag{3}$$

where the resulting 2-D signal $\mathrm{PD}_{m^{(q)}}[v(n,m)]$ exists for $n = 0,1,\ldots,N-1$, $m = 0,1,\ldots,M-1-q\tau_m$. Similarly

$$\mathrm{PD}_{n^{(0)}}[v(n,m)] \triangleq v(n,m) \ , n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1 \ , \tag{4}$$

$$\begin{aligned} \mathrm{PD}_{n^{(p)}}[v(n,m)] \ &\triangleq \ \mathrm{PD}_{n^{(p-1)}}[v(n,m)]\left(\mathrm{PD}_{n^{(p-1)}}[v(n+\tau_n,m)]\right)^* \ , \\ & \qquad n = 0,1,\ldots,N-1-p\tau_n \ , m = 0,1,\ldots,M-1 \ . \end{aligned} \tag{5}$$

Note that applying any of the operators $\mathrm{PD}_{m^{(1)}}[\cdot]$, or $\mathrm{PD}_{n^{(1)}}[\cdot]$ to a 2-D polynomial phase signal of total-degree $S+1$, results in a 2-D polynomial phase signal of total-degree S.

Assume we have sequentially applied, in some arbitrary sequence, P times the phase difference operator $\mathrm{PD}_{n^{(1)}}$, and $S-P$ times the phase difference operator $\mathrm{PD}_{m^{(1)}}$, to the signal $v(n,m)$. Then, it can be shown [4] that the resulting signal, which we denote by $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]$ is given by

$$\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \prod_{q=0}^{S-P}\left\{\prod_{p=0}^{P}\left\{\left[v(n+p\tau_n, m+q\tau_m)\right]^{(-1)^{p+q}}\right\}^{\binom{P}{p}}\right\}^{\binom{S-P}{q}} \ . \tag{6}$$

Theorem 1: Let $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]$ be the 2-D signal obtained by successively applying in some arbitrary sequence, P times the operator $\mathrm{PD}_{n^{(1)}}[\cdot]$, and $S-P$ times the operator $\mathrm{PD}_{m^{(1)}}[\cdot]$, to the observed signal (1). Then, the signal $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]$ is a 2-D exponential given by

$$\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \exp\left\{j[\omega_S n + \nu_S m + \gamma_S(\tau_n,\tau_m)]\right\} \ ,$$
$$n = 0,1,\ldots,N-1-P\tau_n \ , m = 0,1,\ldots,M-1-(S-P)\tau_m \ , \tag{7}$$

where

$$\omega_S = (-1)^S c(P+1,S-P)(P+1)!(S-P)!\tau_n^P \tau_m^{S-P} \ , \tag{8}$$

$$\nu_S = (-1)^S c(P,S+1-P)P!(S+1-P)!\tau_n^P \tau_m^{S-P} \ , \tag{9}$$

and $\gamma_S(\tau_n,\tau_m)$ is not a function of m nor n.

Note from the definition of the phase difference operators in Definition 1, and Theorem 1 that for a 2-D polynomial phase signal $v(n,m)$ of total-degree S, $\mathrm{PD}_{m^{(S+1-P)},n^{(P)}}[v(n,m)] = 1$, and similarly $\mathrm{PD}_{m^{(S-P)},n^{(P+1)}}[v(n,m)] = 1$. Hence, for all $L > S$ applying in some arbitrary sequence, P times the operator $\mathrm{PD}_{n^{(1)}}$, and $L-P$ times the operator $\mathrm{PD}_{m^{(1)}}$, to a 2-D polynomial phase signal $v(n,m)$ of total-degree S yields a unit amplitude constant signal.

3 The Parameter Estimation Algorithm

Consider the observed signal which is given by (1), where S is a non-negative integer, which initially, we assume to be known. We now present an algorithm for sequentially estimating the parameters $\{c(k,\ell)|\ 0 \le k,\ell;\ 0 \le k+\ell \le S+1\}$ of the observed 2-D polynomial phase signal.

Theorem 1 implies that applying in some arbitrary sequence, P times the operator $\mathrm{PD}_{n^{(1)}}$, and $S-P$ times the operator $\mathrm{PD}_{m^{(1)}}$, to the observed signal $v(n,m)$, the resulting signal is the 2-D exponential $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \exp\left\{j[\omega_S n + \nu_S m + \gamma_S(\tau_n,\tau_m)]\right\}$ where ω_S and ν_S are given by (8) and (9), respectively. We can thus reduce any 2-D nonhomogeneous, constant-amplitude polynomial-phase signal, $v(n,m)$, whose phase is of total-degree $S+1$, to a 2-D single tone signal whose frequency is (ω_S,ν_S).

Hence, estimating (ω_S,ν_S) using any standard frequency estimation technique, results in an estimate of $c(P+1,S-P)$, and $c(P,S+1-P)$. In the present paper we estimate the frequency of the exponential using a search for the maximum of the absolute value of the signal 2-D Discrete Fourier Transform (2-D DFT). More specifically, having estimated ω_S and ν_S in (8) and (9), we have

$$\hat{c}(P+1,S-P) = \frac{\hat{\omega}_S}{(-1)^S(P+1)!(S-P)!\tau_n^P \tau_m^{S-P}} \ , \tag{10}$$

and

$$\hat{c}(P,S+1-P) = \frac{\hat{\nu}_S}{(-1)^S P!(S+1-P)!\tau_n^P \tau_m^{S-P}} \ , \tag{11}$$

which constitutes an estimate of two of the parameters of the highest order 'layer', $S+1$, of the phase model parameters (*i.e.*, those $c(k,\ell)$'s for which $0 \leq k,\ell : k+\ell = S+1$).

Recall however that the highest order 'layer', $S+1$, of the phase model parameters has $S+2$ parameters, which need to be estimated. This can be achieved by repeating the procedure which was described above assuming some arbitrary P, for *all* P such that $0 \leq P \leq S$. Note that this procedure results in repeated estimation of some of the phase parameters.

Multiplying $v(n,m)$ by $\exp\{-j\sum_{k=0}^{S+1} \hat{c}(k, S+1-k) n^k m^{S+1-k}\}$ results in a new polynomial phase signal whose total-degree is S. By applying to the resulting signal a procedure similar to the one used to estimate the parameters $c(k,\ell)$ for $k+\ell = S+1$, we obtain an estimate of the $S+1$ parameters in the S 'layer'. Multiplying the 2-D polynomial phase signal of total-degree S, which was obtained in the previous step by $\exp\{-j\sum_{k=0}^{S} \hat{c}(k, S-k) n^k m^{S-k}\}$ we obtain a new polynomial phase signal whose total-degree is $S-1$.

In general, let $v^{(s+1)}(n,m)$ denote the 2-D signal, where $s+1$ denotes the *current* total-degree of its phase polynomial. By repeating for all $s = S, \ldots, 0$, the two basic steps of estimating the $c(k,\ell)$ parameters of 'layer' $s+1$ through finding the maxima of $\left|\mathrm{DFT}\left(\mathrm{PD}_{m^{(s-P)}}\left[\mathrm{PD}_{n^{(P)}}[v^{(s+1)}(n,m)]\right]\right)\right|$, for all $0 \leq P \leq s$, followed by multiplying the already reduced order 2-D polynomial phase signal by $\exp\{-j\sum_{k=0}^{s+1} \hat{c}(k, s+1-k) n^k m^{s+1-k}\}$ in the next step, we obtain estimates for all the phase parameters except $c(0,0)$. The resulting signal after this processing, $v^{(0)}(n,m)$, is a constant phase 2-D signal. Taking now the average of the imaginary part of the logarithm of this signal we obtain an estimate for $c(0,0)$. We have thus completed the estimation of all the coefficients of the 2-D phase polynomial of total-degree $S+1$.

Once the phase parameters were estimated, the amplitude of the polynomial phase signal is obtained by multiplying the original signal by $e^{-j\hat{\phi}(n,m)}$, where $\hat{\phi}(n,m)$ is the estimated phase. Ideally, the resulting 2-D signal is constant with amplitude A. The algorithm is summarized in Table 1.

It was shown in section 2 that overestimating the order of the phase polynomial yields zero estimated coefficients, for the non-existing coefficients, *i.e.*, for all $L > S$ applying L times, in any order, the operators $\mathrm{PD}_{n^{(1)}}[\cdot]$ and $\mathrm{PD}_{m^{(1)}}[\cdot]$ to a 2-D polynomial phase signal $v(n,m)$ of total-degree S yields a unit amplitude constant signal. This property allows for relatively simple order estimation in cases where the polynomial total-degree S is unknown. We start with an assumed upper bound on the total-degree S. In the presence of observation noise, the decision that $\hat{c}(k,\ell) = 0$ can be based on comparison with the Cramer-Rao lower bound (CRB) on the accuracy of jointly estimating the model parameters, derived in [4]. We will decide that $c(k,\ell) = 0$ whenever $|\hat{c}(k,\ell)|$ is not considerably higher than $\{\mathrm{CRB}[c(k,\ell)]\}^{\frac{1}{2}}$.

Next, we present a numerical example to illustrate the operation of the proposed parameter estimation algorithm. Consider a unit amplitude 2-D polynomial phase signal of total-degree

2. In this example the observed field dimensions are $N = 100$, $M = 100$. The phase coefficients are given by $\mathbf{c} = [1; 4.5 \cdot 10^{-1}, 8.2 \cdot 10^{-1}; -1.5 \cdot 10^{-3}, 16 \cdot 10^{-3}, -2.2 \cdot 10^{-3}]^T$, where all the phase parameters were assembled, 'layer' after 'layer' into a vector $\mathbf{c}$ of the general structure $\mathbf{c} = [c(0,0); c(0,1), c(1,0); c(0,2), c(1,1), c(2,0); \ldots, \ldots; c(0,S), \ldots, c(S,0)]^T$, where we use ';' to distinguish 'layer' from 'layer'.

The image of the real part of the observed field $v(n,m)$ is shown in Figure 1, and a plot of the absolute value of its Fourier Transform is shown on its right. It is clear from these two figures that the observed signal is nonhomogeneous and is of a broad bandwidth.

We next describe the steps of the suggested parameter estimation algorithm. Since the polynomial phase total-degree is 2, we start by estimating the parameters of 'layer' 2. In the first step of the algorithm we have $s = 1$, and $P = 0$. Hence, applying the operator $\text{PD}_{n^{(0)}, m^{(1)}}$ to the observed signal, we obtain the signal $x(n,m)$ which is a 2-D polynomial phase signal of total-degree 1, *i.e.*, a 2-D exponential signal. The real part of this signal is shown in the left image of Figure 2, and the absolute value of its DFT is shown in the right hand side of the same figure. Estimating the spatial frequency of the spectral peak results in the estimates of $c(1,1)$, and $c(0,2)$. We therefore see how a broadband nonhomogeneous 2-D signal has been reduced to a 2-D homogeneous signal, in a way that enables us to estimate two of the parameters of the observed signal.

Repeating now the same procedure for $s = 1$, and $P = 1$, *i.e.*, applying the operator $\text{PD}_{n^{(1)}, m^{(0)}}$ to the observed signal, we obtain another 2-D exponential signal. The real part of this signal is shown in the left image of Figure 3, and the absolute value of its DFT is shown in the right hand side of the same figure. Estimating the spatial frequency of the spectral peak results in the estimates of $c(2,0)$, and $c(1,1)$. We have therefore obtained estimates for all three parameters of 'layer' 2. Note, that an estimate of $c(1,1)$ was obtained twice, however both are essentially identical.

Multiplying $v(n,m)$ by $\exp\{-j \sum_{k+\ell=2} \hat{c}(k,\ell) n^k m^\ell\}$ we obtain a new polynomial phase signal, $v^{(1)}(n,m)$, whose total-degree is 1. Its real part is shown in the left image of Figure 4, and the absolute value of its DFT is shown in the right hand side of the same figure. Since in this iteration $s = 0$, and $P = 0$, we have $x(n,m) = v^{(1)}(n,m)$, and the parameters $c(1,0)$, and $c(0,1)$ of 'layer' 1 are estimated by finding the spatial frequency of the peak of the 2-D signal DFT.

Multiplying $v^{(1)}(n,m)$ by $\exp\{-j \sum_{k+\ell=1} \hat{c}(k,\ell) n^k m^\ell\}$ we obtain the signal, $v^{(0)}(n,m)$, whose total-degree is 0. The coefficient $c(0,0)$ can now be computed as the arithmetic average of the imaginary part of the logarithm of $v^{(0)}(n,m)$. Thus, at this point we have completed the estimation of all the phase parameters of the observed 2-D nonhomogeneous signal.

4 Conclusions

We presented a parametric model which can be used as the basic building block in parametric modeling of a broad class of nonhomogeneous signals. We considered 2-D signals which are represented by a single constant-amplitude polynomial phase function. More complex 2-D signals can be represented by sums of components of this type. Using the Polynomial Difference operator and its properties, we derived a computationally efficient (relative to the maximum likelihood estimator) algorithm for estimating the parameters of the polynomial phase function.

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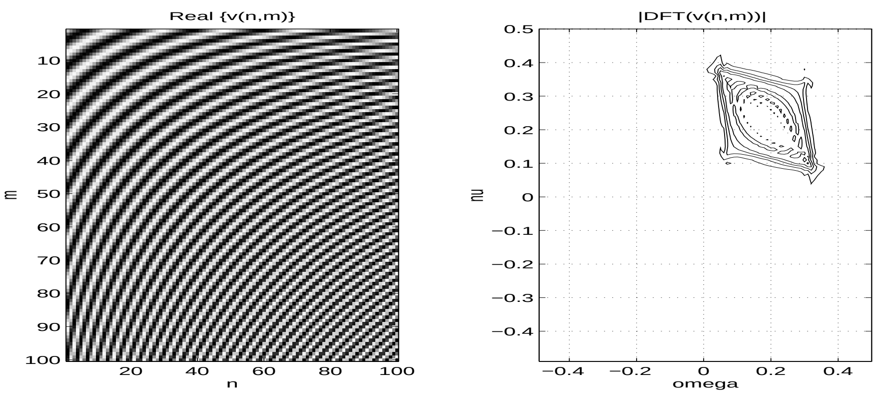

Figure 1: The real part of the observed signal, and the magnitude of the signal DFT.

Table 1: The Estimation Algorithm.

Let $S+1$ denote the total-degree of the observed signal phase.

$s = S,\ v^{(s+1)}(n,m) = v(n,m),\ n = 0, \ldots, N-1,\ m = 0, \ldots, M-1.$

While $s \geq 0$ ($s+1$ is the 'layer' index)

for $P = 0, \ldots, s$ (find all the parameters of the $s+1$ 'layer')

$$x(n,m) = \mathrm{PD}_{m^{(s-P)}}\left[\mathrm{PD}_{n^{(P)}}[v^{(s+1)}(n,m)]\right]$$

$$(\hat{\omega}_s, \hat{\nu}_s) = \underset{(\omega,\nu)}{\mathrm{argmax}} \left|\mathrm{DFT}\Big(x(n,m)\Big)\right|$$

$$\hat{c}(P+1, s-P) = \frac{\hat{\omega}_s}{(-1)^s (P+1)!(s-P)! \tau_n^P \tau_m^{s-P}}$$

$$\hat{c}(P, s+1-P) = \frac{\hat{\nu}_s}{(-1)^s P!(s+1-P)! \tau_n^P \tau_m^{s-P}}$$

end

$$v^{(s)}(n,m) = v^{(s+1)}(n,m) \exp\{-j \textstyle\sum_{\{k+\ell=s+1\}} \hat{c}(k,\ell) n^k m^\ell\}$$

s=s-1

end

$$c(0,0) = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \mathrm{Im}\{\log(v^{(0)}(n,m))\}$$

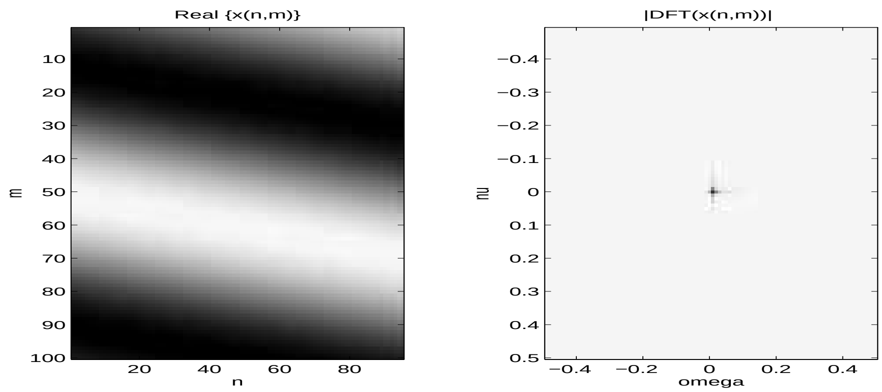

Figure 2: The 2-D polynomial phase signal after applying the operator $\mathrm{PD}_{n^{(0)},m^{(1)}}$ to the observed signal of total-degree 2. (In this iteration, $s = 1$, $p = 0$, and the signal is the observed signal $v^{(2)}(n, m)$). Left: Real part of the resulting 2-D signal. Right: Absolute value of the resulting 2-D signal DFT.

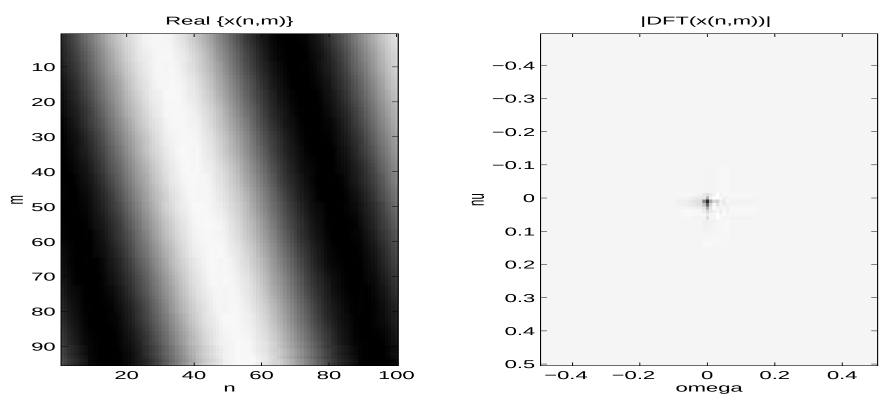

Figure 3: The 2-D polynomial phase signal after applying the operator $\mathrm{PD}_{n^{(1)},m^{(0)}}$ to the observed signal of total-degree 2. (In this iteration, $s = 1$, $p = 1$, and the signal is the observed signal $v^{(2)}(n, m)$). Left: Real part of the resulting 2-D signal. Right: Absolute value of the resulting 2-D signal DFT.

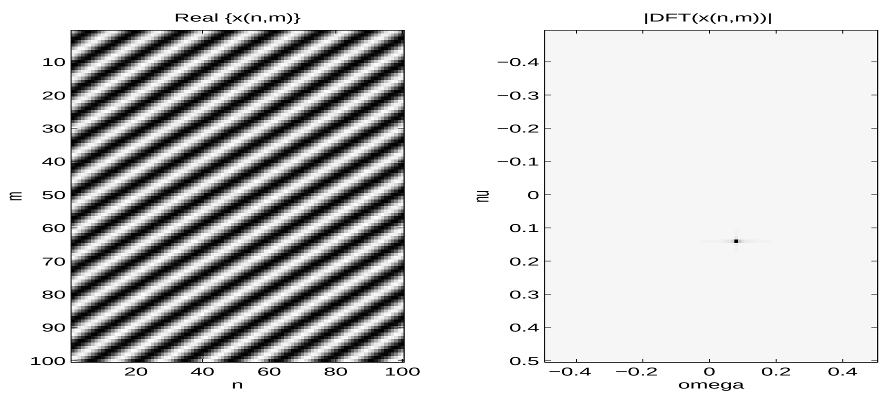

Figure 4: The reduced order 2-D polynomial phase signal $v^{(1)}(n, m)$. (In this iteration, $s = 0$, $p = 0$). Left: Real part of the 2-D signal. Right: Absolute value of the 2-D signal DFT.