Bounds for Estimation of Complex Exponentials in Unknown Colored Noise

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***Abstract*—We consider the problem of estimating the parameters of complex exponentials in the presence of complex additive Gaussian noise with unknown covariance. Bounds are derived for the accuracy of jointly estimating the parameters of the exponentials and the noise. We first present an exact Cramér-Rao bound (CRB) for this problem and specialize it for the cases of circular Gaussian processes and autoregressive processes. We also derive an approximate expression for the CRB, which is related to the conditional likelihood function. Numerical evaluation of these bounds provides some insights on the effect of various signal and noise parameters on the achievable estimation accuracy.**

I. Introduction

THE need to estimate the parameters of sinusoids and complex exponentials in the presence of noise arises in many engineering problems. An extensive amount of work has been done on the development and performance analysis of estimation algorithms for such signals [9], [17]–[19]. The overwhelming majority of this work focuses on the case where the harmonic signals are contaminated by white Gaussian noise. The case where the additive noise is nonwhite but has a *known* correlation function can be reduced to the former case by an operation known as "prewhitening" (with the possible exception of the case where the noise has spectral zeros).

When the additive noise has an unknown covariance function, the problem becomes more difficult. An unbiased estimator of the signal parameters will require, in general, the joint estimation of the parameters of the harmonic component and the parameters of the noise process. In this paper, we develop bounds on the achievable accuracy of jointly estimating the parameters of a complex harmonic process composed of a sum of complex exponentials and the parameters of an additive zero mean Gaussian process.

It should be noted that the Wold decomposition [1] implies that *any* regular discrete and stationary random process can be represented as a sum of two mutually orthogonal components: a *purely indeterministic* process and a *deterministic* one. The spectral measure of the purely indeterministic component is absolutely continuous with respect to the Lebesgue measure. The spectral measure of the deterministic component is singular with respect to the Lebesgue measure, and therefore, it is concentrated on a set of Lebesgue measure zero on the frequency axis. For practical applications, we can exclude singular-continuous spectral measures and distribution functions from the framework of our treatment. Hence, the "spectral density function" of the deterministic component can be represented as a countable sum of delta functions. Under this assumption, the deterministic component is simply the harmonic process $h(n) = \Sigma_{\ell=1}^{L} C_\ell e^{j\omega_\ell n}$, where the C_ℓ's are mutually orthogonal random variables, and ω_ℓ is the frequency of the ℓth harmonic. To put this in other words, the class of signals consisting of exponentials in additive noise is quite general and encompasses essentially all stationary processes. Since, in general, only a single realization of the random process is observed, we cannot infer anything about the variation of the C_ℓ's over different realizations. The best we can do is to estimate the particular values that these coefficients take for the given realization; in other words, we might just as well treat the C_ℓ's as unknown constants.

The problem of analyzing mixed spectrum processes has received some attention in the past. Priestley [2] describes Whittle's and Bartlett's periodogram-based tests for detecting harmonic components in colored noise, as well as a sequential, periodogram-based estimation method for analyzing the long-term sample covariances of the observed data. The estimation method is based on the assumption that for long lags, the contribution of the noise to the covariance function is insignificant. The idea is to first test for the existence of harmonic components. If such components are detected, their parameters are estimated, and their contribution to the sample covariance is removed. Next, the spectral density function of the noise is estimated from the "corrected" sample covariances. More recently, a conditional maximum likelihood algorithm for estimating the parameters of exponentials in colored AR noise was suggested in [4] and [5]. In [6], the asymptotic properties of the AR and Capon's spectral estimators are employed for mixed spectrum identification in situations where a large number of correlation lags are available. However, all the above works are concerned with real valued processes. The algorithm [4] was extended to the case of circular Gaussian complex-valued processes in [7]. Beyond that, little information seems to be available on the problem of estimating mixed-spectrum complex processes.

In this paper, we show that for Gaussian noise, the estimation of the noise and harmonic components are decoupled, regardless of the parametric model used for the noise. We derive the results both for the general case where the noise is a complex-valued Gaussian process and for the special case of

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a circular Gaussian process. For the case of an autoregressive noise process, we present an analytic expression for the exact Cramér-Rao bound (CRB) on the noise and harmonic signal parameters. We also derive the conditional CRB, which corresponds to the estimation algorithm proposed in [7].

The paper is organized as follows. We first derive an exact CRB for the joint estimation problem in terms of the noise covariance function without assuming any specific model for this component. In Section III, we give results for the special case of circular Gaussian processes. In Section IV, we present the results for the case in which the noise is an autoregressive Gaussian process. In Section V, we derive an approximate expression for the CRB on the autoregressive noise and the harmonic signal by computing the CRB from the conditional likelihood function of the observed data, rather than the exact likelihood function. The results show that for a conditional ML estimator, the bounds on both the amplitude and frequency parameters of the harmonic components are functions of the frequency response of the colored noise model at the frequencies of the harmonic components and of the derivative of the frequency response at these frequencies. The relation of this approximation to the exact CRB is also studied, and it is shown how the conditional bound can be derived as a special case of the exact bound. A conditional bound on the frequency parameters of exponentials in AR noise has been recently stated (with no proof) in [10]. In Section VI, we present numerical examples in order to provide some insights on the effect of various signal and noise parameters. Some concluding remarks are presented in Section VII.

II. The CRB on the Harmonic and Noise Components

In this section, we present the CRB formulas for the joint estimation problem for the most general case. We start by formulating the problem and introducing some of the necessary notation.

We are given measurements from a single realization of a process $\{y(n)\}_{n=0}^{N-1}$, where

$$y(n) = \mu(n) + w(n), \qquad n = 0, 1, \cdots, (N-1) \tag{1}$$

is the sum of a zero-mean, stationary, complex Gaussian noise process $w(n)$ and a complex harmonic mean. Let us rewrite the equation above using real variables:

$$\tilde{y}(n) = \tilde{\mu}(n) + \tilde{w}(n) \tag{2}$$

where $\tilde{y}(n) = [\text{Re}\{y(n)\}, \text{Im}\{y(n)\}]^T, \tilde{w}(n) = [\text{Re}\{w(n)\}, \text{Im}\{w(n)\}]^T, \tilde{\mu}(n) = [u(n), v(n)]^T$. In other words, the real and imaginary parts of the harmonic process are defined by

$$\mu(n) = u(n) + jv(n) \tag{3}$$

where

$$u(n) = \text{Re}\{\mu(n)\} = \sum_{\ell=1}^{L} C_\ell^R \cos\omega_\ell n - \sum_{\ell=1}^{L} C_\ell^I \sin\omega_\ell n, \tag{4}$$

$$v(n) = \text{Im}\{\mu(n)\} = \sum_{\ell=1}^{L} C_\ell^I \cos\omega_\ell n + \sum_{\ell=1}^{L} C_\ell^R \sin\omega_\ell n \tag{5}$$

and $C_\ell^R = \text{Re}\{C_\ell\}, C_\ell^I = \text{Im}\{C_\ell\}$.

Let $\boldsymbol{t} = [0, 1, \cdots, (N-1)]^T$ be the time index, and let

$$\boldsymbol{A} = [\cos\omega_1\boldsymbol{t} \cdots \cos\omega_L\boldsymbol{t}] \tag{6}$$

where $\cos\omega_k\boldsymbol{t}$ denotes a column vector whose elements are $\cos\omega_k t$, where t are the elements of $\boldsymbol{t}$. Similarly

$$\boldsymbol{B} = [\sin\omega_1\boldsymbol{t} \cdots \sin\omega_L\boldsymbol{t}] \tag{7}$$

where $\sin\omega_k\boldsymbol{t}$ denotes a column vector whose elements are $\sin\omega_k t$, where t are the elements of $\boldsymbol{t}$.

Next, we make the following definitions:

$$\boldsymbol{\omega} = [\omega_1 \cdots \omega_L]^T, \tag{8}$$

$$\boldsymbol{c}_R = [C_1^R, \cdots, C_L^R]^T, \tag{9}$$

$$\boldsymbol{c}_I = [C_1^I, \cdots, C_L^I]^T. \tag{10}$$

In addition, let

$$\boldsymbol{y} = [y(0), y(1), \cdots, y(N-1)]^T. \tag{11}$$

The vectors $\boldsymbol{w}, \boldsymbol{\mu}, \boldsymbol{u}, \boldsymbol{v}$ are similarly defined.

Let $\tilde{\boldsymbol{\mu}}$ be the real valued vector $\tilde{\boldsymbol{\mu}} = [\boldsymbol{u}^T\boldsymbol{v}^T]^T$. Similarly, let $\tilde{w} = [\text{Re}\{\boldsymbol{w}^T\}\ \text{Im}\{\boldsymbol{w}^T\}]^T, \tilde{\boldsymbol{y}} = [\text{Re}\{\boldsymbol{y}^T\}\ \text{Im}\{\boldsymbol{y}^T\}]^T$, and $\boldsymbol{c} = [\boldsymbol{c}_R^T\boldsymbol{c}_I^T]^T$. Thus

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\rho}\boldsymbol{c} \tag{12}$$

where

$$\boldsymbol{\rho} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}. \tag{13}$$

Let $\boldsymbol{\theta} = \{\boldsymbol{\alpha}, \boldsymbol{c}, \boldsymbol{\omega}\}$ be the vector of unknown parameters. Here, we assume that the noise component w has some known parametric model, where the real vector $\boldsymbol{\alpha}$ is the parameter vector. At the moment, we will not specify the model but rather leave it implicit. In Section IV, we specialize our results for the case in which $\boldsymbol{\alpha}$ is the parameter vector of an autoregressive process.

The general expression for the Fisher information matrix (FIM) of a real Gaussian process is given by (e.g., [15])

$$\tilde{\boldsymbol{J}}_{k,\ell}(\boldsymbol{\theta}) = \frac{\partial\tilde{\boldsymbol{\mu}}^T}{\partial\boldsymbol{\theta}_k}\tilde{\boldsymbol{\Gamma}}^{-1}\frac{\partial\tilde{\boldsymbol{\mu}}}{\partial\boldsymbol{\theta}_\ell} + \frac{1}{2}\text{tr}\left\{\tilde{\boldsymbol{\Gamma}}^{-1}\frac{\partial\tilde{\boldsymbol{\Gamma}}}{\partial\boldsymbol{\theta}_k}\tilde{\boldsymbol{\Gamma}}^{-1}\frac{\partial\tilde{\boldsymbol{\Gamma}}}{\partial\boldsymbol{\theta}_\ell}\right\} \tag{14}$$

where $\tilde{\boldsymbol{\Gamma}}$ is the covariance matrix of $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{\mu}}$ is its mean. To compute the elements of the FIM, we need to evaluate the derivatives of the covariance matrix and the mean with respect to the various unknown parameters.

Taking the partial derivatives of $\tilde{\boldsymbol{\mu}}$ we get

$$\frac{\partial\tilde{\boldsymbol{\mu}}}{\partial\boldsymbol{c}_\ell} = \boldsymbol{\rho}\boldsymbol{e}_\ell = \boldsymbol{\rho}_\ell \tag{15}$$

where $\boldsymbol{e}_\ell$ is a column vector whose ℓth element is 1, and all its other elements are zero, and $\boldsymbol{\rho}_\ell$ is the ℓth column of $\boldsymbol{\rho}$. Since the noise component w is a zero mean process, $\tilde{\boldsymbol{\mu}}$ is independent of the parameters of the noise component, and hence

$$\frac{\partial\tilde{\boldsymbol{\mu}}}{\partial\alpha_k} = 0. \tag{16}$$

Therefore

$$\tilde{J}^{\alpha,\alpha}_{k,\ell}(\theta) = \frac{1}{2}\mathrm{tr}\left\{\tilde{\Gamma}^{-1}\frac{\partial\tilde{\Gamma}}{\partial\alpha_k}\tilde{\Gamma}^{-1}\frac{\partial\tilde{\Gamma}}{\partial\alpha_\ell}\right\}. \tag{17}$$

Note also that since the process covariance function $\tilde{\Gamma}$ is independent of the mean

$$\frac{\partial\tilde{\Gamma}}{\partial c_k} = 0. \tag{18}$$

Hence, the $\frac{1}{2}\mathrm{tr}\{\cdot\}$ term in (14) vanishes for all the FIM entries that correspond to the amplitude parameters of the mean, and it follows that $\tilde{J}^{\alpha,c} = 0$.

The FIM elements that correspond to the amplitude parameters of the harmonic component are given by

$$\tilde{J}^{c,c}_{k,\ell} = \rho_k^T\tilde{\Gamma}^{-1}\rho_\ell. \tag{19}$$

Taking the partial derivative with respect to the harmonic frequencies, we get

$$\frac{\partial u}{\partial\omega_k} = -T(C_k^R\sin\omega_k t + C_k^I\cos\omega_k t), \tag{20}$$

$$\frac{\partial v}{\partial\omega_k} = T(C_k^T\cos\omega_k t - C_k^I\sin\omega_k t) \tag{21}$$

where $T = \mathrm{diag}\{t\}$. Hence

$$\frac{\partial\tilde{\mu}}{\partial\omega_k} = \begin{bmatrix}\frac{\partial u}{\partial\omega_k}\\ \frac{\partial v}{\partial\omega_k}\end{bmatrix}. \tag{22}$$

Note also that since the process covariance function $\tilde{\Gamma}$ is independent of the mean of $y(n)$

$$\frac{\partial\tilde{\Gamma}}{\partial\omega_k} = 0. \tag{23}$$

Hence, the $\frac{1}{2}\mathrm{tr}\{\cdot\}$ term in (14) vanishes for all the FIM entries that correspond to parameters of the mean of $y(n)$, and we have $\tilde{J}^{\alpha,\omega} = 0$, and

$$\tilde{J}^{c,\omega}_{k,\ell} = \rho_k^T\tilde{\Gamma}^{-1}\frac{\partial\tilde{\mu}}{\partial\omega_\ell}. \tag{24}$$

Finally

$$\tilde{J}^{\omega,\omega}_{k,\ell} = \left(\frac{\partial\tilde{\mu}}{\partial\omega_k}\right)^T\tilde{\Gamma}^{-1}\frac{\partial\tilde{\mu}}{\partial\omega_\ell}. \tag{25}$$

Hence, we conclude that the bound on the achievable accuracy in estimating the parameters of the noise component is decoupled from the bound on the mean parameters. Therefore, the bound on the noise component is found by inverting (17), and it is independent of the mean parameters. Hence, this bound is identical to the one obtained for a zero mean process. A similar conclusion was derived in [12] for real-valued processes.

III. The CRB for Circular Gaussian Processes

The results in the previous section were derived for a general complex stationary Gaussian process. A special class of complex Gaussian processes that arises in many engineering applications is the class of circular Gaussian processes [16]. In this section, we derive briefly the CRB for this important class of processes.

An N-dimensional complex valued random vector y is circular Gaussian if the $2N$ vector

$$\begin{bmatrix}\mathrm{Re}\{y\}\\ \mathrm{Im}\{y\}\end{bmatrix} \tag{26}$$

is normally distributed with mean

$$\begin{bmatrix}\mathrm{Re}\{\mu\}\\ \mathrm{Im}\{\mu\}\end{bmatrix} \tag{27}$$

and covariance matrix

$$\frac{1}{2}\begin{bmatrix}\mathrm{Re}\{\Gamma\} & -\mathrm{Im}\{\Gamma\}\\ \mathrm{Im}\{\Gamma\} & \mathrm{Re}\{\Gamma\}\end{bmatrix} \tag{28}$$

where

$$\Gamma = E\{(y-\mu)(y-\mu)^H\}. \tag{29}$$

The probability density function of a circular Gaussian process is given by

$$P(y) = \frac{1}{\pi^N\det\Gamma}\exp\{-(y-\mu)^H\Gamma^{-1}(y-\mu)\}. \tag{30}$$

The FIM of a circular Gaussian process is given by

$$J_{k,\ell}(\theta) = 2\,\mathrm{Re}\left\{\frac{\partial\mu^H}{\partial\theta_k}\Gamma^{-1}\frac{\partial\mu}{\partial\theta_\ell}\right\} + \mathrm{tr}\left\{\Gamma^{-1}\frac{\partial\Gamma}{\partial\theta_k}\Gamma^{-1}\frac{\partial\Gamma}{\partial\theta_\ell}\right\} \tag{31}$$

where θ is a real-valued parameter vector (e.g., [11]). Since

$$\frac{\partial\mu}{\partial\omega_k} = \frac{\partial u}{\partial\omega_k} + j\frac{\partial v}{\partial\omega_k} \tag{32}$$

we obtain,using (20) and (21)

$$\begin{aligned}\frac{\partial\mu}{\partial\omega_k} &= -T(C_k^R\sin\omega_k t + C_k^I\cos\omega_k t)\\ &\quad + jT(C_k^R\cos\omega_k t - C_k^I\sin\omega_k t)\\ &= jC_kT\exp(j\omega_k t).\end{aligned} \tag{33}$$

Similarly, using (15), we conclude that

$$\frac{\partial\mu}{\partial c_\ell} = \rho_\ell^u + j\rho_\ell^v \tag{34}$$

where

$$\rho_\ell^u = [\rho_\ell(0),\cdots,\rho_\ell(N-1)]^T \tag{35}$$

and

$$\rho_\ell^v = [\rho_\ell(N),\cdots,\rho_\ell(2N-1)]^T. \tag{36}$$

Following the same arguments as in the previous section, we finally conclude that the FIM of y is given by

$$J = \begin{bmatrix}J^{\alpha,\alpha} & 0 & 0\\ 0 & J^{c,c} & J^{c,\omega}\\ 0 & (J^{c,\omega})^T & J^{\omega,\omega}\end{bmatrix} \tag{37}$$

where

$$J^{\alpha,\alpha}_{k,\ell} = \mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial \alpha_k}\Gamma^{-1}\frac{\partial \Gamma}{\partial \alpha_\ell}\right\}, \tag{38}$$

$$J^{c,c}_{k,\ell} = 2\,\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial c_k}\Gamma^{-1}\frac{\partial \mu}{\partial c_\ell}\right\}, \tag{39}$$

$$J^{c,\omega}_{k,\ell} = 2\,\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial c_k}\Gamma^{-1}\frac{\partial \mu}{\partial \omega_\ell}\right\}, \tag{40}$$

$$J^{\omega,\omega}_{k,\ell} = 2\,\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial \omega_k}\Gamma^{-1}\frac{\partial \mu}{\partial \omega_\ell}\right\}. \tag{41}$$

Note that for the special case of a single exponential in circular complex white Gaussian noise of variance σ^2, (39)–(41) reduce to

$$J^{c,c}_{k,\ell} = \frac{2}{\sigma^2}\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial c_k}\frac{\partial \mu}{\partial c_\ell}\right\}, \tag{42}$$

$$J^{c,\omega}_{k,\ell} = \frac{2}{\sigma^2}\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial c_k}\frac{\partial \mu}{\partial \omega_1}\right\}, \tag{43}$$

$$J^{\omega,\omega}_{k,\ell} = \frac{2}{\sigma^2}\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial \omega_1}\frac{\partial \mu}{\partial \omega_1}\right\} \tag{44}$$

with

$$\frac{\partial \mu}{\partial \omega_1} = jC_1 T \exp(j\omega_1 t), \tag{45}$$

$$\frac{\partial \mu}{\partial c_1} = \exp(j\omega_1 t), \tag{46}$$

$$\frac{\partial \mu}{\partial c_2} = j\exp(j\omega_1 t). \tag{47}$$

Since $c_1 = C_1^R$ and $c_2 = C_1^I$, the FIM block that corresponds to the parameters of the exponential is given in this case by

$$J_1 = \frac{2}{\sigma^2}\begin{bmatrix} N & 0 & 0 \\ 0 & N & C_1^R\sum_{n=0}^{N-1} n \\ 0 & C_1^R\sum_{n=0}^{N-1} n & |C_1|^2\sum_{n=0}^{N-1} n^2 \end{bmatrix} \tag{48}$$

which is a well known result [8], [9].

IV. The Exact CRB for Autoregressive Noise

In the previous sections, we have derived the CRB for a general noise process, whose covariance matrix is parameterized in some unspecified way. It is well known that the most general model for the noise component w is that of a complex moving average model with a possibly infinite order. In this section, we consider the special case in which the noise component is a Pth-order autoregressive (AR) process. The autoregressive process is defined by

$$w(n) = -\sum_{i=1}^{P} a_i w(n-i) + u(n) \tag{49}$$

with $u(n) = u^R(n) + ju^I(n)$, where $\{u^R(n)\}$ and $\{u^I(n)\}$ are independent real Gaussian white noise processes each with zero mean and variance $\sigma^2_{\mathrm{AR}}/2$. The process (49) is a circular Gaussian process, and therefore, the results of Section III are applicable to this case.

Since Γ is Toeplitz–Hermitian, it can be shown (e.g., [13]) that the inverse covariance matrix Γ^{-1} of a Pth-order AR process $(N \geq P)$ is given by

$$\Gamma^{-1} = \frac{1}{\sigma^2_{\mathrm{AR}}}(A_1A_1^H - A_2A_2^H) \tag{50}$$

where A_1 and A_2 are lower triangular Toeplitz matrices such that

$$(A_1)_{i,j} = \begin{cases} 1, & i=j \\ a_{i-j}, & i>j \\ 0, & i<j; \end{cases} \tag{51}$$

$$(A_2)_{i,j} = \begin{cases} a^*_{N-i+j}, & i \geq j \\ 0, & i<j \end{cases} \tag{52}$$

and $a_k = 0$ for $k<0$ and $k>P$. Hence

$$\Gamma = \sigma^2_{\mathrm{AR}}(A_1A_1^H - A_2A_2^H)^{-1}. \tag{53}$$

In the present case, the parameter vector of the noise process α is the $2P+1$ dimensional vector $\alpha = [\sigma^2_{\mathrm{AR}}, a_1^R, a_2^R, \cdots, a_P^R, a_1^I, a_2^I, \cdots, a_P^I]^T$, where $a_n^R = \mathrm{Re}\{a_n\}$ and $a_n^I = \mathrm{Im}\{a_n\}$. Taking the partial derivatives of Γ^{-1} with respect to σ^2_{AR}, using (50), we get

$$\frac{\partial \Gamma^{-1}}{\partial \sigma^2_{\mathrm{AR}}} = -\frac{1}{\sigma^4_{\mathrm{AR}}}(A_1A_1^H - A_2A_2^H) = -\frac{1}{\sigma^2_{\mathrm{AR}}}\Gamma^{-1}. \tag{54}$$

Taking the partial derivatives of Γ^{-1} with respect to the real and imaginary components of a_n, we get

$$\begin{aligned}\frac{\partial \Gamma^{-1}}{\partial a_n^R} &= \frac{1}{\sigma^2_{\mathrm{AR}}}\left(\frac{\partial A_1}{\partial a_n^R}A_1^H + A_1\frac{\partial A_1^H}{\partial a_n^R} - \frac{\partial A_2}{\partial a_n^R}A_2^H - A_2\frac{\partial A_2^H}{\partial a_n^R}\right) \\ &= \frac{1}{\sigma^2_{\mathrm{AR}}}(Z_nA_1^H + A_1Z_n^T - Z_{N-n}A_2^H - A_2Z_{N-n}^T) \\ &\quad n = 1, \cdots, P \end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\partial \Gamma^{-1}}{\partial a_n^I} &= \frac{j}{\sigma^2_{\mathrm{AR}}}(Z_nA_1^H - A_1Z_n^T + Z_{N-n}A_2^H - A_2Z_{N-n}^T) \\ &\quad n = 1, \cdots, P \end{aligned} \tag{56}$$

where Z_n is the down shift matrix

$$(Z_n)_{i,j} = \begin{cases} 1, & i-j=n \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

Substituting (50) and (54)–(56) into the general expression of $J^{\alpha,\alpha}_{k,\ell}$ in (38), we obtain a relatively simple expression for $J^{\alpha,\alpha}$. Similarly, substituting (50) into (39)–(41), expressions for $J^{c,c}$, $J^{c,\omega}$, and $J^{\omega,\omega}$ are obtained. Hence, we have obtained a simple expression for the CRB on the parameters of the harmonic signal and on the colored AR noise in which it is embedded.

Following (37) and the general conclusions of the previous section, we conclude that if the noise process is an AR process, the bound on the AR process parameters is obtained by inverting (38) after the above substitutions were made. This bound is independent of the mean parameters and is identical to the one obtained for the same AR process with a mean component, which is identically zero.

In many cases, we are interested not in the mean or the noise component parameters but in estimating some function of these parameters. For example, having estimated $\boldsymbol{\alpha}$, we can estimate the spectral density function of the process by using the estimated parameters. The CRB on the spectral density function $S(e^{j\omega})$ of the noise component is given by (see e.g., [14])

$$\text{CRB}(S(e^{j\omega})) = \boldsymbol{W}\,\text{CRB}(\boldsymbol{\alpha})\boldsymbol{W}^T \tag{58}$$

where

$$\begin{aligned}\boldsymbol{W} &= \left[\frac{\partial S(e^{j\omega})}{\partial \sigma_{\text{AR}}^2}, \frac{\partial S(e^{j\omega})}{\partial a_1^R}, \cdots, \frac{\partial S(e^{j\omega})}{\partial a_P^R}, \frac{\partial S(e^{j\omega})}{\partial a_1^I}, \cdots, \frac{\partial S(e^{j\omega})}{\partial a_P^I}\right] \\ &= 2S(e^{j\omega})\left[\frac{1}{2\sigma_{\text{AR}}^2}, -\text{Re}\left\{\frac{e^{j\omega(P-1)}}{A(e^{j\omega})}\right\}, -\text{Re}\left\{\frac{e^{j\omega(P-2)}}{A(e^{j\omega})}\right\}, \cdots, \text{Re}\left\{\frac{1}{A(e^{j\omega})}\right\}, \text{Im}\left\{\frac{e^{j\omega(P-1)}}{A(e^{j\omega})}\right\}, \cdots, \text{Im}\left\{\frac{1}{A(e^{j\omega})}\right\}\right]\end{aligned} \tag{59}$$

and

$$A(e^{j\omega}) = e^{j\omega P} + a_1 e^{j\omega(P-1)} + \cdots + a_P. \tag{60}$$

V. The Conditional CRB for Autoregressive Noise

A conditional maximum likelihood (CML) algorithm for joint estimation of the parameters of exponentials in circular Gaussian AR noise, and the AR model parameters, was suggested in [7]. In this section, we derive the performance bound for this algorithm and compare it with the exact bound that was derived in the previous section.

The conditional CRB is derived by using the conditional probability density function of the observed process.

$$\begin{aligned}P(\boldsymbol{y}|y(0)\cdots y(P-1);\boldsymbol{\theta}) &= \frac{1}{(\pi\sigma_{\text{AR}}^2)^{N-P}} \\ &\cdot \exp\left\{-\frac{1}{\sigma_{\text{AR}}^2}\sum_{n=P}^{N-1}\left|\sum_{i=0}^{P} a_i[y(n-i)-\mu(n-i)]\right|^2\right\}\end{aligned} \tag{61}$$

where $a_0 = 1$.

Let us denote by $\bar{\boldsymbol{\theta}}$ the mean component parameter vector, i.e., $\bar{\boldsymbol{\theta}} = [\boldsymbol{c}^T\ \boldsymbol{\omega}^T]^T$. Taking now the partial derivatives of the conditional log likelihood function with respect to the mean parameters, we have

$$\begin{aligned}\frac{\partial \ln P}{\partial \bar{\boldsymbol{\theta}}_k} = \frac{1}{\sigma_{\text{AR}}^2}\Bigg\{&\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i \frac{\partial \mu(n-i)}{\partial \bar{\boldsymbol{\theta}}_k}\right) \cdot \left(\sum_{i=0}^{P} a_i^*[y^*(n-i)-\mu^*(n-i)]\right) \\ &+ \sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i[y(n-i)-\mu(n-i)]\right) \cdot \left(\sum_{i=0}^{P} a_i^* \frac{\partial \mu^*(n-i)}{\partial \bar{\boldsymbol{\theta}}_k}\right)\Bigg\}.\end{aligned} \tag{62}$$

Taking the partial derivative with respect to the AR process driving noise variance parameter yields

$$\begin{aligned}-E\left\{\frac{\partial^2 \ln P}{\partial \sigma_{\text{AR}}^2 \partial \bar{\boldsymbol{\theta}}_\ell}\right\} = \frac{1}{\sigma_{\text{AR}}^4} E\Bigg\{&\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i \frac{\partial \mu(n-i)}{\partial \bar{\boldsymbol{\theta}}_k}\right) \cdot \left(\sum_{i=0}^{P} a_i^*[y^*(n-i)-\mu^*(n-i)]\right) \\ &+ \sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i[y(n-i)-\mu(n-i)]\right) \cdot \left(\sum_{i=0}^{P} a_i^* \frac{\partial \mu^*(n-i)}{\partial \bar{\boldsymbol{\theta}}_k}\right)\Bigg\} = 0.\end{aligned} \tag{63}$$

Similarly, taking the partial derivatives with respect to the real and imaginary parts of the AR process parameters $\{a_\ell^R\}_{\ell=1}^P$ and $\{a_\ell^I\}_{\ell=1}^P$, we find that $-E\{\partial^2 \ln P/\partial\bar{\boldsymbol{\theta}}_k \partial a_\ell^R\} = 0$ and $-E\{\partial^2 \ln P/\partial\bar{\boldsymbol{\theta}}_k \partial a_\ell^I\} = 0$.

Thus, the conditional FIM is block diagonal. Hence, the conditional CRB on the parameters of the harmonic mean is decoupled from the bound for the autoregressive parameters, as was the case for the exact CRB. Therefore, the conditional CRB is obtained by inverting the FIM blocks that correspond to the mean and to the AR parameters, respectively.

Taking the partial derivatives with respect to the parameters of the harmonic mean, we get

$$\begin{aligned}-E\left\{\frac{\partial^2 \ln P}{\partial \bar{\boldsymbol{\theta}}_k \partial \bar{\boldsymbol{\theta}}_\ell}\right\} &= \frac{1}{\sigma_{\text{AR}}^2}\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i \frac{\partial \mu(n-i)}{\partial \bar{\boldsymbol{\theta}}_k}\right) \cdot \left(\sum_{i=0}^{P} a_i^* \frac{\partial \mu^*(n-i)}{\partial \bar{\boldsymbol{\theta}}_\ell}\right) \\ &\quad + \frac{1}{\sigma_{\text{AR}}^2}\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i \frac{\partial \mu(n-i)}{\partial \bar{\boldsymbol{\theta}}_\ell}\right) \cdot \left(\sum_{i=0}^{P} a_i^* \frac{\partial \mu^*(n-i)}{\partial \bar{\boldsymbol{\theta}}_k}\right) \\ &= 2\,\text{Re}\left\{\frac{1}{\sigma_{\text{AR}}^2}\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i^* \frac{\partial \mu^*(n-i)}{\partial \bar{\boldsymbol{\theta}}_k}\right) \cdot \left(\sum_{i=0}^{P} a_i \frac{\partial \mu(n-i)}{\partial \bar{\boldsymbol{\theta}}_\ell}\right)\right\}.\end{aligned} \tag{64}$$

Thus, the derivation of the conditional FIM reveals that the bounds on both the amplitude and the frequency parameters of the harmonic components are functions of the frequency response of the AR model transfer function at the frequencies of the harmonic components and of the *derivative* of the frequency response at these frequencies.

Finally, writing (64) in matrix form, we get

$$-E\left\{\frac{\partial^2 \ln P}{\partial \bar{\boldsymbol{\theta}}_k \partial \bar{\boldsymbol{\theta}}_\ell}\right\} = \frac{2}{\sigma_{\text{AR}}^2}\text{Re}\left\{\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \bar{\boldsymbol{\theta}}_k}\right)^H \bar{\boldsymbol{A}}_1 \bar{\boldsymbol{A}}_1^H \left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \bar{\boldsymbol{\theta}}_\ell}\right)\right\}. \tag{65}$$

$\overline{A}_1$ is the $N \times (N-P)$ matrix of the conjugates of the first $N-P$ columns of A_1

$$\overline{A}_1 = \begin{bmatrix} 1 & & & \\ a^*(1) & 1 & & \\ a^*(2) & a^*(1) & 1 & \\ \vdots & & \ddots & \\ a^*(p) & a^*(p-1) & \cdots & 1 \\ & a^*(p) & \cdots & a^*(1) \\ & & \ddots & \vdots \\ & & & a^*(p) \end{bmatrix} \tag{66}$$

and

$$K = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \tag{67}$$

is the exchange matrix.

Next, we will study the relationship of the conditional CRB to the exact CRB.

A. Analysis of the Conditional CRB

Let $\overline{y}$ be the reverse-order version of y, i.e. $\overline{y} = Ky$. Let $\overline{\Gamma}$ be the covariance matrix of $\overline{y}$. Since Γ is Hermitian and Toeplitz

$$\overline{\Gamma} = E[Kyy^H K^H] = K\Gamma K = \Gamma^T \tag{68}$$

where we have used the Hermitian property of K. Hence, in our problem

$$\begin{aligned} J_{k,\ell}(\theta) &= 2\,\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial \theta_k}\Gamma^{-1}\frac{\partial \mu}{\partial \theta_\ell}\right\} + \mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial \theta_k}\Gamma^{-1}\frac{\partial \Gamma}{\partial \theta_\ell}\right\} \\ &= 2\,\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial \theta_k}K^H K\Gamma^{-1}K^H K\frac{\partial \mu}{\partial \theta_\ell}\right\} \\ &\quad + \mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial \theta_k}\Gamma^{-1}\frac{\partial \Gamma}{\partial \theta_\ell}\right\} \\ &= 2\,\mathrm{Re}\left\{\left(K\frac{\partial \mu}{\partial \theta_k}\right)^H (\Gamma^{-1})^T \left(K\frac{\partial \mu}{\partial \theta_\ell}\right)\right\} \\ &\quad + \mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial \theta_k}\Gamma^{-1}\frac{\partial \Gamma}{\partial \theta_\ell}\right\} \end{aligned} \tag{69}$$

where the last equality results from the fact that $K\Gamma^{-1}K^H = K^{-1}\Gamma^{-1}K^{-1} = (K\Gamma K)^{-1} = (\Gamma^T)^{-1}$. Substituting (50) into the mean dependent part of (69) yields for the mean parameters

$$\begin{aligned} &2\,\mathrm{Re}\left\{\left(K\frac{\partial \mu}{\partial \overline{\theta}_k}\right)^H (\Gamma^{-1})^T \left(K\frac{\partial \mu}{\partial \overline{\theta}_\ell}\right)\right\} \\ &= \frac{2}{\sigma_{\mathrm{AR}}^2}\mathrm{Re}\left\{\left(K\frac{\partial \mu}{\partial \overline{\theta}_k}\right)^H (A_1 A_1^H)^T \left(K\frac{\partial \mu}{\partial \overline{\theta}_\ell}\right)\right\} \\ &\quad - \frac{2}{\sigma_{\mathrm{AR}}^2}\mathrm{Re}\left\{\left(K\frac{\partial \mu}{\partial \overline{\theta}_k}\right)^H (A_2 A_2^H)^T \left(K\frac{\partial \mu}{\partial \overline{\theta}_\ell}\right)\right\} \\ &= \frac{2}{\sigma_{\mathrm{AR}}^2}\mathrm{Re}\left\{\left(K\frac{\partial \mu}{\partial \overline{\theta}_k}\right)^H A_1^* A_1^T \left(K\frac{\partial \mu}{\partial \overline{\theta}_\ell}\right)\right\} \\ &\quad - \frac{2}{\sigma_{\mathrm{AR}}^2}\mathrm{Re}\left\{\left(K\frac{\partial \mu}{\partial \overline{\theta}_k}\right)^H A_2^* A_2^T \left(K\frac{\partial \mu}{\partial \overline{\theta}_\ell}\right)\right\} \end{aligned} \tag{70}$$

TABLE I
Three First-Order AR Models

Test Case	σ_{AR}^2	a_1
Narrow Band AR	1	$-0.975e^{j\frac{\pi}{4}}$
Medium Bandwidth AR	1	$-0.85e^{j\frac{\pi}{4}}$
Wideband AR	1	$-0.35e^{j\frac{\pi}{4}}$

where A_1^* denotes the conjugate of A_1, i.e., $A_1^* = (A_1^H)^T$ and similarly for A_2^*. Note that $(K(\partial\mu/\partial\overline{\theta}_k))^H A_2^*$ is a function of only the last P elements of $K(\partial\mu/\partial\overline{\theta}_k)$, which are the P initial values of $\partial\mu/\partial\overline{\theta}_k$. In addition, note that the entries of A_2 are all zeros except for the lower $P \times P$ triangular block.

For $N \gg P$, we can ignore the contributions to (70) of the term that is a function of A_2 as well as the contribution of the P right-most columns of A_1 (which is also a function of only the P initial values of $\partial\mu/\partial\overline{\theta}_k$), in which case, the part of (69) depending on the mean can be approximated by

$$\begin{aligned} &2\,\mathrm{Re}\left\{\frac{\partial \mu^H}{\partial \overline{\theta}_k}\Gamma^{-1}\frac{\partial \mu}{\partial \overline{\theta}_\ell}\right\} \\ &\approx \frac{2}{\sigma_{\mathrm{AR}}^2}\mathrm{Re}\left\{\left(K\frac{\partial \mu}{\partial \overline{\theta}_k}\right)^H \overline{A}_1\overline{A}_1^H \left(K\frac{\partial \mu}{\partial \overline{\theta}_\ell}\right)\right\} \end{aligned} \tag{71}$$

which is identical to (65). Hence, the derivation of the conditional CRB through the use of the conditional likelihood is equivalent to the approximation (71) of the exact CRB. Note that this approximation holds for any mean function $\mu(n)$ and is independent of the parametric model selected for the harmonic mean.

VI. Numerical Examples

To gain more insight into the behavior of the different bounds, we resort to numerical evaluation of some specific examples. In this section we present several such examples, which illustrate the dependence of the bounds on various signal and noise parameters.

A. The Bound for the AR Parameters

In the first part of this section we investigate the behavior of the exact CRB for autoregressive noise. We consider three different first-order AR models: narrow band, wide band, and medium band. All three models have their spectral peak at 0.25π. The parameters of the AR processes are listed in Table I.

Since we have shown that the bound for the noise is independent of the bound for the harmonic mean, the same results hold for any choice of harmonic component. The data length in these experiments is $N = 256$.

Figs. 1–3 depict the spectral density function of the AR component. The mean value of the spectrum (dashed line) and the confidence bounds (solid lines), i.e., the mean plus and minus the standard deviation computed from the CRB, are shown.

B. The Effect of the Data Record Length

Next, we investigate the relations between the exact and conditional bounds for the parameters of the harmonic component as a function of the data length for different combinations

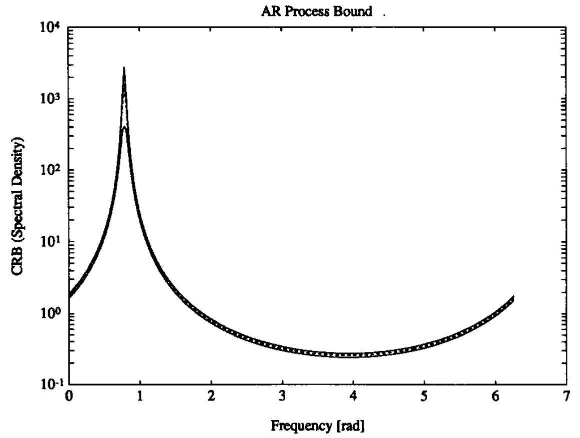

Fig. 1. Spectral density and confidence bounds for the narrowband Gaussian AR component.

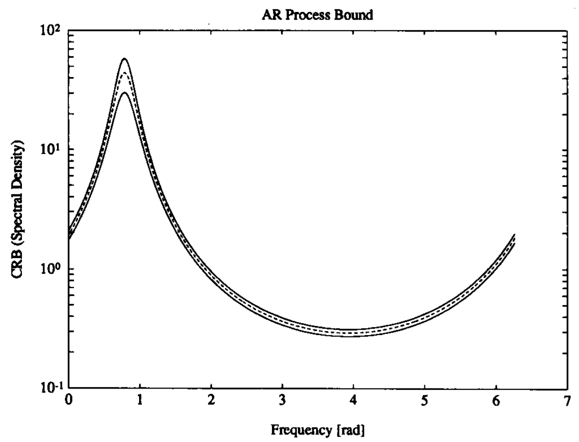

Fig. 2. Spectral density and confidence bounds for the medium bandwidth Gaussian AR component.

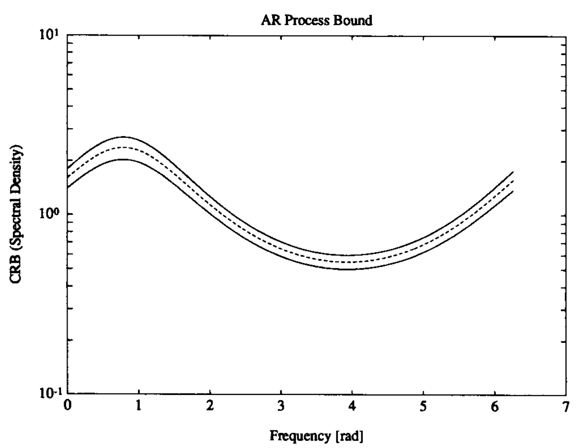

Fig. 3. Spectral density and confidence bounds for the wideband Gaussian AR component.

of the harmonic and noise components. For the colored noise, we consider the three AR models of Table I.

The harmonic component is comprised of two exponential components ($L = 2$). The exponential frequencies are 0.24π and 0.26π for one set of experiments and 0.49π and 0.51π for a second set. Here $C_1 = 1, C_2 = 1 \cdot e^{j(\pi/18)}$. For the exponentials, we have plotted for each test case both the exact (solid line) and the conditional (dashed line) CRB's on the frequencies and the ratio of the two bounds (i.e., the results of the conditional bound are normalized with respect to the exact bound).

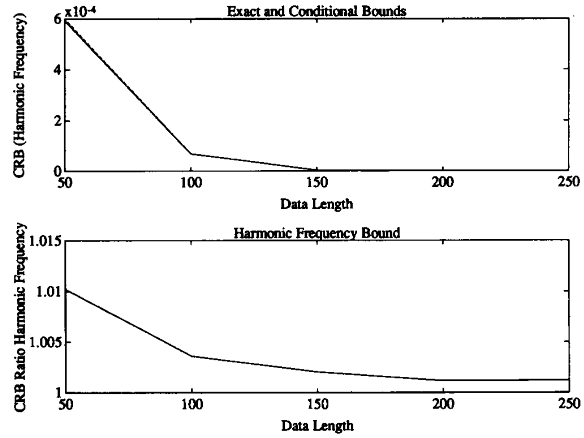

Fig. 4. CRB for the frequency of one of two exponentials in a narrowband Gaussian AR component. The exponentials' frequencies are far from the peak of the noise spectral density.

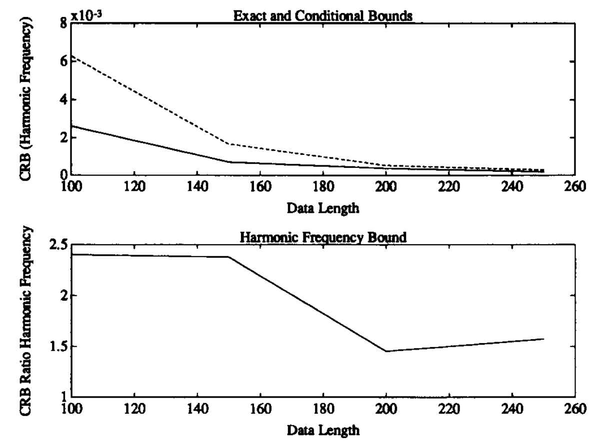

Fig. 5. CRB for the frequency of one of two exponentials in a narrowband Gaussian AR component. The exponentials' frequencies are close to the peak of the noise spectral density.

The simulation results indicate that for the case where the AR process is narrowband and the frequencies of the exponentials are far from the frequency of the peak of the noise spectrum, the conditional CRB and the exact CRB are very close, even for relatively short ($N < 100$) data records (Fig. 4). Larger deviation from the exact bound is observed as the bandwidth of the colored noise spectral density increases (Figs. 6 and 8).

For the case in which the exponentials frequencies are close to the peak of the colored noise spectrum (Figs. 5, 7, and 9), we see that the conditional CRB usually has a large deviation from the exact bound for short data records, although it converges to the exact bound as the data length increases.

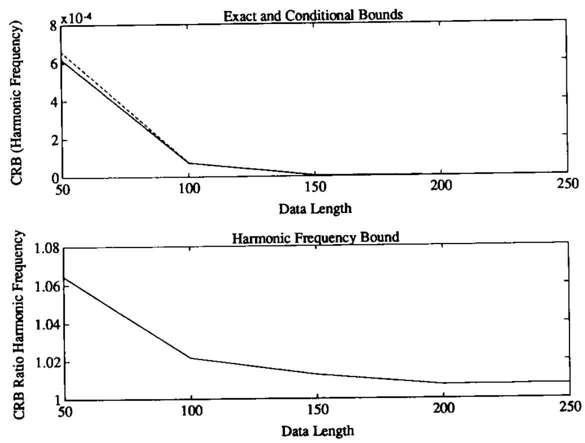

Fig. 6. CRB for the frequency of one of two exponentials in a medium bandwidth Gaussian AR component. The exponentials' frequencies are far from the peak of the noise spectral density.

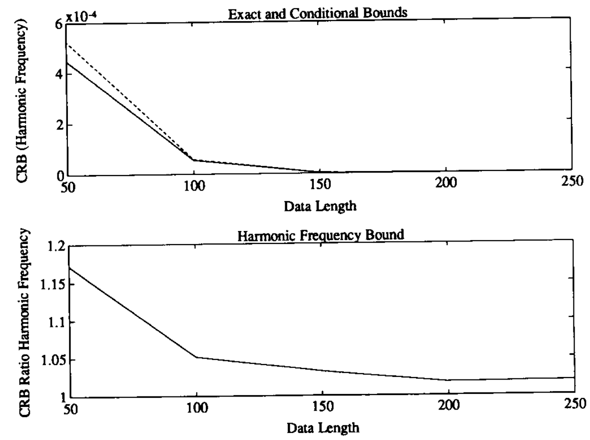

Fig. 8. CRB for the frequency of one of two exponentials in a wideband Gaussian AR component. The exponentials' frequencies are far from the peak of the noise spectral density.

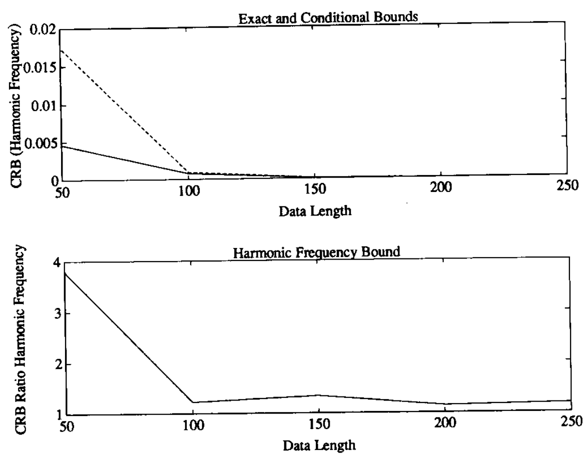

Fig. 7. CRB for the frequency of one of two exponentials in a medium bandwidth Gaussian AR component. The exponentials' frequencies are close to the peak of the noise spectral density.

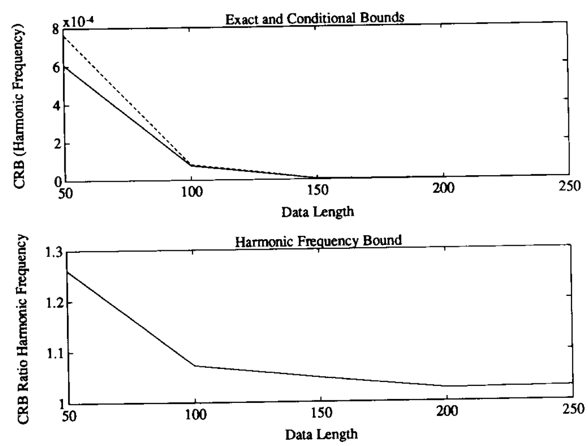

Fig. 9. CRB for the frequency of one of two exponentials in a wideband Gaussian AR component. The exponentials' frequencies are close to the peak of the AR noise spectral density.

In some cases, the conditional bound is considerably higher than the exact bound. In fact, for the case depicted in Fig. 5, for data length $N = 50$, the conditional bound is 100 times higher than the exact bound (and hence not depicted in the figure itself).

C. The Bounds as a Function of Exponential Frequency

Next, we investigate the exact and conditional bounds on the harmonic component parameters, as a function of frequency, for constant local SNR and a fixed data length. The harmonic component is comprised of a single exponential. For each of the three different noise models listed in Table I, the frequency of the exponential is varied in the interval $(0, 2\pi)$, while the local SNR, which is given by

$$\mathrm{SNR}_{\omega_k} = \frac{|C_k|^2}{S(e^{j\omega_k})} \tag{72}$$

is held constant at a level of $\mathrm{SNR}_{\omega_1} = 10$ dB. Here, $S(e^{j\omega}) = \sigma^2_{\mathrm{AR}}/|A(e^{j\omega})|^2$ denotes the spectral density function of the AR process. In this set of examples, the data length is chosen to be relatively short ($N = 100$), in order to not to have "asymptotic" results.

The results depicted in Fig. 10 (a solid line denotes the exact CRB, and a dashed line denotes the conditional CRB) indicate that the performance of the conditional ML estimator is very close to the exact bound for the parameters of the harmonic component, as long as the exponential frequency is not too close to the spectral peak of the noise. At these frequencies, the conditional CRB is larger than the exact CRB, especially for the narrowband case.

D. The Bounds as a Function of the Spectral Slope

In this example, the harmonic component is comprised of a single exponential, and the data length is chosen to be

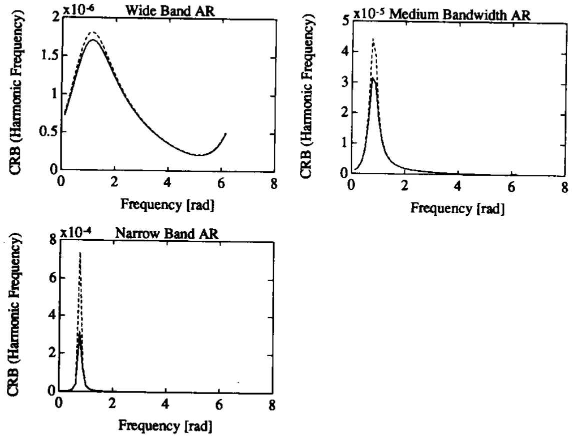

Fig. 10. CRB on the frequency of a single exponential in narrow, medium, and wideband Gaussian AR noise as a function of the exponential frequency.

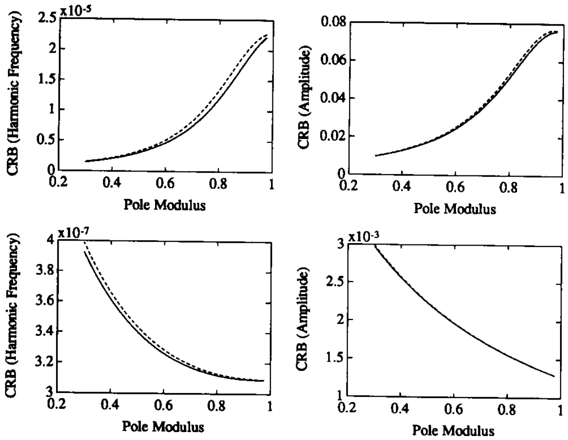

Fig. 11. CRB for the frequency and amplitude of a single exponential in Gaussian AR noise versus noise bandwidth. The exponential frequency is $\pi/3$ for the upper pair of figures and $5\pi/4$ for the lower pair.

$N = 100$ and is held fixed throughout the experiments. The local SNR is also held constant. However, the bandwidth of the noise and, hence, the derivative of the AR model transfer function at the exponential's frequency, are changed. The exact and conditional bounds on the parameters of the harmonic component are investigated under these conditions.

The harmonic component is comprised of a single exponential with frequency $\omega_1 = \pi/3$ and amplitude $C_1 = 1$ in the first experiment, and $\omega_1 = 5\pi/4, C_1 = 1$ in the second. In each experiment, the bandwidth of the noise is varied by changing the modulus of the pole of the first-order AR model (all models have their spectral peak at 0.25π). The local SNR (see (72)) is held constant at a level of $\mathrm{SNR}_{\omega_1} = 10$ dB. Varying the noise bandwidth while holding the local SNR fixed has the effect of varying only the derivative (slope) of the noise spectrum at the frequency of the exponential. The results depicted in Fig. 11 indicate that both the exact and the conditional bounds for short data records are affected significantly by the slope of the noise spectrum. Here, a solid line denotes the exact CRB, and a dashed line denotes the conditional CRB. It is also interesting to note that in these examples, the conditional bound is very close to the exact bound.

VII. Conclusions

In this paper, we derived the exact CRB for the joint estimation of the parameters of complex exponentials and complex colored additive Gaussian noise. These results were specialized to the cases of circular Gaussian noise and complex autoregressive noise. An approximate bound, which we refer to as the conditional CRB, was derived as well, and it was shown that the conditional CRB can be obtained as a special case of the exact bound. We have shown that both the exact and conditional CRB's for the noise and harmonic mean parameters are decoupled. In most cases, the conditional bound is quite close to the exact bound. However, when the number of data points is small, they may be significantly different.

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Benjamin Friedlander, (S'74–M'76–SM'82–F'87) for photgraph and biography, please see p. 2094 of this issue of this TRANSACTIONS.