On the Accuracy of Estimating the Parameters of a Regular Stationary Process

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***Abstract*— Any regular stationary random processes can be represented as the sum of a purely indeterministic process and a deterministic one. This paper considers the achievable accuracy in the joint estimation of the parameters of these two components, from a single observed realization of the process. An exact form of the Cramèr–Rao Bound (CRB) is derived, as well as a conditional CRB. The relationships between these bounds, and their relations to the previously derived asymptotic bound, are explored by analysis and numerical examples.**

***Index Terms*—Stationary process, Wold decomposition, purely indeterministic process, deterministic process, Cramer–Rao bound.**

I. Introduction

In THIS paper we address the general problem of establishing bounds on the achievable estimation accuracy of the parameters of a *regular, stationary* process, from a single observed realization of this process. The Wold decomposition [1] implies that any regular, discrete, stationary random process can be represented as a sum of two mutually orthogonal components: a *purely indeterministic* process and a *deterministic* one. The purely indeterministic process has a unique white innovations driven moving average representation (of possibly infinite order). Its spectral measure is absolutely continuous with respect to the Lebesgue measure, while the spectral measure of the deterministic component is singular with respect to the Lebesgue measure. Since for practical applications we can exclude singular-continuous spectral measures and distribution functions from the framework of our treatment, the deterministic component becomes the harmonic process

$$h(n) = C_0 + \sum_{\ell=1}^{L} C_\ell \cos\omega_\ell \cdot n + \sum_{\ell=1}^{L} D_\ell \sin\omega_\ell \cdot n \quad (1)$$

where the C_ℓ's and D_ℓ's are mutually orthogonal random variables, $E[C_\ell]^2 = E[D_\ell]^2 = \sigma_\ell^2$, and ω_ℓ is the frequency of the ℓth harmonic. In general, L is infinite.

Manuscript received November 6, 1993; revised May 19, 1995. This work was supported by the Army Research Office under Contract DAAL03-91-C-0022, sponsored by the U.S. Army Communications Electronics Command, Center for Signals Warfare.

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Publisher Item Identifier S 0018-9448(96)03643-7.

Let $\{y(n)\}$ be the observed regular process. The foregoing discussion implies that $y(n)$ is uniquely represented by $y(n) = w(n)+h(n)$, where $\{w(n)\}$ denotes the purely indeterministic component. Since, in general, only a single realization of the random process is observed, we cannot hope to infer anything about the statistics of the C_ℓ's, and D_ℓ's over different realizations. The best we can do is to estimate the particular values which these coefficients take for the given realization. In other words, we might just as well treat these as unknown constants, and the deterministic component as the unknown mean of the observed realization. In the present framework we assume the number L of harmonic components to be known and finite.

The problem of analyzing mixed-spectrum processes has received some attention in the past. Priestley [2] describes Whittle's and Bartlett's periodogram-based tests for detecting harmonic components in colored noise, as well as a sequential, periodogram-based estimation method for analyzing the long-term sample covariances of the observed data. More recently, a conditional maximum-likelihood algorithm for estimating the parameters of sinusoids in colored autoregressive (AR) noise was suggested in [4] and [5], where the model order selection problem was considered as well. In [5], an approximation to the conditional Fisher Information Matrix (FIM) is presented.

The special case of estimating the parameters of harmonic components in the presence of noise with a *known* spectrum or covariance function has been studied quite extensively [7]–[9]. Most of this work assumes that the noise is white. Relatively little work seems to have been done for the case of harmonic components in *unknown* noise. Asymptotic results on the achievable accuracy of estimating the parameters of sinusoidal signals in colored noise were given in [3]. However, as we show in this paper, for short data records (and in some cases even for relatively long data records) the asymptotic bound demonstrates large deviation from the exact one.

In the present paper we concentrate on a solution to the problem of the achievable accuracy in *jointly* estimating the parameters of the harmonic (deterministic) and purely indeterministic components of the process, based on a *finite-length, single observed realization* of this process. In Section II we derive an exact Cramèr–Rao Bound (CRB) expression for the estimation problem in terms of the covariance function of the purely indeterministic component, without assuming any specific model for this component. We show that for a Gaussian, purely indeterministic component, the bounds on the purely indeterministic and harmonic components are decoupled, regardless of the parametric model for the purely indeterministic component. In Section III, by assuming an AR model for the purely indeterministic component, we find a closed-form exact CRB on the achievable accuracy of jointly estimating the parameters of the harmonic and autoregressive components of the process. Next we derive an approximate expression of the CRB by computing it from the conditional-likelihood function of the observed data, rather than by using the exact-likelihood function. The results show that the conditional bounds on both the amplitude and frequency parameters of the harmonic components are functions of the frequency response of the colored noise model at the frequencies of the harmonic components, and of the derivative of the frequency response at these frequencies. Finally, the exact and conditional bounds are compared with the asymptotic bound of [3] which shows dependence *only* on the local SNR at the frequencies of the harmonic components, but not on the derivatives.

II. A General Form of the CRB

We start by assuming that the purely indeterministic component is a general Gaussian process. Hence, the observed process $\{y(n)\}_{n=0}^{N-1}$ which is given by

$$y(n) = \mu(n) + w(n), \quad n = 0, 1, \cdots, (N-1) \tag{2}$$

is the sum of a real, zero-mean, stationary, purely indeterministic, Gaussian process $w(n)$ and a real harmonic mean, which can be written as

$$\mu(n) = \sum_{k=0}^{K} c_k \rho_k(n) \tag{3}$$

where $\rho_k(n)$ are sine or cosine functions with different frequencies, and $K = 2L + 1$.

Next we will rewrite (1) in a generalized matrix form. To do that, we introduce some notation. Let $\boldsymbol{t} = [0, 1, \cdots, (N-1)]^T$ be the time index vector, and let

$$\boldsymbol{\rho} = [\mathbf{1} \quad \cos\omega_1 \boldsymbol{t} \cdots \cos\omega_L \boldsymbol{t} \quad \sin\omega_1 \boldsymbol{t} \cdots \sin\omega_L \boldsymbol{t}] \tag{4}$$

where $\mathbf{1}$ is an $N \times 1$ vector of 1's, and where $\cos\omega_k \boldsymbol{t}$ denotes a column vector whose elements are $\cos\omega_k t$, where t are the elements of $\boldsymbol{t}$. Next we define the amplitude and frequency vectors for the sine and cosine functions by

$$\boldsymbol{c} = [C_0, C_1, \cdots, C_L, D_1, \cdots, D_L]^T \tag{5}$$

$$\boldsymbol{\omega} = [\omega_1, \cdots, \omega_L]^T. \tag{6}$$

Finally, we assemble the elements of the observed process into vector form,

$$\boldsymbol{y} = [y(0), y(1), \cdots, y(N-1)]^T. \tag{7}$$

The vectors $\boldsymbol{w}$ and $\boldsymbol{\mu}$ are similarly defined. Thus we can rewrite (3) in matrix form as

$$\boldsymbol{\mu} = \boldsymbol{\rho} \boldsymbol{c}. \tag{8}$$

Assuming that the purely indeterministic component, $\{w(n)\}$, admits some finite-order parametric model, let $\boldsymbol{\alpha}$ be its parameter vector. Let $\boldsymbol{\theta} = \{\boldsymbol{\alpha}, \boldsymbol{c}, \boldsymbol{\omega}\}$ be the parameter vector of the process $\{y(n)\}$. The general expression for the Fisher Information Matrix (FIM) of a real Gaussian process is given by (e.g., [12])

$$\boldsymbol{J}_{k,\ell}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\mu}^T}{\partial \theta_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell} + \frac{1}{2} \operatorname{tr}\left\{ \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}}{\partial \theta_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}}{\partial \theta_\ell} \right\} \tag{9}$$

where $\boldsymbol{\mu}$ is the mean of the observation vector, and $\boldsymbol{\Gamma}$ its covariance matrix. Since the purely indeterministic component $\{w(n)\}$ is a zero-mean process, $\boldsymbol{\mu}$ is independent of the parameters of the purely indeterministic component, and hence $\partial \boldsymbol{\mu} / \partial \alpha_k = 0$. Therefore

$$\boldsymbol{J}_{k,\ell}^{\alpha,\alpha}(\boldsymbol{\theta}) = \frac{1}{2} \operatorname{tr}\left\{ \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}}{\partial \alpha_k} \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}}{\partial \alpha_\ell} \right\}. \tag{10}$$

Note also that since the process covariance function $\boldsymbol{\Gamma}$ is independent of the mean

$$\frac{\partial \boldsymbol{\Gamma}}{\partial c_k} = 0. \tag{11}$$

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \omega_k} = 0. \tag{12}$$

Hence, the $\frac{1}{2} \operatorname{tr}\{\cdot\}$ term in (9) vanishes for all the FIM entries that correspond to parameters of the mean. Therefore, $\boldsymbol{J}^{\alpha,c} = 0$, and $\boldsymbol{J}^{\alpha,\omega} = 0$. Taking the partial derivatives of $\boldsymbol{\mu}$ we get

$$\frac{\partial \boldsymbol{\mu}}{\partial \omega_k} = \boldsymbol{T}(D_k \cos\omega_k \boldsymbol{t} - C_k \sin\omega_k \boldsymbol{t}) \tag{13}$$

where $\boldsymbol{T} = \operatorname{diag} \boldsymbol{t}$, and

$$\frac{\partial \boldsymbol{\mu}}{\partial c_\ell} = \boldsymbol{\rho}_\ell \tag{14}$$

where $\boldsymbol{\rho}_\ell$ is the ℓth column of $\boldsymbol{\rho}$. Using (14) and (11) we conclude that the FIM elements which correspond to the amplitude parameters of the mean component are given by

$$\boldsymbol{J}_{k,\ell}^{c,c} = \boldsymbol{\rho}_k^T \boldsymbol{\Gamma}^{-1} \boldsymbol{\rho}_\ell. \tag{15}$$

Substituting (14), (11), (13), and (12) into (9) yields

$$\boldsymbol{J}_{k,\ell}^{c,\omega} = \boldsymbol{\rho}_k^T \boldsymbol{\Gamma}^{-1} \boldsymbol{T}(D_\ell \cos\omega_\ell \boldsymbol{t} - C_\ell \sin\omega_\ell \boldsymbol{t}). \tag{16}$$

Finally, substituting (13), and (12) into (9) we get

$$\begin{aligned} \boldsymbol{J}_{k,\ell}^{\omega,\omega} = & \boldsymbol{T}(D_k \cos\omega_k \boldsymbol{t} - C_k \sin\omega_k \boldsymbol{t})^T \boldsymbol{T} \boldsymbol{\Gamma}^{-1} \\ & \cdot \boldsymbol{T}(D_\ell \cos\omega_\ell \boldsymbol{t} - C_\ell \sin\omega_\ell \boldsymbol{t}). \end{aligned} \tag{17}$$

Since $\boldsymbol{J}^{\alpha,c} = 0$ and $\boldsymbol{J}^{\alpha,\omega} = 0$ we conclude that the estimation problems of the purely indeterministic and deterministic components are decoupled. Hence, the bound on the purely indeterministic component is found by inverting (10), and it is independent of the mean parameters. Therefore, this bound is identical to the one obtained for a zero-mean process.

III. The CRB for an AR Purely Indeterministic Component

As stated earlier, the most general model for the purely indeterministic component $\{w(n)\}$ is the moving average (MA) model. In this section we analyze the special case in which the purely indeterministic component is a Pth-order AR process. For this special case, we derive closed-form formulas for the exact CRB. Next we derive the conditional and asymptotic CRB's and compare them to the exact bound.

A. The Exact CRB

A real Gaussian AR process is defined by

$$w(n) = -\sum_{i=1}^{P} a_i w(n-i) + u(n)$$

where $\{u(n)\}$ is a stationary, zero-mean Gaussian white noise with variance σ_{AR}^2. It can be shown [11] that the inverse covariance matrix $\boldsymbol{\Gamma}^{-1}$ of a Pth-order AR process ($N \geq P$) is given by

$$\boldsymbol{\Gamma}^{-1} = \frac{1}{\sigma_{\mathrm{AR}}^2}\left(\boldsymbol{A}_1\boldsymbol{A}_1^T - \boldsymbol{A}_2\boldsymbol{A}_2^T\right) \tag{18}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are lower triangular Toeplitz matrices such that

$$(\boldsymbol{A}_1)_{i,j} = \begin{cases} 1, & i = j \\ a_{i-j}, & i > j \\ 0, & i < j \end{cases} \tag{19}$$

$$(\boldsymbol{A}_2)_{i,j} = \begin{cases} a_{N-i+j}, & i \geq j \\ 0, & i < j \end{cases} \tag{20}$$

and $a(k) = 0$ for $k < 0$ and $k > P$.

For an AR modeled purely indeterministic component, the parameter vector $\boldsymbol{\alpha}$ is the $P+1$-dimensional vector $\boldsymbol{\alpha} = [\sigma_{\mathrm{AR}}^2, a(1), a(2), \cdots, a(P)]^T$. Taking the partial derivatives of $\boldsymbol{\Gamma}^{-1}$ in (18) with respect to $\boldsymbol{\alpha}$, followed by substituting these derivatives and (18) itself into the previously derived general expressions of $\boldsymbol{J}^{\alpha,\alpha}, \boldsymbol{J}^{c,c}, \boldsymbol{J}^{c,\omega}$, and $\boldsymbol{J}^{\omega,\omega}$ we obtain closed-form expression for the CRB on the parameters of the deterministic component and on the colored AR process in which it is embedded. Hence we conclude that if the purely indeterministic process is an AR process, the bound on the AR parameters is obtained by inverting (10), after the substitutions above are made. This bound is identical to the one obtained for the same AR process in the case where the mean component is identically zero [10].

B. The Conditional CRB

A conditional ML algorithm for estimating the parameters of sinusoids in AR process, jointly with estimating the AR model parameters, was suggested in [4]. In this section we derive the performance bound for this algorithm.

The conditional CRB is derived using the conditional probability density function of the observed process. In order to simplify and unify the notations we assume here that $C_0 = 0$. The joint probability density function of the observations, conditioned on the vector of initial conditions (the first P data samples) is given by

$$\begin{aligned} P(\boldsymbol{y}|y(0)\cdots y(P-1);\boldsymbol{\theta}) &= \frac{1}{(\sqrt{2\pi}\sigma_{\mathrm{AR}})^{N-P}} \exp\Bigg\{-\frac{1}{2\sigma_{\mathrm{AR}}^2}\sum_{n=P}^{N-1} \\ &\cdot \Bigg[\sum_{i=0}^{P} a_i\Bigg(y(n-i) - \sum_{\ell=1}^{L}[C_\ell\cos\omega_\ell(n-i) \\ &+ D_\ell \sin\omega_\ell(n-i)]\Bigg)\Bigg]^2\Bigg\} \end{aligned} \tag{21}$$

where $a_0 = 1$. Let

$$\mu_k(n) = C_k\cos\omega_k n + D_k \sin\omega_k n \tag{22}$$

be the kth component of the mean $\mu(n)$. Hence, for $\theta_k = C_k, D_k$, or $\omega_k, \partial\boldsymbol{\mu}/\partial\theta_k = \partial\boldsymbol{\mu}_k/\partial\theta_k$, where

$$\boldsymbol{\mu}_k = [\mu_k(0), \mu_k(1), \cdots, \mu_k(N-1)]^T. \tag{23}$$

Taking the partial derivatives of the conditional log-likelihood function with respect to the AR process parameters, we have

$$\begin{aligned} \frac{\partial \ln P}{\partial a_k} &= -\frac{1}{\sigma_{\mathrm{AR}}^2}\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i[y(n-i)-\mu(n-i)]\right) \\ &\cdot [(y(n-k) - \mu(n-k)] \end{aligned} \tag{24}$$

and

$$\begin{aligned} \frac{\partial \ln P}{\partial \sigma_{\mathrm{AR}}^2} &= -\frac{N-P}{2\sigma_{\mathrm{AR}}^2} + \frac{1}{2\sigma_{\mathrm{AR}}^4}\sum_{n=P}^{N-1} \\ &\cdot \left(\sum_{i=0}^{P} a_i[y(n-i)-\mu(n-i)]\right)^2. \end{aligned} \tag{25}$$

Taking now the second derivative with respect to the mean component parameters θ_{μ_ℓ} yields

$$\begin{aligned} -E\left\{\frac{\partial^2 \ln P}{\partial a_k \partial\theta_{\mu_\ell}}\right\} &= -\frac{1}{\sigma_{\mathrm{AR}}^2}\sum_{n=P}^{N-1} E\Bigg\{\left(\sum_{i=0}^{P} a_i \frac{\partial\mu(n-i)}{\partial\theta_{\mu_\ell}}\right) \\ &\cdot [y(n-k) - \mu(n-k)] \\ &+ \left(\sum_{i=0}^{P} a_i[y(n-i)-\mu(n-i)]\right) \\ &\cdot \frac{\partial\mu(n-k)}{\partial\theta_{\mu_\ell}}\Bigg\} = 0 \end{aligned} \tag{26}$$

and

$$\begin{aligned} -E&\left\{\frac{\partial^2 \ln P}{\partial\sigma_{\mathrm{AR}}^2 \partial\theta_{\mu_\ell}}\right\} \\ &= \frac{1}{\sigma_{\mathrm{AR}}^4} E\Bigg\{\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i[y(n-i)-\mu(n-i)]\right) \\ &\cdot \left(\sum_{i=0}^{P} a_i \frac{\partial\mu(n-i)}{\partial\mu_\ell}\right)\Bigg\} = 0. \end{aligned} \tag{27}$$

Thus the conditional FIM is block-diagonal, and the conditional CRB for the parameters of the mean is decoupled from the bound for the autoregressive parameters. This is the same situation as in the case of the exact CRB which was presented in the previous section. Therefore, the conditional CRB for the parameters of the mean can be obtained by inverting the corresponding block of the FIM.

Let us define

$$A(e^{j\omega}) = \sum_{t=0}^{P} a_t e^{-j\omega t} \qquad A(e^{j\omega_k}) = A(e^{j\omega})|_{\omega=\omega_k}$$

and let

$$N_1 = \sum_{n=P}^{N-1} e^{j(\omega_k+\omega_\ell)n}$$
$$N_2 = \sum_{n=P}^{N-1} e^{j(\omega_k-\omega_\ell)n}$$
$$N_3 = \sum_{n=P}^{N-1} n e^{j(\omega_k-\omega_\ell)n}$$
$$N_4 = \sum_{n=P}^{N-1} n e^{j(\omega_k+\omega_\ell)n}$$
$$N_5 = \sum_{n=P}^{N-1} n^2 e^{j(\omega_k+\omega_\ell)n}$$
$$N_6 = \sum_{n=P}^{N-1} n^2 e^{j(\omega_k-\omega_\ell)n}.$$

Taking the partial derivatives with respect to the mean component parameters, we find that

$$\begin{aligned} -E\left\{\frac{\partial^2 \ln P}{\partial\theta_{\mu_k}\partial\theta_{\mu_\ell}}\right\} &= \frac{1}{\sigma_{\text{AR}}^2}\sum_{n=P}^{N-1}\left(\sum_{i=0}^{P} a_i \frac{\partial\mu(n-i)}{\partial\theta_{\mu_k}}\right)\left(\sum_{i=0}^{P} a_i \frac{\partial\mu(n-i)}{\partial\theta_{\mu_\ell}}\right) \\ &= \frac{1}{\sigma_{\text{AR}}^2}\sum_{n=P}^{N-1}\frac{\partial}{\partial\theta_{\mu_k}}\left(\sum_{i=0}^{P} a_i\mu_k(n-i)\right)\frac{\partial}{\partial\theta_{\mu_\ell}} \\ &\quad\cdot\left(\sum_{i=0}^{P} a_i\mu_\ell(n-i)\right). \end{aligned} \tag{28}$$

Substituting (22) into (28) we find after some algebraic manipulations that

$$\begin{aligned} -E\left(\frac{\partial^2 \ln P}{\partial\omega_k\partial C_\ell}\right) &= \frac{1}{2\sigma_{\text{AR}}^2}\,\text{Re}\left\{N_1 A(e^{j\omega_\ell})\frac{\partial}{\partial\omega_k}A(e^{j\omega_k})(C_k - jD_k)\right. \\ &\quad + N_2 A(e^{j\omega_\ell})\frac{\partial}{\partial\omega_k}A^*(e^{j\omega_k})(C_k + jD_k) \\ &\quad + N_3 A(e^{j\omega_\ell})A^*(e^{j\omega_k})(D_k - jC_k) \\ &\quad \left. + N_4 A(e^{j\omega_\ell})A(e^{j\omega_k})(D_k + jC_k)\right\} \end{aligned} \tag{29}$$

$$\begin{aligned} -E\left(\frac{\partial^2 \ln P}{\partial\omega_\ell\partial\omega_k}\right) &= \frac{1}{2\sigma_{\text{AR}}^2}\,\text{Re}\left\{\left[N_5 A(e^{j\omega_k})A(e^{j\omega_\ell}) - N_1\frac{\partial}{\partial\omega_k}A(e^{j\omega_k})\right.\right. \\ &\quad\left.\cdot\frac{\partial}{\partial\omega_\ell}A(e^{j\omega_\ell})\right][(-C_kC_\ell + D_kD_\ell) + j(C_kD_\ell + D_kC_\ell)] \\ &\quad + N_4\left[A(e^{j\omega_\ell})\frac{\partial}{\partial\omega_k}A(e^{j\omega_k}) + A(e^{j\omega_k})\frac{\partial}{\partial\omega_\ell}\right. \\ &\quad\left.\cdot A(e^{j\omega_\ell})\right][(C_kD_\ell + D_kC_\ell) + j(C_kC_\ell - D_kD_\ell)] \\ &\quad + \left[N_6 A^*(e^{j\omega_k})A(e^{j\omega_\ell}) + N_2\frac{\partial}{\partial\omega_k}A^*(e^{j\omega_k})\right. \\ &\quad\left.\cdot\frac{\partial}{\partial\omega_\ell}A(e^{j\omega_\ell})\right][(C_kC_\ell + D_kD_\ell) - j(C_kD_\ell - D_kC_\ell)] \\ &\quad + N_3\left[\frac{\partial}{\partial\omega_k}A^*(e^{j\omega_k})A(e^{j\omega_\ell}) - A^*(e^{j\omega_k})\frac{\partial}{\partial\omega_\ell}A(e^{j\omega_\ell})\right] \\ &\quad\left.\cdot[(C_kD_\ell + D_kC_\ell) + j(C_kC_\ell + D_kD_\ell)]\right\}. \end{aligned} \tag{30}$$

A similar result to (30) is also stated in [6].

Also

$$-E\left(\frac{\partial^2 \ln P}{\partial C_k\partial C_\ell}\right) = \frac{1}{2\sigma_{\text{AR}}^2}\,\text{Re}\left\{N_2 A(e^{j\omega_\ell})A(e^{j\omega_k}) + N_4 A(e^{j\omega_k})A^*(e^{j\omega_\ell})\right\} \tag{31}$$

and, in a similar manner,

$$-E\left(\frac{\partial^2 \ln P}{\partial D_k\partial D_\ell}\right) = \frac{1}{2\sigma_{\text{AR}}^2}\,\text{Re}\left\{N_4 A(e^{j\omega_\ell})A^*(e^{j\omega_k}) - N_2 A(e^{j\omega_k})A(e^{j\omega_\ell})\right\} \tag{32}$$

$$-E\left\{\frac{\partial^2 \ln P}{\partial D_k\partial C_\ell}\right\} = \frac{1}{2\sigma_{\text{AR}}^2}\,\text{Im}\left\{N_2 A(e^{j\omega_\ell})A(e^{j\omega_k}) - N_4 A(e^{j\omega_\ell})A^*(e^{j\omega_k})\right\}. \tag{33}$$

The derivation of the conditional FIM reveals that the bounds on both the amplitude and the frequency parameters of the harmonic components are functions of the frequency response of the colored noise model at the frequencies of the harmonic components, and of the *derivative* of the frequency response at these frequencies. Next we study the relationship between the conditional CRB and the exact bound.

C. Analysis of the Conditional CRB

Let

$$\boldsymbol{K} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \tag{34}$$

be the exchange matrix, and let $\boldsymbol{x}$ be some vector $\boldsymbol{x} = [x(0), x(1), \cdots, x(N-1)]^T$ and let $\boldsymbol{z}$ be its "flipped around" version, i.e., $\boldsymbol{z} = \boldsymbol{K}\boldsymbol{x}$. Let $\boldsymbol{R}_z$ and $\boldsymbol{R}_x$ be the covariance matrices of $\boldsymbol{z}$ and $\boldsymbol{x}$, respectively. Note that since $\boldsymbol{R}_x$ is Toeplitz

$$\boldsymbol{R}_z = E[\boldsymbol{K}\boldsymbol{x}\boldsymbol{x}^T\boldsymbol{K}^T] = \boldsymbol{K}\boldsymbol{R}_x\boldsymbol{K} = \boldsymbol{R}_x \tag{35}$$

where we have used the symmetric property of $\boldsymbol{K}$. Hence, in our problem

$$\begin{aligned} \boldsymbol{J}_{k,\ell}(\boldsymbol{\theta}) &= \frac{\partial \boldsymbol{\mu}^T}{\partial \theta_k}\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell} + \frac{1}{2}\operatorname{tr}\left\{\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_k}\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_\ell}\right\} \\ &= \left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_k}\right)^T\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell}\right) \\ &\quad + \frac{1}{2}\operatorname{tr}\left\{\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_k}\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \theta_\ell}\right\}. \end{aligned} \tag{36}$$

Substituting (18) into the mean dependent part of (36), we have

$$\begin{aligned} \left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_k}\right)^T\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell}\right) &= \frac{1}{\sigma_{\mathrm{AR}}^2}\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_k}\right)^T\boldsymbol{A}_1\boldsymbol{A}_1^T\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell}\right) \\ &\quad - \frac{1}{\sigma_{\mathrm{AR}}^2}\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_k}\right)^T\boldsymbol{A}_2\boldsymbol{A}_2^T\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell}\right). \end{aligned} \tag{37}$$

Note, that $(\boldsymbol{K}(\partial\boldsymbol{\mu}/\partial\theta_k))^T\boldsymbol{A}_2$ is a function of only the last P elements of $\boldsymbol{K}(\partial\boldsymbol{\mu}/\partial\theta_k)$ which are the P initial values of $\partial\boldsymbol{\mu}/\partial\theta_k)$. Also note that the entries of $\boldsymbol{A}_2$ are all zeros except for the lower $P \times P$ triangular block.

In the case where $N \gg P$ we can neglect the second term in (37) (which is a function of $\boldsymbol{A}_2$), as well as the contribution of the P rightmost columns of $\boldsymbol{A}_1$ (which is a function only of the P initial values of $\partial\boldsymbol{\mu}/\partial\theta_k$). In that case, (37) can be approximated by

$$\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_k}\right)^T\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell}\right) \approx \frac{1}{\sigma_{\mathrm{AR}}^2}\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_k}\right)^T\overline{\boldsymbol{A}}_1\overline{\boldsymbol{A}}_1^T\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell}\right) \tag{38}$$

where $\overline{\boldsymbol{A}}_1$ is the $N \times (N-P)$ matrix of the first $N-P$ columns of A_1

$$\overline{\boldsymbol{A}}_1 = \begin{bmatrix} 1 & & & \\ a(1) & 1 & & \\ a(2) & a(1) & 1 & \\ \vdots & & \ddots & \\ a(p) & a(p-1) & \cdots & 1 \\ & a(p) & \cdots & a(1) \\ & & \ddots & \vdots \\ & & & a(p) \end{bmatrix}. \tag{39}$$

Let $\overline{\boldsymbol{\mu}} = \boldsymbol{K}\boldsymbol{\mu}$ and $\overline{\boldsymbol{\mu}}_\ell = \boldsymbol{K}\boldsymbol{\mu}_\ell$. Then (38) yields

$$\begin{aligned} &\frac{1}{\sigma_{\mathrm{AR}}^2}\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_{\mu_k}}\right)^T\overline{\boldsymbol{A}}_1\overline{\boldsymbol{A}}_1^T\left(\boldsymbol{K}\frac{\partial \boldsymbol{\mu}}{\partial \theta_{\mu_\ell}}\right) \\ &= \frac{1}{\sigma_{\mathrm{AR}}^2}\frac{\partial \overline{\boldsymbol{\mu}}}{\partial \theta_{\mu_k}}^T\overline{\boldsymbol{A}}_1\overline{\boldsymbol{A}}_1^T\frac{\partial \overline{\boldsymbol{\mu}}}{\partial \mu_{\mu_\ell}} \\ &= \frac{1}{\sigma_{\mathrm{AR}}^2}\frac{\partial \overline{\boldsymbol{\mu}}_k}{\partial \theta_{\mu_k}}^T\overline{\boldsymbol{A}}_1\overline{\boldsymbol{A}}_1^T\frac{\partial \overline{\boldsymbol{\mu}}_\ell}{\partial \theta_{\mu_\ell}} \end{aligned} \tag{40}$$

which is identical to (28). We conclude that the derivation of the conditional CRB through the use of the conditional likelihood is equivalent to an approximation (38) of the exact CRB. Note that this approximation holds for any mean function $\mu(n)$ and is independent of the parametric model for the mean.

D. The Asymptotic Bound

In the previous sections we derived a closed-form exact CR bound for the case of an AR-modeled purely indeterministic component, and gave an approximation to this bound through the conditional-likelihood function. In this section, we show how the asymptotic results of [3] can be derived as a special case of the bounds which were presented in the previous sections.

Let $S(e^{j\omega}) = \sigma_{\mathrm{AR}}^2/|A(e^{j\omega})|^2$ denote the spectral density function of the AR process. Also, let

$$\mathrm{SNR}_{\omega_k} = \frac{C_k^2 + D_k^2}{2S(e^{j\omega_k})}. \tag{41}$$

Returning now to the results presented in Section III-B and letting $N \to \infty$ while using the formula [13]

$$\lim_{N\to\infty}\frac{1}{N^{k+1}}\sum_{n=1}^{N} n^k\cos(\omega n) = \begin{cases} \dfrac{1}{k+1}, & \omega = 0 \\ 0, & \omega \neq 0 \end{cases} \tag{42}$$

we find that

$$\lim_{N\to\infty}\sum_{n=1}^{N} n^2 = N^3\lim_{N\to\infty}\frac{1}{N^3}\sum_{n=1}^{N} n^2 = \frac{N^3}{3}. \tag{43}$$

Therefore, $\lim_{N\to\infty} N_6 = N^3/3$ for elements on the diagonal of the FIM block which corresponds to the sinusoids frequency parameters. The remaining coefficients in (30) either tend to zero or are functions of N^2 and N. If we now neglect all terms in (30) which grow slower than N^3, we have that

$$\begin{aligned} -\lim_{N\to\infty} E\left(\frac{\partial^2 \ln P}{\partial \omega_k^2}\right) &= \frac{C_k^2 + D_k^2}{2\sigma_{\mathrm{AR}}^2}\frac{N^3}{3}|A(e^{j\omega_k})|^2 \\ &= N^3\frac{C_k^2 + D_k^2}{6S(e^{j\omega_k})}. \end{aligned} \tag{44}$$

Since for $k \neq \ell, N_1, N_2, N_3, N_4, N_5, N_6$, all tend to zero as $N \to \infty$, off-diagonal terms of the FIM blocks tend to zero as $N \to \infty$. By using similar arguments to those we have applied to obtain the asymptotic FIM block, which corresponds to the sinusoids frequency parameters, all the blocks of asymptotic FIM are obtained. Taking the inverse of the asymptotic FIM while using the diagonality of its blocks, it can be shown that the block of the asymptotic CRB matrix, which corresponds to the sinusoids frequency parameters, is diagonal with the diagonal elements given by

$$\mathrm{CRB}(\omega_k) = \frac{12}{N^3\,\mathrm{SNR}_{\omega_k}}. \tag{45}$$

The asymptotic CRB on the amplitude parameter C_k is given by

$$\mathrm{CRB}(C_k) = \frac{C_k^2 + 4D_k^2}{N\,\mathrm{SNR}_{\omega_k}} \tag{46}$$

and the asymptotic CRB on the amplitude parameter D_k is similar. These results are identical to those in [3]. Hence, the

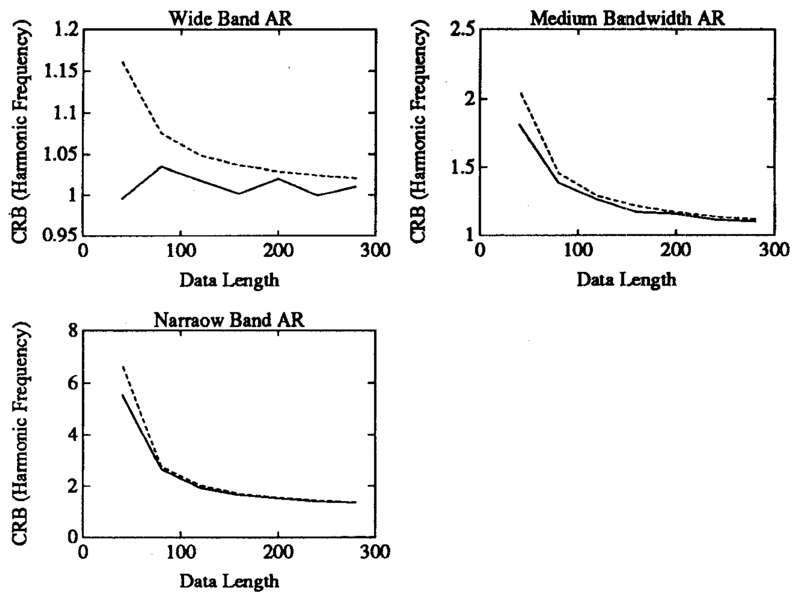

Fig. 1. The CRB on the frequency of a single sinusoid in narrow, medium, and wideband Gaussian AR noise. The sinusoid frequency is near the peak of the AR spectrum.

TABLE I
THREE SECOND-ORDER AR MODELS

Test Case	σ^2_{AR}	α_1	α_2
Narrowband AR	1	-1.378	0.95
Medium-bandwidth AR	1	-1.183	0.7
Wideband AR	1	-0.447	0.1

asymptotic bound of [3] is a special case of the conditional CRB derived in Section III-B, which is itself an approximation to the exact bound presented in Section III-A.

IV. Numerical Examples

To gain more insight into the behavior of the different bounds, we resort to numerical evaluation of some specific examples. In the first part of this section we investigate the relations between the three bounds derived in the previous sections, as a function of the data length, for different combinations of the deterministic and purely indeterministic components. For colored noise, we consider three different second-order AR models: narrowband, wideband, and a medium band. All three models have their spectral peaks at 0.25π. The parameters of the AR processes are listed in Table I.

In the second part, the data length is chosen to be relatively short ($N = 100$) and is held fixed throughout the experiment. The deterministic component is comprised of a single sinusoid. For each of the three different noise models listed in Table I, the frequency of the sinusoid is changed, while the local SNR (41) is held constant.

In the third set of experiments, the deterministic component is comprised of a single sinusoid, the data length is chosen to be $N = 100$ and is held fixed throughout the experiments. The local SNR is also held constant. However, the bandwidth of the noise, and hence the derivative of its spectral density at the sinusoidal frequency, is changed.

A. The Bounds as a Function of the Data Record Length

Let the deterministic component be a single sinusoid (i.e., $L = 1$ in (1)). We perform two experiments: the first with sinusoidal frequency of 0.24π, and the second with sinusoidal frequency of 0.5π. In both cases $C_1 = D_1 = 1$.

In the second set of examples the same second-order AR processes are used. However, the deterministic component is now comprised of two sinusoidal components ($L = 2$). The sinusoidal frequencies are 0.24π and 0.26π for one set of experiments and 0.49π and 0.51π for a second set. Here $C_1 = C_2 = D_1 = D_2 = 1$.

In all these examples, the results are normalized with respect to the exact bound, i.e., the plots show the ratio of the asymptotic CRB and the conditional CRB to the exact CRB. In all the plots, the solid line denotes the ratio of the asymptotic CRB to the exact CRB while the dashed line denotes the ratio of the conditional CRB to the exact CRB.

The simulation results indicate that in the case where the AR process is narrow band, and the harmonic frequencies are far from the frequency of its spectral peak, the conditional CRB and the exact CRB are very close, even for relatively short ($N < 100$) data records—see Figs. 2 and 4. Larger deviations from the exact bound are observed as the bandwidth of the colored noise increases. We can therefore conclude that the conditional CRB is close to the optimal performance bound of an unbiased estimator (which is given by the exact CRB) as long as the harmonic frequencies are far from the spectral peaks of the noise. Note also, that as expected, the asymptotic bound approaches the exact CRB as the length of the data record increases.

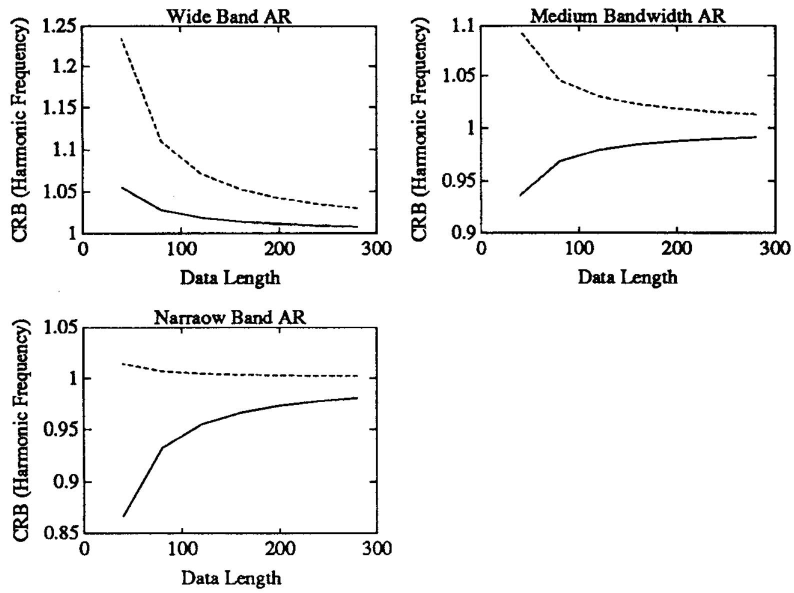

Fig. 2. The CRB on the frequency of a single sinusoid in narrow, medium, and wideband Gaussian AR noise. The sinusoid frequency is far from the peak of the AR spectrum.

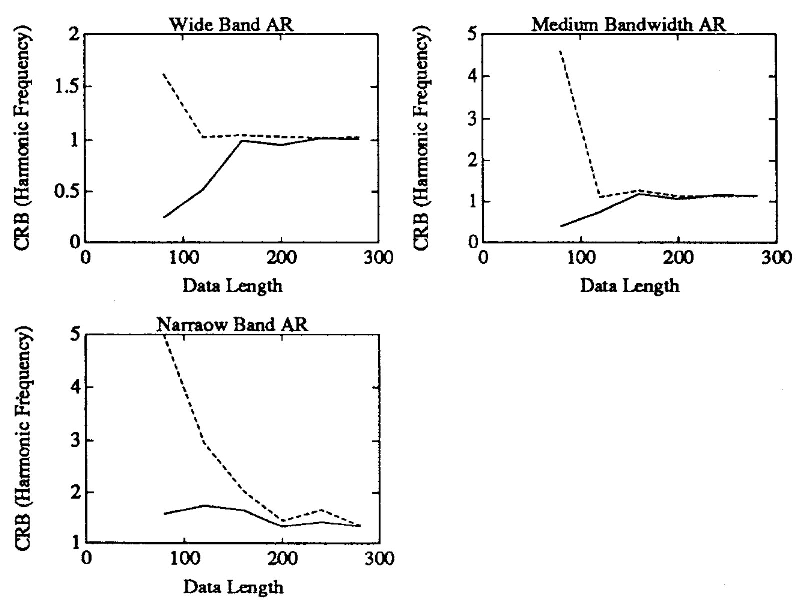

Fig. 3. The CRB on the frequency of one of two sinusoids in narrow, medium, and wideband Gaussian AR noise. The sinusoids frequencies are near the peak of the AR spectrum.

In the case where the sinusoidal frequencies are near the peak of the colored noise spectrum, see figs. 1 and 3, we note that the conditional CRB usually has a larger deviation from the exact bound than that of the asymptotic CRB, although they both converge to the exact bound as the data length increases. These results imply that for short and medium length data records, when the harmonic frequencies are close to the peaks of the noise spectrum, the conditional bound may be far from the optimal bound.

B. The Bounds as a Function of the Sinusoidal Frequency

In this set of examples the data length is chosen to be relatively short ($N = 100$), so as not to have "asymptotic" results. This data length is held fixed throughout the experiment. The deterministic component is comprised of a single sinusoid. For each of the three different noise models which are listed in Table I, the frequency of the sinusoid is varied in the interval $(0, \pi)$, while the local SNR (41) is held constant at a level of $\text{SNR}_{\omega_1} = 10$ dB. Here, the solid line denotes the exact CRB, the dashed line denotes the conditional CRB, and the dotted line denotes the asymptotic bound.

The results depicted in Figs. 5 and 6 indicate that for medium- and narrowband noise the conditional CRB is very close to the exact bound, as long as the sinusoidal frequencies are not too close to the spectral peaks of the noise. At those

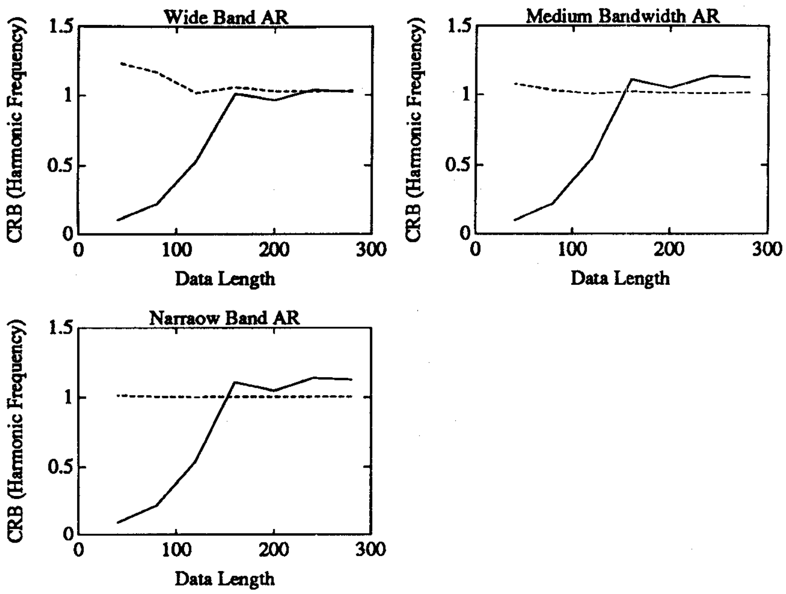

Fig. 4. The CRB on the frequency of one of two sinusoids in narrow, medium, and wideband Gaussian AR noise. The sinusoids frequencies are far from the peak of the AR spectrum.

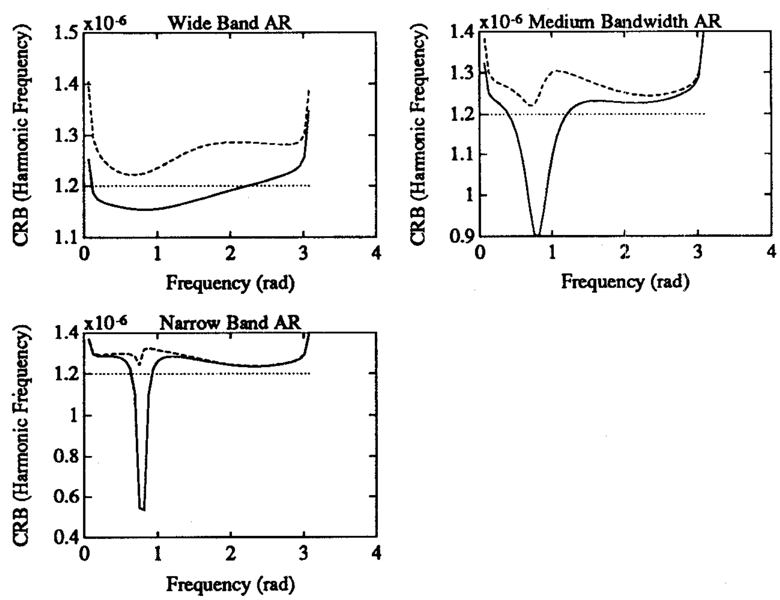

Fig. 5. The CRB on the frequency of a single sinusoid in narrow, medium, and wideband Gaussian AR noise, as a function of the sinusoidal frequency.

frequencies, the conditional CRB is significantly larger than the exact CRB. For the wideband case, it can be seen that the difference between the conditional and exact bounds is almost constant, in most of the interval $(0, \pi)$.

Note that all the experimental results in Sections IV-A and IV-B indicate that, while the asymptotic bounds on the frequencies and amplitudes of the sinusoidal components are functions only of the local SNR, for finite-length data both the exact and conditional bounds depend on the locations of the sinusoidal frequencies with respect to the spectral peaks of the noise. This dependence is quite strong even when the local SNR is held constant.

C. The Bounds as a Function of the Spectral Slope

In this section the data length is chosen to be $N = 100$ and is held fixed throughout the experiment. The deterministic component is comprised of a single sinusoid with frequency of 0.2π and amplitude parameters $C_1 = D_1 = 1$ in the first experiment, and $0.5\pi, C_1 = D_1 = 1$ in the second. In each experiment, the bandwidth of the noise is varied by varying the modulus of the poles of the second-order AR model (all

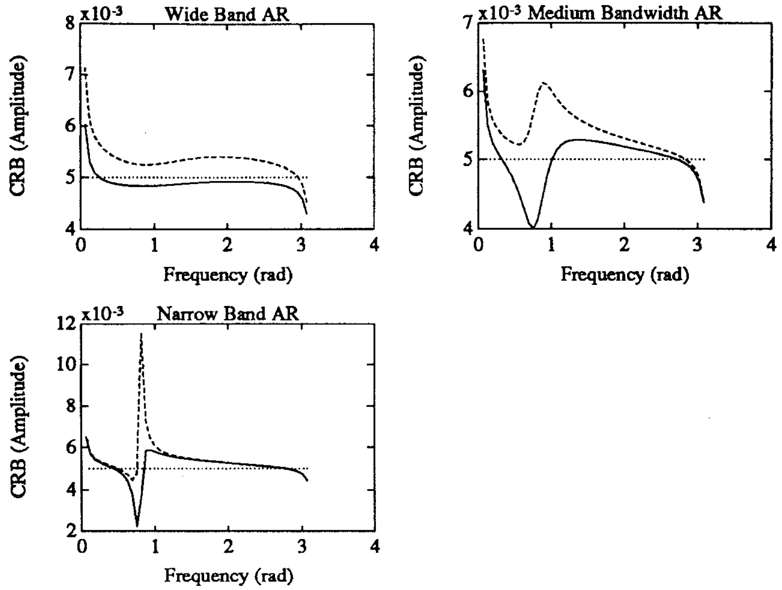

Fig. 6. The CRB on C_1 for a single sinusoid in narrow, medium and wideband Gaussian AR noise, as a function of the sinusoidal frequency.

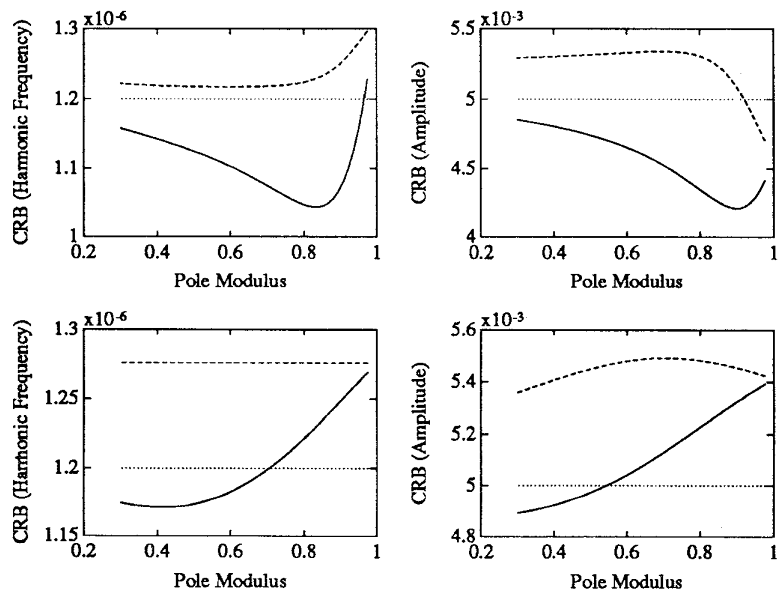

Fig. 7. The CRB on the frequency and amplitude of a single sinusoid in Gaussian AR noise versus noise bandwidth. The sinusoidal frequency is 0.2π for the upper pair of figures, and 0.5π for the lower pair.

models have their spectral peak at 0.25π). The local SNR, (41), is held constant at a level of $\mathrm{SNR}_{\omega_1} = 10$ dB. Varying the noise bandwidth while holding the local SNR fixed has the effect of varying only the derivative (slope) of the noise spectrum at the frequency of the sinusoid. The results, depicted in Fig. 7, indicate that while asymptotically the bound is not affected by the slope of the noise spectrum, it does affect the exact and conditional bounds for short data records. Here, the solid line denotes the exact CRB, the dashed line denotes the conditional CRB, and the dotted line denotes the asymptotic bound.

V. Conclusions

We presented an exact form of the CRB for estimating the parameters of a general regular stationary process, and specialized it for the case where the purely indeterministic component is an autoregressive process. Comparison of the exact CRB with the conditional and asymptotic bounds shows that the approximations deviate significantly from the exact bound in many cases. It is therefore recommended that the exact form of the bound be used unless the data length is sufficiently large.

The results presented here are useful for assessing the achievable accuracy in estimating the amplitudes and frequencies of sinusoidal signals in the presence of colored noise with unknown characteristics. This type of estimation problem arises in many engineering applications in the areas of communications, array processing, and sonar.

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