The Annals of Applied Probability
1995, Vol. 5, No. 1, 248–260

A WOLD-LIKE DECOMPOSITION OF TWO-DIMENSIONAL DISCRETE HOMOGENEOUS RANDOM FIELDS

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Imposing a *total order* on a regular two-dimensional discrete random field induces an orthogonal decomposition of the random field into two components: a *purely indeterministic* field and a *deterministic* field. The deterministic component is further orthogonally decomposed into a *half-plane deterministic* field and a countable number of mutually orthogonal *evanescent* fields. Each of the evanescent fields is generated by the column-to-column innovations of the deterministic field with respect to a different nonsymmetrical-half-plane total-ordering definition. The half-plane deterministic field has no innovations, nor column-to-column innovations, with respect to any nonsymmetrical-half-plane total-ordering definition. This decomposition results in a corresponding decomposition of the spectral measure of the regular random field into a countable sum of mutually singular spectral measures.

1. Introduction. In this paper we consider the structure of two-dimensional (2-D) discrete homogeneous random fields. We extend the results of Helson and Lowdenslager (1962), Korezlioglu and Loubaton (1986), Kallianpur, Miamee and Niemi (1990) and Chiang (1991) to show that the two-, three- and four-fold Wold-type decompositions are special cases of the countably-infinite-fold decomposition presented in this paper. The countably-infinite-fold decomposition arises from a set of new total-order and nonsymmetrical half-plane (NSHP) definitions imposed on the random field. These order definitions are obtained by rotating the NSHP support by angles of rational tangent, rather than considering only the vertical and horizontal orientations.

A family of real, zero-mean, random variables $\{y(n,m),\ (n,m) \in \mathcal{Z}^2\}$ is called a *discrete homogeneous random field* if $E[y^2(n,m)] < \infty$, and if $r(k,l) = E[y(n+k, m+l)y(n,m)]$ is independent of n and m, where $(k,l) \in \mathcal{Z}^2$. Let $\hat{y}(n,m)$ be the projection of $y(n,m)$ on the Hilbert space spanned by those samples of the field that are in the "past" of the (n,m)th sample, where the "past" is defined with respect to the *totally ordered, nonsymmetrical-half-plane support*, that is,

$$(i,j) \prec (s,t) \quad \text{iff} \quad (i,j) \in \{(k,l) \mid k = s,\ l < t\} \cup \{(k,l) \mid k < s, -\infty < l < \infty\}. \tag{1}$$

Received August 1992; revised January 1994.

AMS 1991 *subject classifications.* Primary 60G60; secondary 60G25.

Key words and phrases. Two-dimensional random fields, two-dimensional Wold decomposition, purely indeterministic random fields, deterministic random fields, evanescent random fields, two-dimensional spectral measures.

Because in this paper we consider other total-order definitions as well, we shall denote this order definition by $o = (1,0)$. The reason for this notation is explained in Section 2. The results given in this section are with respect to $o = (1,0)$. The *innovation* with respect to the defined support and total order is given by $u(n,m) = y(n,m) - \hat{y}(n,m)$ and its variance is denoted by σ^2. If $E[y(n,m) - \hat{y}(n,m)]^2 = \sigma^2 > 0$, the field $\{y(n,m)\}$ is called *regular*. The field is called *deterministic* if $E[y(n,m) - \hat{y}(n,m)]^2 = 0$. A regular field $\{y(n,m)\}$ is called *purely indeterministic* if $y(n,m) \in \overline{\text{Sp}}\{u(s,t) \mid (s,t) \preceq (n,m)\}$. In the following text, all spectral measures are defined on the square region $K = [-1/2, 1/2] \times [-1/2, 1/2]$. The spectral representation of $y(n,m)$ is given by $y(n,m) = \int_K \exp[2\pi j(n\omega + m\nu)]\, dZ(\omega,\nu)$, where $Z(\omega,\nu)$ is a doubly orthogonal increments process, such that $dF_y(\omega,\nu) = E[dZ(\omega,\nu)\, dZ^*(\omega,\nu)]$. $F_y(\omega,\nu)$ is the *spectral distribution function* of $\{y(n,m)\}$. Let $f(\omega,\nu)$ be the corresponding *spectral density function*, which is the Lebesgue 2-D derivative of $F_y(\omega,\nu)$. $F^s(\omega,\nu)$ denotes the singular part in the Lebesgue decomposition of $F_y(\omega,\nu)$. Let L be a set of Lebesgue measure zero in K, such that the measure defined by $F^s(\omega,\nu)$ is concentrated on L.

THEOREM 1 [Helson and Lowdenslager (1962)]. *Let $\{y(n,m), (n,m) \in \mathcal{Z}^2\}$ be a 2-D regular and homogeneous random field. Then $y(n,m)$ can be uniquely represented by the orthogonal decomposition*

$$y(n,m) = w(n,m) + v(n,m), \tag{2}$$

where

$$w(n,m) = \sum_{(0,0) \preceq (k,l)} a(k,l) u(n-k, m-l) \tag{3}$$

and $\sum_{(0,0) \preceq (k,l)} a^2(k,l) < \infty$, $a(0,0) = 1$. The field $\{w(n,m)\}$ is purely-indeterministic and regular. The field $\{v(n,m)\}$ is a deterministic random field. The innovation field $\{u(n,m)\}$ is a white noise field. The fields $\{w(n,m)\}$ and $\{v(s,t)\}$ are mutually orthogonal for all (n,m) and (s,t). The spectral representations of the purely indeterministic and deterministic components are given by $w(n,m) = \int_{K \setminus L} \exp[2\pi j(n\omega + m\nu)]\, dZ(\omega,\nu)$ and $v(n,m) = \int_L \exp[2\pi j(n\omega + m\nu)]\, dZ(\omega,\nu)$, respectively. Hence, $F_y(\omega,\nu)$ can be written uniquely as $F_y(\omega,\nu) = F_w(\omega,\nu) + F_v(\omega,\nu)$. The spectral distribution function $F_w(\omega,\nu)$ of the purely-indeterministic component is absolutely continuous, and $F_v(\omega,\nu) = F^s(\omega,\nu)$, where $F_v(\omega,\nu)$ is the spectral distribution function of the deterministic field.

Let $\mathcal{H}$ be the Hilbert space formed by the random variables $y(n,m)$ such that $(n,m) \in \mathcal{Z}^2$. Define $\overset{o}{\mathcal{H}}{}^y_{(n,m)} = \overline{\text{Sp}}\{y(s,t) \mid (s,t) \preceq (n,m)\} \subset \mathcal{H}$; $\overset{o}{\mathcal{H}}{}^u_{(n,m)}$, $\overset{o}{\mathcal{H}}{}^v_{(n,m)}$ are similarly defined. Using these notations, we have from Theorem 1 that $\overset{o}{\mathcal{H}}{}^y_{(n,m)} = \overset{o}{\mathcal{H}}{}^u_{(n,m)} \oplus \overset{o}{\mathcal{H}}{}^v_{(n,m)}$. Define $\overset{o}{\mathcal{H}}{}^v_{(n,-\infty)} = \bigcap_{m=-\infty}^{\infty} \overset{o}{\mathcal{H}}{}^v_{(n,m)}$. Using Theorem 1 it can also be shown that for all m, $\overset{o}{\mathcal{H}}{}^v_{(n,m)} = \overset{o}{\mathcal{H}}{}^v_{(n,-\infty)}$.

Define $\overset{o}{\mathscr{H}}{}^{y}_{(-\infty,-\infty)} = \bigcap_{(n,m)\in\mathscr{Z}^2} \overset{o}{\mathscr{H}}{}^{y}_{(n,m)}$. The Hilbert space $\overset{o}{\mathscr{H}}{}^{y}_{(-\infty,-\infty)}$ is called the *remote past space w.r.t. the NSHP total-order definition o*. It is spanned by the intersection of *all* the Hilbert spaces spanned by samples of the regular field $\{y(n,m)\}$ at all (n,m), with respect to the specific order definition denoted by o.

Let $\overset{o}{\mathscr{H}}{}^{v}_{n} = \overline{\mathrm{Sp}}\{v \mid v \in \overset{o}{\mathscr{H}}{}^{v}_{(n,-\infty)},\ v \perp \overset{o}{\mathscr{H}}{}^{v}_{(n-1,-\infty)}\}$. We thus have that $\overset{o}{\mathscr{H}}{}^{v}_{(n,-\infty)} = \overset{o}{\mathscr{H}}{}^{v}_{(n-1,-\infty)} \oplus \overset{o}{\mathscr{H}}{}^{v}_{n}$. Hence, as was shown by Helson and Lowdenslager (1962), Korezlioglu and Loubaton (1986) and Chiang (1991),

$$\overset{o}{\mathscr{H}}{}^{v}_{(n,m)} = \overset{o}{\mathscr{H}}{}^{v}_{(n,-\infty)} = \overset{o}{\mathscr{H}}{}^{y}_{(-\infty,-\infty)} \oplus \bigoplus_{l=-\infty}^{n} \overset{o}{\mathscr{H}}{}^{v}_{l}. \tag{4}$$

The subspace $\oplus_{l=-\infty}^{n} \overset{o}{\mathscr{H}}{}^{v}_{l}$ is spanned by the *column-to-column innovations* of the regular field *deterministic* component.

DEFINITION 1. A 2-D deterministic random field $\{e_o(n,m)\}$ is called *evanescent w.r.t. the NSHP total order o* if it spans a Hilbert space identical to the one spanned by its column-to-column innovations at each coordinate (n,m) (w.r.t. the total order o).

In the following sections we introduce the concept of multiple NSHP total-ordering definitions. Using this new approach, we derive a countably-infinite-fold decomposition and show that the preceding decomposition is a special case of the countably-infinite-fold decomposition.

2. Multiple order definitions and the evanescent fields. The NSHP support definition that results from the total-order definition (1) is not the only possible definition of that type on the 2-D lattice. Korezlioglu and Loubaton (1986) define "horizontal" and "vertical" total orders and describe the horizontally and vertically evanescent components of homogeneous random fields. Kallianpur, Miamee and Niemi (1990), as well as Chiang (1991), employ similar techniques to obtain four-fold orthogonal decompositions of regular and homogeneous random fields. In the following text, we shall generalize the idea of multiple order definitions by introducing a family of NSHP total-ordering definitions in which the boundary line of the NSHP is of rational slope. Note that it is only the total order imposed on the random field that is changed, but not the 2-D discrete grid itself. We show that by using multiple total-order definitions the regular field deterministic component can be decomposed into a countably infinite number of mutually orthogonal components, rather than the two components that result from the three-fold decomposition of Helson and Lowdenslager (1962) and Kallianpur, Miamee and Niemi (1990) or the three components that result from the four-fold decompositions of Chiang (1991), Kallianpur, Miamee and Niemi (1990) and Korezlioglu and Loubaton (1986).

DEFINITION 2. Let α, β be two coprime integers, such that $\alpha \neq 0$. Let us define a new NSHP total ordering by rotating the NSHP support, which was defined with respect to (1), through a counterclockwise angle θ about the origin of its coordinate system, such that $\tan\theta = \beta/\alpha$.

Let the coordinates (n^*, m^*) be defined by

$$\begin{pmatrix} n^* \\ m^* \end{pmatrix} = \begin{pmatrix} \sqrt{\alpha^2+\beta^2} & 0 \\ 0 & 1/\sqrt{\alpha^2+\beta^2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}, \tag{5}$$

where (n, m) are the original coordinates and

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \begin{pmatrix} \sqrt{\alpha^2+\beta^2} & 0 \\ 0 & 1/\sqrt{\alpha^2+\beta^2} \end{pmatrix}$$

are the rotation transformation matrix and the normalization matrix, respectively. The normalization matrix is such that the indices n^* of the "columns" under the new total-order definition are consecutive integers and the distance between two neighboring samples on the same "column" is 1. Thus, the new coordinates $(n^{(\alpha,\beta)}, m^{(\alpha,\beta)})$ of the original point (n, m) are given by

$$\begin{aligned} n^{(\alpha,\beta)} &= n^*, \\ m^{(\alpha,\beta)} &= m^* - c(n^{(\alpha,\beta)}). \end{aligned} \tag{6}$$

$c(n^{(\alpha,\beta)})$ is a correction term that guarantees that $m^{(\alpha,\beta)}$ is an integer as well. For each fixed column index $n^{(\alpha,\beta)}$ of the new total order, $c(n^{(\alpha,\beta)})$ is determined by $c(n^{(\alpha,\beta)}) = \arg\min_{(n^*,m^*)}\{|m^*|\}$, that is, $c(n^{(\alpha,\beta)})$ is set equal to the m^* of the lowest absolute value in the $n^{(\alpha,\beta)}$ column. For $\theta = \pi/2$ the transformation is obtained by interchanging the roles of columns and rows. The total order in the rotated system is defined similarly to (1), that is,

$$\begin{aligned} &(i^{(\alpha,\beta)}, j^{(\alpha,\beta)}) \prec (s^{(\alpha,\beta)}, t^{(\alpha,\beta)}) \\ &\quad \text{iff} \quad (i^{(\alpha,\beta)}, j^{(\alpha,\beta)}) \in \{(k,l) \mid k = s^{(\alpha,\beta)},\ l < t^{(\alpha,\beta)}\} \\ &\qquad\qquad\qquad\qquad \cup \{(k,l) \mid k < s^{(\alpha,\beta)},\ -\infty < l < \infty\}. \end{aligned} \tag{7}$$

Let us denote by O the above-defined set of all possible NSHP total-ordering definitions on the 2-D lattice, in which the boundary line of the NSHP is of rational slope, that is, $O = \{(\alpha,\beta) \mid \alpha,\ \beta \text{ are coprime integers}\}$. We shall call such support *rational nonsymmetrical half-plane* (RNSHP). An example is illustrated in Figure 1. Note the way the column is defined.

THEOREM 2. *The regularity property of a homogeneous random field is* NSHP total-ordering invariant: *if a homogeneous random field is regular with respect to one NSHP total-ordering definition, then it is regular with respect to any other NSHP total-ordering definition.*

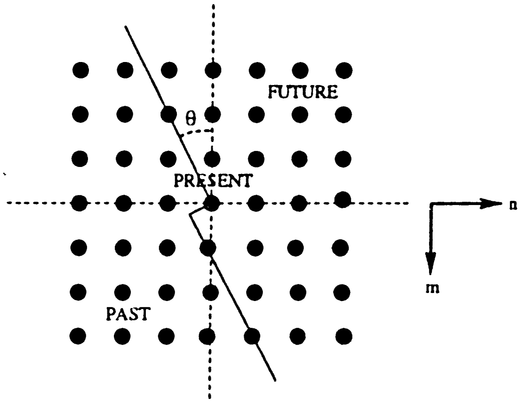

FIG. 1. *RNSHP total-order definition.*

PROOF. Because a 2-D homogeneous random field $\{y(n,m)\}$ is regular if and only if $f(\omega,\nu) > 0$ almost everywhere in K (Lebesgue measure) and $\int_K \log f(\omega,\nu)\,d\omega\,d\nu > -\infty$, [Helson and Lowdenslager (1958)], and because this result is independent of the chosen total-order definition, the proof follows. □

THEOREM 3. *The decomposition of the regular random field into purely indeterministic and deterministic random fields is* unique *and is* NSHP total-ordering invariant: *The purely indeterministic (deterministic) component obtained with respect to one NSHP total-ordering definition is identical to the purely indeterministic (deterministic) component obtained with respect to any other NSHP total-ordering.*

PROOF. The proof readily follows from the foregoing one-to-one correspondence between the decomposition of the regular random field into purely indeterministic and deterministic components, and the unique decomposition of the regular field spectral measure into two mutually singular spectral measures, which are concentrated on the sets $K \setminus L$ and L, respectively. □

Note that Theorems 2 and 3 are valid for any NSHP total-ordering definition and it is not required that the support be RNSHP.

As was shown in (4) for a specific total-order, under each order definition $o \in O$, only a single evanescent field can be resolved: The field that generates the column-to-column innovations of the deterministic component. Next, we shall study the family of total orders defined by Definition 2 to gain further insight into the structure of the deterministic component of the decomposition (2).

Define $\mathscr{H}^u_{(\infty,\infty)} = \overline{\mathrm{Sp}}\{u(n,m) \mid (n,m) \in \mathscr{Z}^2\}$, to be the Hilbert space spanned by the purely indeterministic component of the regular field and similarly define $\mathscr{H}^{v}_{(\infty,\infty)} = \overline{\mathrm{Sp}}\{v(n,m) \mid (n,m) \in \mathscr{Z}^2\}$ to be the Hilbert space spanned by the deterministic component of the regular field.

LEMMA 1. *Interchanging of roles of "past" and "future" in any RNSHP total-ordering definition imposed on a regular field results in identical evanescent components.*

PROOF. Let P be some RNSHP total order. The Hilbert space spanned by the corresponding evanescent component is given by

$$\bigoplus_{n=-\infty}^{\infty} \overset{P}{\mathscr{H}}{}^{v}_{n} = \bigoplus_{n=-\infty}^{\infty} \overline{\mathrm{Sp}}\left\{v \mid v \in \overset{P}{\mathscr{H}}{}^{v}_{(n,-\infty)},\ v \perp \overset{P}{\mathscr{H}}{}^{v}_{(n-1,-\infty)}\right\}. \tag{8}$$

Let F denote the total order obtained by rotating the order P by $\theta = \pi$. The Hilbert space spanned by the corresponding evanescent component is given by

$$\bigoplus_{n'=-\infty}^{\infty} \overset{F}{\mathscr{H}}{}^{v}_{n'} = \bigoplus_{n'=-\infty}^{\infty} \overline{\mathrm{Sp}}\left\{v \mid v \in \overset{F}{\mathscr{H}}{}^{v}_{(n',-\infty)},\ v \perp \overset{F}{\mathscr{H}}{}^{v}_{(n'-1,-\infty)}\right\}. \tag{9}$$

Let (n,m) and (n',m') be the indices of the same grid point under the two different order definitions. The properties of the deterministic random field imply that for any (s,t) and for any fixed $o \in O$, $\overset{o}{\mathscr{H}}{}^{v}_{(s,t)} = \overset{o}{\mathscr{H}}{}^{v}_{(s,-\infty)} = \overset{o}{\mathscr{H}}{}^{v}_{(s,\infty)}$. Because for any $o' \in O$, $\overset{o'}{\mathscr{H}}{}^{v}_{(n',m')} = \overline{\mathrm{Sp}}\{v(s',t') \mid (s',t') \preceq (n',m')\}$ and because the deterministic component of the random field is unique and NSHP total-ordering invariant,

$$\overset{F}{\mathscr{H}}{}^{v}_{(n',-\infty)} = \overset{F}{\mathscr{H}}{}^{v}_{(n',m')} = \mathscr{H}^{v}_{(\infty,\infty)} \ominus \overset{P}{\mathscr{H}}{}^{v}_{(n,m-1)} = \mathscr{H}^{v}_{(\infty,\infty)} \ominus \overset{P}{\mathscr{H}}{}^{v}_{(n,-\infty)}. \tag{10}$$

Due to the reversed order of indexing induced by the two total-order definitions P and F, when $m \to \infty$, $m' \to -\infty$. By the same argument,

$$\overset{F}{\mathscr{H}}{}^{v}_{(n'-1,-\infty)} = \mathscr{H}^{v}_{(\infty,\infty)} \ominus \overset{P}{\mathscr{H}}{}^{v}_{(n+1,-\infty)}. \tag{11}$$

Hence,

$$\begin{aligned}\bigoplus_{n'=-\infty}^{\infty} \overset{F}{\mathscr{H}}{}^{v}_{n'} &= \bigoplus_{n=-\infty}^{\infty} \overline{\mathrm{Sp}}\left\{v \mid v \in \mathscr{H}^{v}_{(\infty,\infty)} \ominus \overset{P}{\mathscr{H}}{}^{v}_{(n,-\infty)},\ v \perp \mathscr{H}^{v}_{(\infty,\infty)} \ominus \overset{P}{\mathscr{H}}{}^{v}_{(n+1,-\infty)}\right\} \\ &= \bigoplus_{n=-\infty}^{\infty} \overline{\mathrm{Sp}}\left\{v \mid v \in \overset{P}{\mathscr{H}}{}^{v}_{(n+1,-\infty)},\ v \perp \overset{P}{\mathscr{H}}{}^{v}_{(n,-\infty)}\right\} = \bigoplus_{n=-\infty}^{\infty} \overset{P}{\mathscr{H}}{}^{v}_{n}.\end{aligned} \tag{12}$$

□

Define $\mathscr{H}^{y}_{(-\infty,-\infty)} = \bigcap_{o\in O} \overset{o}{\mathscr{H}}{}^{y}_{(-\infty,-\infty)}$. Note that $\mathscr{H}^{y}_{(-\infty,-\infty)}$ is the Hilbert space spanned by the intersection of all Hilbert spaces spanned by the regular field samples $\{y(n,m)\}$ for all (n,m) and w.r.t. all possible RNSHP total-order definitions.

DEFINITION 3. A 2-D deterministic random field $\{p(n,m)\}$ is called *half-plane deterministic* if it has no column-to-column innovations w.r.t. any RNSHP total-ordering definition.

THEOREM 4. *Let $\{v(n,m)\}$ be the deterministic component of a 2-D regular and homogeneous random field. Then $\{v(n,m)\}$ can be uniquely represented by the countably infinite orthogonal decomposition*

$$v(n,m) = p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \tag{13}$$

The random field $\{p(n,m)\}$ is *half-plane deterministic*. The field $\{e_{(\alpha,\beta)}(n,m)\}$ is the evanescent component that generates the column-to-column innovations of the deterministic field w.r.t. the RNSHP total-ordering definition $(\alpha,\beta)\in O$.

PROOF OF THEOREM 4. We first show that for any pair of RNSHP total-ordering definitions o, $o' \in O$ such that o' is not obtained by rotating o by $\theta = \pi$,

$$\left(\bigoplus_{n=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}_n^v\right) \perp \left(\bigoplus_{n'=-\infty}^{\infty} \overset{o'}{\mathscr{H}}{}_{n'}^v\right). \tag{14}$$

Using Lemma 1, we conclude that it is sufficient to consider only $-\pi < \theta < 0$. Let $v(n,m)$ be the deterministic component of the (n,m)th sample of the regular field, where the indexing is w.r.t. the order definition o. Let also $\overset{o}{\mathscr{H}}_n(q) \triangleq \overline{\mathrm{Sp}}\{v(k,l) \mid k = n,\ l \le q\}$ (the indexing is w.r.t. o). Let u be a vector such that $u \in \overset{o}{\mathscr{H}}{}_n^v$ for some fixed n. Using the definition of $\overset{o}{\mathscr{H}}{}_n^v$ and because for all m, $\overset{o}{\mathscr{H}}{}_{(n,m)}^v = \overset{o}{\mathscr{H}}{}_{(n,-\infty)}^v$, we conclude that for all m,

$$\overset{o}{\mathscr{H}}{}_n^v = \overset{o}{\mathscr{H}}{}_{(n,m)}^v \ominus \overset{o}{\mathscr{H}}{}_{(n-1,\infty)}^v \subset \overset{o}{\mathscr{H}}_n(m). \tag{15}$$

Assume $u \in \overset{o'}{\mathscr{H}}{}_{n'}^v$ for some fixed n'. Therefore, $u \in \overset{o'}{\mathscr{H}}{}_{(n',-\infty)}^v$, $u \perp \overset{o'}{\mathscr{H}}{}_{(n'-1,-\infty)}^v$. Because any support o' considered here will contain an infinite number of samples $\{v(n,m)\}_{m=-\infty}^{t}$, from the nth column defined w.r.t. the RNSHP order definition o, we have that for any o', $\overset{o}{\mathscr{H}}_n(t) \subset \overset{o'}{\mathscr{H}}{}_{(n'-1,-\infty)}^v$. Because (15) holds for all m, we have $\overset{o}{\mathscr{H}}{}_n^v \subset \overset{o}{\mathscr{H}}_n(t)$. Hence, $\overset{o}{\mathscr{H}}{}_n^v \subset \overset{o'}{\mathscr{H}}{}_{(n'-1,-\infty)}^v$ and $u = 0$. Because the preceding argument holds for all n', we conclude that $\overset{o}{\mathscr{H}}{}_n^v \perp \oplus_{n'=-\infty}^{\infty} \overset{o'}{\mathscr{H}}{}_{n'}^v$. Repeating the same arguments for each n, we obtain (14). Hence, the evanescent fields are mutually orthogonal.

The deterministic component of the random field is unique and NSHP total-ordering invariant. We can therefore rewrite (4) for any total-order definition

$o \in O$, while letting $n, m \to \infty$

$$\mathscr{H}^v_{(\infty,\infty)} = \overset{o}{\mathscr{H}}{}^v_{(\infty,\infty)} = \overset{o}{\mathscr{H}}{}^y_{(-\infty,-\infty)} \oplus \bigoplus_{l=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}^v_l. \tag{16}$$

For any $o \in O$, $\oplus_{l=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}^v_l \subset \mathscr{H}^v_{(\infty,\infty)}$. Also, for any two total-order definitions $o, o' \in O$ such that o' is not obtained by rotating o by $\theta = \pi$, $(\oplus_{l=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}^v_l) \perp (\oplus_{k'=-\infty}^{\infty} \overset{o'}{\mathscr{H}}{}^v_{k'})$. Hence, we conclude using (16) that for any two such total-order definitions $o, o' \in O$,

$$\left(\bigoplus_{k'=-\infty}^{\infty} \overset{o'}{\mathscr{H}}{}^v_{k'} \right) \subset \overset{o}{\mathscr{H}}{}^y_{(-\infty,-\infty)}. \tag{17}$$

Using (16) together with the uniqueness and NSHP total-ordering invariance of the deterministic component, we conclude that

$$\begin{aligned} \mathscr{H}^v_{(\infty,\infty)} &= \bigcap_{o \in O} \overset{o}{\mathscr{H}}{}^v_{(\infty,\infty)} = \bigcap_{o \in O} \left(\overset{o}{\mathscr{H}}{}^y_{(-\infty,-\infty)} \oplus \bigoplus_{l=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}^v_l \right) \\ &= \mathscr{H}^y_{(-\infty,-\infty)} \oplus \bigoplus_{o \in O} \bigoplus_{l=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}^v_l, \end{aligned} \tag{18}$$

where the last equality results from the definition of $\mathscr{H}^y_{(-\infty,-\infty)}$, from (14), which results in the elimination of the cross terms that involve the intersection of more than one Hilbert space of the type $\oplus_{l=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}^v_l$, and from (17). $\oplus_{o \in O} \oplus_{l=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}^v_l$ is the Hilbert space spanned by all the evanescent components of the regular field. Because O is a countable set, the number of evanescent components of a regular field is countable. By (18) and Definition 3, $\mathscr{H}^y_{(-\infty,-\infty)}$ is spanned by a half-plane deterministic field. □

The result in (17) implies that for each RNSHP total-order definition $o \in O$, all subspaces spanned by the evanescent components $e_{o'}$, where $o' \neq o$, remain in the *remote past space* $\overset{o}{\mathscr{H}}{}^y_{(-\infty,-\infty)}$, which corresponds to the definition o. Hence, from (18) we conclude that in order to resolve all the evanescent components of a regular field, the field has to be tested against all the possible RNSHP total-ordering definitions in O. Note also that because column-to-column innovations are found only when RNSHP total-ordering definitions are imposed on the field, and because the half-plane deterministic component of the decomposition is deterministic by definition, we conclude that the half-plane deterministic field has no innovations nor column-to-column innovations w.r.t. any NSHP total ordering.

Because the purely-indeterministic component is unique and RNSHP total-ordering invariant, the same subspace $\mathscr{H}^u_{(\infty,\infty)}$ is obtained w.r.t. any order definition and hence we omit the order notation. Let $\mathscr{H}^{e_o}$ be the Hilbert space spanned by the evanescent field that corresponds to the RNSHP total-ordering $o \in O$, that is,

$$\mathscr{H}^{e_o} = \bigoplus_{n=-\infty}^{\infty} \overset{o}{\mathscr{H}}{}_n^v. \tag{19}$$

COROLLARY 5. *The following orthogonal decomposition holds:*

$$\mathscr{H} = \mathscr{H}^u_{(\infty,\infty)} \oplus \mathscr{H}^y_{(-\infty,-\infty)} \oplus \bigoplus_{o \in O} \mathscr{H}^{e_o}. \tag{20}$$

PROOF. Using the direct sum representation (18) and Theorem 3, (20) results. □

Hence, if $\{y(n,m)\}$ is a 2-D regular and homogeneous random field, then $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + p(n,m) + \sum_{(\alpha,\beta) \in O} e_{(\alpha,\beta)}(n,m). \tag{21}$$

THEOREM 6. *The spectral measures of the decomposition components in* (21) *are mutually singular.*

PROOF. Let P_u denote the projection operator of $\mathscr{H}$ onto $\mathscr{H}^u_{(\infty,\infty)}$, and let P_{e_o} denote the projection operator of $\mathscr{H}$ onto $\mathscr{H}^{e_o}$. Similarly, let us denote by P_p the projection operator of $\mathscr{H}$ onto $\mathscr{H}^y_{(-\infty,-\infty)}$. All subspaces in the right-hand side of (20) are mutually orthogonal and it is readily verified that they are all invariant to the vertical and horizontal shift operators. Hence for any pair of these subspaces, say A and B, we have, using Chiang (1991), Lemma 7, that $P_A P_B = P_B P_A = P_{A \cap B}$ and for any $h(\omega,\nu) \in L^2_{dF_y}$,

$$P_{A\cap B}\left\{\int_K h(\omega,\nu)\,dZ(\omega,\nu)\right\} = \int_{A^* \cap B^*} h(\omega,\nu)\,dZ(\omega,\nu), \tag{22}$$

where A^*, B^* are two-dimensional Borel sets in K, such that

$$P_A\left\{\int_K h(\omega,\nu)\,dZ(\omega,\nu)\right\} = \int_{A^*} h(\omega,\nu)\,dZ(\omega,\nu)$$

and

$$P_B\left\{\int_K h(\omega,\nu)\,dZ(\omega,\nu)\right\} = \int_{B^*} h(\omega,\nu)\,dZ(\omega,\nu).$$

Because A and B are mutually orthogonal, the left-hand side of (22) is identically 0 for any $h(\omega,\nu) \in L^2_{dF_y}$. Hence A^*, B^* are disjoint, except maybe on a set of dF_y measure zero. □

Let F_p be the spectral distribution function of the half-plane deterministic component of the regular and homogeneous random field, and let $F_{e_{(\alpha,\beta)}}(\omega, \nu)$ be the spectral distribution function of the evanescent component that generates the column-to-column innovations of the deterministic field w.r.t. the RNSHP total-ordering (α, β). Hence, Theorem 6 implies that the spectral distribution function of the regular field deterministic component is uniquely represented by

$$F_v(\omega, \nu) = F_p(\omega, \nu) + \sum_{(\alpha,\beta)\in O} F_{e_{(\alpha,\beta)}}(\omega, \nu) \tag{23}$$

and that the spectral measures defined by F_p and $F_{e_{(\alpha,\beta)}}$, $(\alpha, \beta) \in O$, are all mutually singular.

Thus, the decomposition of the deterministic component of a regular field into a half-plane deterministic field and a countable number of evanescent fields corresponds in terms of spectral measures to the representation of the spectral measure of the deterministic component as a countable sum of mutually singular spectral measures. However, contrary to the separation of the absolutely continuous component of the regular field spectral distribution from the singular component, which can be accomplished by a linear operation on the "past" defined with respect to *any* RNSHP total ordering, the decomposition of Theorem 6 is attained only by using a countable number of total-order definitions, while performing a linear operation on the "past" defined with respect to *each one* of these definitions. Note that because both the half-plane deterministic field and all the evanescent fields in the decomposition (21) are components of the deterministic component of the regular field, their spectral measures are concentrated on subsets of the set L. Hence, the spectral decomposition in Theorem 6 yields a decomposition of a spectral measure which is concentrated on a set of Lebesgue measure zero.

Using Theorem 3 we have that the decomposition of the regular random field into purely indeterministic and deterministic random fields is NSHP total-ordering invariant. It is therefore invariant to the interchange of past and future definitions. We have also shown that interchanging the roles of past and future in any total-order definition results in identical evanescent components. Hence, using (16), we conclude that if a total-order definition o' is obtained by rotating some other total-order definition o by $\theta = \pi$, then $\overset{o}{\mathscr{H}}{}^y_{(-\infty,-\infty)} = \overset{o'}{\mathscr{H}}{}^y_{(-\infty,-\infty)}$. Therefore, the regular field decomposition w.r.t. the total-order definition in which the roles of past and future were interchanged is identical to the one obtained under the original RNSHP total-ordering, and no additional components of the random field can be found in this way. Hence, in Definition 2, it is sufficient to consider only $0 \leq \theta < \pi$. By similar arguments it can be shown that for any given RNSHP total-ordering, no new components of the random field are found when a new RNSHP total-order is defined by reflecting the order on one of the axes.

Finally, all the subspaces $\mathop{\mathscr{H}}\limits^{o}{}_l^v$, $l \in \mathscr{Z}$, spanned by the column-to-column innovations of the deterministic field with respect to some RNSHP total-ordering $o \in O$ have the same dimension. Let us denote this dimension by $\overset{o}{M}$. We shall call $\overset{o}{M}$ the *multiplicity* of the random field with respect to the RNSHP total-ordering o. Thus the multiplicity properties defined by Chiang (1991) and by Kallianpur, Miamee and Niemi (1990), with respect only to the horizontal and vertical NSHP definitions, can be naturally defined with respect to any RNSHP definition.

3. Approximations and applications. The definition of the evanescent field and Theorem 6 imply that the spectral measure of the evanescent component that generates the column-to-column innovations for $(\alpha, \beta) = (1, 0)$ is a linear combination of spectral measures of the form

$$dF_{e_{(1,0)}}(\omega, \nu) = k(\omega)\, d\omega\, dF^s(\nu), \tag{24}$$

where $F^s(\nu)$ is a one-dimensional singular spectral distribution function and $k(\omega)$ is a one-dimensional spectral density function. In other words, the spectral distribution function of each evanescent component is separable: it is absolutely continuous in one dimension and singular in the orthogonal one (or a linear combination of such separable distribution functions).

According to Theorem 1, the spectral measure of the deterministic component is concentrated on a set of Lebesgue measure zero. For practical applications we can exclude singular-continuous spectral distributions from the framework of our treatment. Hence, the "spectral density function" of the evanescent field $e_{(1,0)}$ has the countable sum form $f_{e_{(1,0)}}(\omega, \nu) = \sum_i k_i(\omega)\{\delta(\nu - \nu_i) + \delta(\nu + \nu_i)\}$. A model for this evanescent field is given by

$$e_{(1,0)}(n, m) = \sum_i s_i(n) \cos 2\pi m \nu_i + t_i(n) \sin 2\pi m \nu_i, \tag{25}$$

where the 1-D purely-indeterministic processes $\{s_i(n)\}$, $\{s_j(n)\}$, $\{t_k(n)\}$, $\{t_l(n)\}$ are mutually orthogonal for all $i, j, k, l, i \neq j, k \neq l$, and for all i the processes $\{s_i(n)\}$ and $\{t_i(n)\}$ have an identical spectral density function, $2k_i(\omega)$.

Similarly, for any $(\alpha, \beta) \in O$,

$$\begin{aligned} &e_{(\alpha,\beta)}(n^{(\alpha,\beta)}, m^{(\alpha,\beta)}) \\ &\quad = \sum_i s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) \cos 2\pi m^{(\alpha,\beta)} \nu_i^{(\alpha,\beta)} + t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) \sin 2\pi m^{(\alpha,\beta)} \nu_i^{(\alpha,\beta)}, \end{aligned} \tag{26}$$

where the 1-D purely-indeterministic processes $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$, $\{s_j^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$, $\{t_k^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$, $\{t_l^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ are mutually orthogonal for all $i, j, k, l, i \neq j, k \neq l$, and for all i the processes $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ have an identical autocorrelation function. Using the transformation (5), we can rewrite (26) in terms of the original indices:

$$
\begin{aligned}
e_{(\alpha,\beta)}(n,m) = \sum_i s_i^{(\alpha,\beta)}(n\alpha - m\beta)\cos\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha)\right) \\
+ t_i^{(\alpha,\beta)}(n\alpha - m\beta)\sin\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha)\right).
\end{aligned}
\tag{27}
$$

Hence, the "spectral density function" of each evanescent component has the form of a countable sum of 1-D delta functions that are supported on lines of rational slope in the 2-D spectral domain.

One component of the half-plane-deterministic component, which is often found in physical problems, is the harmonic random field $\{h(n,m)\}$. This component generates the 2-D delta functions of the "spectral density." The harmonic field has the countable sum representation

$$
h(n,m) = \sum_p \{C_p \cos 2\pi(n\omega_p + m\nu_p) + D_p \sin 2\pi(n\omega_p + m\nu_p)\}, \tag{28}
$$

where the C_p's and D_p's are mutually orthogonal random variables, $E[C_p^2] = E[D_p^2] = \sigma_p^2$, and (ω_p, ν_p) are the spatial frequencies of the pth harmonic.

The foregoing analysis establishes a basis for solving problems that require the modeling and parameter estimation of 2-D homogeneous random fields with mixed spectral distributions. One such problem is that of texture modeling in 2-D images, [Francos, Meiri and Porat (1993) and Francos, Narasimhan and Woods (1994)]. The texture field is assumed to be a realization of a regular homogeneous random field. On the basis of the 2-D Wold-like decomposition, the texture field is decomposed into a sum of mutually orthogonal components: a purely indeterministic component, a harmonic component and a countable number of evanescent components. The resulting model, which is applicable to a wide variety of texture types found in natural images, leads to the derivation of texture analysis and synthesis algorithms designed to estimate the texture parameters and to reconstruct the original texture field from the estimated parameters. The model is very efficient in terms of the number of parameters required to faithfully represent textures. The reconstructed textures are practically indistinguishable from the originals.

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