Orthogonal Decompositions of 2-D Nonhomogeneous Discrete Random Fields

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Abstract

Imposing a *total-order* on a 2-D discrete random field induces an orthogonal decomposition of the random field into two components: A *purely-indeterministic* field and a *deterministic* one. The purely-indeterministic component is shown to have a 2-D white-innovations driven moving-average representation. The 2-D deterministic random field can be perfectly predicted from the field's "past" with respect to the imposed total order definition. The deterministic field is further orthogonally decomposed into an *evanescent* field, and a *remote past* field. The evanescent field is generated by the column-to-column innovations of the deterministic field with respect to the imposed non-symmetrical-half-plane total-ordering definition. The presented decomposition can be obtained with respect to any non-symmetrical-half-plane total-ordering definition, for which the non-symmetrical-half-plane boundary line has rational slope.

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1. Introduction

Recently, there has been a growing interest in nonstationary one-dimensional and multi-dimensional processes. In the present paper we study the structure of 2-D nonhomogeneous discrete random fields, and show that any 2-D regular random field can be represented as a sum of three mutually orthogonal components: purely indeterministic, evanescent, and a remote past component. This study generalizes the Wold decomposition of nonstationary random processes which was derived by Cramer [C2], to the case of 2-D nonhomogeneous discrete random fields. The results proven in this paper establish a formalism for analysis and parameter estimation methods of such fields. The analysis is carried out in the context of the 2-D linear prediction problem for a non-symmetrical-half-plane (NSHP) support. For homogeneous random fields analysis, a similar type of support was used by Whittle [W], as well as by Ekstrom and Woods [EW], to develop the concept of 2-D spectral factorization; by Marzetta [M], to describe a theoretical solution of the 2-D normal equations by a 2-D Levinson-type algorithm; and in [FMP2], [FNW] to implement an analysis/synthesis procedure for homogeneous texture fields.

Helson and Lowdenslager [HL2] proved some of the results contained in sections 3 and 4 for the case of homogeneous random fields using the character group approach. However, frequency domain analysis is applicable only to homogeneous random fields, since it relies upon a spectral representation in the form of a Fourier-Stieltjes integral, both for the field variables and for the associated covariance functions. In this paper we use constructions in the *spatial domain*, so the theorems and their proofs are applicable to nonhomogeneous as well as to homogeneous 2-D discrete random fields. Thus the known results on the 2-D Wold decomposition for homogeneous random fields become a special case of those presented here for nonhomogeneous random fields. In [KL], Korezlioglu and Loubaton presented a spatial domain reformulation to Helson and Lowdenslager results on the decomposition of homogeneous random fields, using Hilbert space representations. They define “horizontal” and “vertical” total-orders and derive the corresponding decompositions of the homogeneous random field.

In section 5 we define a set of NSHP total-ordering definitions and show that the results of sections 2,3,4 hold for any definition in the set. These NSHP total-ordering definitions are obtained by rotating the "conventional" NSHP support by angles having rational tangent, rather than considering only the vertical and horizontal orientations. Thus, the random field decomposition can be obtained with respect to *any* non-symmetrical-half-plane total-ordering definition, for which the non-symmetrical-half-plane boundary line has rational slope. However, contrary to the homogeneous case [FMP1], the orthogonal decomposition of the field into purely-indeterministic and deterministic components is not unique, but NSHP total-ordering dependent.

2. Definitions and Fundamental Properties

In the sequel we shall assume the 2-D random field $\{y(n,m)\}$ to be real, with zero mean. We shall also assume that the random field has finite second-order moments, i.e.,

$$\sup_{(n,m)\in\mathbb{Z}^2} E[y^2(n,m)] < \infty \tag{1}$$

and that $E[y^2(n,m)] > 0$ for at least one $(n,m) \in \mathbb{Z}^2$.

Let $\mathcal{H}$ be the Hilbert space formed by the random variables $y(n,m)$, such that $(n,m) \in \mathbb{Z}^2$, with the inner product of any two random variables x, y being defined by $E[xy]$. Let $\hat{y}(n,m)$ be the projection of $y(n,m)$ on the Hilbert space spanned by those samples of the field that are in the "past" of the (n,m)th sample, where the "past" is defined with respect to the *totally ordered, non-symmetrical-half-plane support*, i.e.,

$$(i,j) \prec (s,t) \text{ iff } (i,j) \in \Big\{(k,\ell)|k=s, \ell<t\Big\} \bigcup \Big\{(k,\ell)|k<s, -\infty<\ell<\infty\Big\} . \tag{2}$$

Since in this paper we consider other total-order definitions as well, we shall denote this order definition by $o = (1,0)$. The reason for this notation is explained in Section 5. The results given in this section, as well as in sections 3 and 4, are with respect to $o = (1,0)$. Let $\overset{o}{\mathcal{H}}{}^Y{}_{(n,m)} = \overline{Sp}\{y(s,t)|(s,t) \preceq (n,m)\} \subset \mathcal{H}$. This definition implies the *nesting property* of the Hilbert spaces, i.e., whenever $(s,t) \preceq (n,m)$, $\overset{o}{\mathcal{H}}{}^Y{}_{(s,t)} \subset \overset{o}{\mathcal{H}}{}^Y{}_{(n,m)}$. The *innovation* with respect to the defined support and total order is given by $u(n,m) = y(n,m) - \hat{y}(n,m)$. By the orthogonal projection theorem, $u(n,m)$ is orthogonal to every vector in $\overset{o}{\mathcal{H}}{}^Y{}_{(n,m-1)}$.

We show first that the projection of $y(n,m)$ on the Hilbert space $\overset{o}{\mathcal{H}}^{Y}{}_{(n,m-1)}$ can be approximated by a predictor which is based on a *finite half-plane support*. Let the finite support $S_{N,M}$ be defined by

$$S_{N,M} = \{(k,l)|k=0, \quad 0<l\leq M\} \cup \{(k,l)|1\leq k\leq N, \quad -M\leq l\leq M\} \tag{3}$$

where N and M are positive integers. Let also the support S_N be defined by

$$S_N = \{(k,l)|k=0, \quad 0<l<\infty\} \cup \{(k,l)|1\leq k\leq N, \quad -\infty<l<\infty\}\ . \tag{4}$$

Correspondingly, let $\overset{o}{\mathcal{H}}^{Y}{}_{(n,m);S_{N,M}} = Sp\,\{y(n-k,m-l)|(k,l)\in\{S_{N,M}\cup\{(0,0)\}\}\,\}$.
$\hat{y}_{S_{N,M}}(n,m)$, the projection of $y(n,m)$ on $\overset{o}{\mathcal{H}}^{Y}{}_{(n,m-1);S_{N,M}}$ is given by

$$\hat{y}_{S_{N,M}}(n,m) = \sum_{(k,l)\in S_{N,M}} g_{(n,m)}(k,l)y(n-k,m-l) \quad . \tag{5}$$

$\hat{y}_{S_N}(n,m)$ is the projection of $y(n,m)$ on $\overset{o}{\mathcal{H}}^{Y}{}_{(n,m-1);S_N}$, where
$\overset{o}{\mathcal{H}}^{Y}{}_{(n,m);S_N} = \overline{Sp}\,\{y(n-k,m-l)|(k,l)\in\{S_N\cup\{(0,0)\}\}\,\}$.

Theorem 1:

$$\lim_{N\to\infty}\lim_{M\to\infty} E[\hat{y}(n,m)-\hat{y}_{S_{N,M}}(n,m)]^2 = 0 \quad . \tag{6}$$

Proof: See Appendix A.

Definition 1: A 2-D random field is called *regular* if there exists at least one $(n,m)\in \mathbb{Z}^2$ such that $E[y(n,m)-\hat{y}(n,m)]^2>0$. Hence, a discrete 2-D random field is regular if its innovation field $\{u(n,m)\}$ does not vanish.

3. The 2-D Wold-Like Decomposition

Theorem 2: Let $\{y(n,m)\}$ be a 2-D regular random field. Then $\{y(n,m)\}$ can be uniquely represented by the following orthogonal decomposition

$$y(n,m) = w(n,m) + v(n,m) \tag{7}$$

where

$$w(n,m) = \sum_{(0,0)\preceq(k,l)} a_{(n,m)}(k,l)u(n-k,m-l) \tag{8}$$

$$= \sum_{l=0}^{\infty} a_{(n,m)}(0,l)u(n,m-l) + \sum_{k=1}^{\infty}\sum_{l=-\infty}^{\infty} a_{(n,m)}(k,l)u(n-k,m-l) \quad .$$

$a_{(k,l)}(s,t)$ is given by:

$$a_{(k,l)}(s,t) = \begin{cases} \frac{E[y(k,l)u(k-s,l-t)]}{E[u^2(k-s,l-t)]} & \text{if} \quad E[u^2(k-s,l-t)] > 0 \\ \\ 0 & \text{if} \quad E[u^2(k-s,l-t)] = 0 \end{cases} \tag{9}$$

where if $E[u^2(k-s,l-t)] = 0$, $a_{(k,l)}(s,t)$ is arbitrarily set to zero.

Also,

(a) $\sum_{(0,0)\preceq(k,l)} a^2_{(n,m)}(k,l)E[u^2(n-k,m-l)] < \infty$

(b) $E[v(n,m)] = 0$

(c) $E[u(n,m)u(s,t)] = 0 \quad ; \qquad (n,m) \neq (s,t)$

(d) $E[u(n,m)v(s,t)] = 0 \quad ; \qquad \forall\, (n,m),(s,t)$

(e) $w(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$

(f) $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$ where the Hilbert space $\overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$ is defined by $\overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)} = \bigcap_{m=-\infty}^{\infty} \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$.

(g) If for all $(n,m) \in \mathbb{Z}^2$, $E[u^2(n,m)] > 0$, the sequences $\{u(n,m)\}$ and $\{a_{(s,t)}(n,m)\}$ are unique, i.e., there is only one 2-D sequence of random variables $\{u(n,m)\}$ and only one 2-D sequence of constants $\{a_{(s,t)}(n,m)\}$ satisfying the previously stated results. However, if there are $(n,m) \in \mathbb{Z}^2$, such that $E[u^2(n,m)] = 0$, the uniqueness of the sequence $\{a_{(s,t)}(n,m)\}$ may be achieved by the arbitrary setting of the corresponding elements of the sequence $a_{(s,t)}(n,m)$ to zero.

Proof: We shall first prove (c). From the orthogonal projection theorem, $u(n,m)$ is orthogonal to every vector in $\overset{o}{\mathcal{H}}^Y{}_{(n,m-1)}$. Using the nesting property we deduce that $u(n,m) \perp \overset{o}{\mathcal{H}}^Y{}_{(s,t)}$ for all $(s,t) \prec (n,m)$. Because $u(s,t) \in \overset{o}{\mathcal{H}}^Y{}_{(s,t)}$ for all (s,t), we conclude that $u(n,m) \perp u(s,t)$ for all $(s,t) \prec (n,m)$. By interchanging the role of indices we also have that $u(s,t) \perp u(n,m)$ for all $(n,m) \prec (s,t)$, and this completes the proof of (c).

We shall now prove (a). Let the support S' be defined by $S' = S_{N,M} \cup \{(0,0)\}$. We shall look at the following expression:

$$\begin{aligned} 0 \;&\leq\; E\left[y(n,m) - \sum_{S'} a_{(n,m)}(k,l)u(n-k,m-l)\right]^2 \qquad (10) \\ &= E[y^2(n,m)] - 2\sum_{S'} a_{(n,m)}(k,l)E[y(n,m)u(n-k,m-l)] \\ &+ \sum_{S'}\sum_{S'} a_{(n,m)}(k,l)a_{(n,m)}(s,t)E[u(n-k,m-l)u(n-s,m-t)] \\ &= E[y^2(n,m)] - 2\sum_{S'} a^2_{(n,m)}(k,l)E[u^2(n-k,m-l)] + \sum_{S'} a^2_{(n,m)}(k,l)E[u^2(n-k,m-l)] \\ &= E[y^2(n,m)] - \sum_{S'} a^2_{(n,m)}(k,l)E[u^2(n-k,m-l)] \quad . \end{aligned}$$

By assumption $\sup_{(n,m)\in\mathbb{Z}^2} E[y^2(n,m)] < \infty$. We therefore conclude that

$$\sum_{S'} a^2_{(n,m)}(k,l)E[u^2(n-k,m-l)] \leq \sup_{(n,m)\in\mathbb{Z}^2} E[y^2(n,m)] < \infty \ . \qquad (11)$$

This sum is bounded for any N and M by an expression which is neither a function of N nor M. Therefore, the positive series converges since the sequence of its partial sums is bounded. This completes the proof of (a).

In (8) $w(n,m)$ was defined as: $w(n,m) = \sum_{(0,0)\preceq(k,l)} a_{(n,m)}(k,l)u(n-k,m-l)$. Using (a) and (c) we conclude that

$$E[w^2(n,m)] = \sum_{(0,0)\preceq(k,l)} a^2_{(n,m)}(k,l)E[u^2(n-k,m-l)] < \infty \quad . \qquad (12)$$

From (8), $w(n,m)$ is in the linear manifold spanned by $u(i,j)$ such that $(i,j) \preceq (n,m)$. Because $u(s,t) \in \overset{o}{\mathcal{H}}^Y{}_{(s,t)}$ for all (s,t) and $\overset{o}{\mathcal{H}}^Y{}_{(s,t)} \subset \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$ whenever $(s,t) \preceq (n,m)$, $w(n,m)$ is a linear combination of elements in $\overset{o}{\mathcal{H}}^Y{}_{(n,m)}$. Since its second moment is finite, $w(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$, as stated in (e).

The proof of (b) follows immediately from the definition of $v(n,m)$ as

$$v(n,m) = y(n,m) - w(n,m) \ . \qquad (13)$$

We shall now turn to prove (d). For every $(s,t) \preceq (n,m)$ such that $E[u^2(s,t)] > 0$ we have

$$E[v(n,m)u(s,t)] = E[y(n,m)u(s,t)] - \sum_{(0,0)\preceq(k,l)} a_{(n,m)}(k,l)E[u(n-k,m-l)u(s,t)] \tag{14}$$

where it follows from the definition of $a_{(n,m)}(n-s,m-t)$ that $E[y(n,m)u(s,t)] = a_{(n,m)}(n-s,m-t)E[u^2(s,t)]$. Since

$$\sum_{(0,0)\preceq(k,l)} a_{(n,m)}(k,l)E[u(n-k,m-l)u(s,t)] = a_{(n,m)}(n-s,m-t)E[u^2(s,t)] \tag{15}$$

we have that $E[v(n,m)u(s,t)] = 0$.

For the case in which $E[u^2(s,t)] = 0$, we have using the Cauchy-Schwarz inequality that

$$0 \leq |E[v(n,m)u(s,t)]|^2 \leq E[v^2(n,m)]E[u^2(s,t)] = 0. \tag{16}$$

Hence, $E[v(n,m)u(s,t)] = 0$ in this case as well.

For $(n,m) \prec (s,t)$ we have by using (13), and since both $w(n,m)$ and $y(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$, that $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$. Since $u(s,t) \perp \overset{o}{\mathcal{H}}^Y{}_{(s,t-1)}$ and since $\overset{o}{\mathcal{H}}^Y{}_{(n,m)} \subset \overset{o}{\mathcal{H}}^Y{}_{(s,t-1)}$ whenever $(n,m) \preceq (s,t-1)$, we have that for every $(n,m) \prec (s,t), u(s,t) \perp v(n,m)$. Combining the two cases, we conclude that for every two pairs of indices (s,t) and (n,m), $u(n,m) \perp v(s,t)$.

In order to prove (f), define $Sp\{u(n,m)\}$ as the subspace of $\overset{o}{\mathcal{H}}^Y{}_{(n,m)}$ spanned by the vector $u(n,m)$. From the orthogonal projection theorem, $u(n,m) \perp \overset{o}{\mathcal{H}}^Y{}_{(n,m-1)}$ and therefore $\overset{o}{\mathcal{H}}^Y{}_{(n,m)} = \overset{o}{\mathcal{H}}^Y{}_{(n,m-1)} \oplus Sp\{u(n,m)\}$. Since $v(n,m) \perp u(n,m)$ and $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$ it follows that $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,m-1)}$. By induction $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$.

Let us now prove (g). From the orthogonal projection theorem it follows that $u(n,m)$ is unique. This holds for every (n,m) and therefore the field $\{u(n,m)\}$ is unique. If for all $(n,m) \in \mathbb{Z}^2$, $E[u^2(n,m)] > 0$, then since for every (n,m) and (s,t) such that $(s,t) \succeq (0,0)$, $y(n,m)$ and $u(n-s,m-t)$ are elements in the Hilbert space $\overset{o}{\mathcal{H}}^Y{}_{(n,m)}$ where the inner product is defined as $E[xy]$, the uniqueness of $\{u(n,m)\}$ implies the uniqueness of $\{a_{(s,t)}(n,m)\}$. However, if there are $(n,m) \in \mathbb{Z}^2$, such that $E[u^2(n,m)] = 0$, the uniqueness of the sequence $a_{(s,t)}(n,m)$ may be achieved by the arbitrary setting of the corresponding elements of the sequence $a_{(s,t)}(n,m)$ to zero. $\square$

4. Properties Of The 2-D Wold-Like Decomposition

Definition 2: A field $\{y(n,m)\}$ is called *deterministic* if for all $(n,m) \in \mathbb{Z}^2$, $E[y(n,m) - \hat{y}(n,m)]^2 = 0$. This means that for all (n,m), $y(n,m)$ can be perfectly predicted as a linear combination of elements of its past (or as a limit of such), i.e., elements of $\overset{o}{\mathcal{H}}{}^{Y}{}_{(n,m-1)}$.

Define $\overset{o}{\mathcal{H}}{}^{U}{}_{(n,m)} = \overline{Sp}\,\{u(s,t) | (s,t) \preceq (n,m)\}$, $\overset{o}{\mathcal{H}}{}^{V}{}_{(n,m)} = \overline{Sp}\,\{v(s,t) | (s,t) \preceq (n,m)\}$, $\overset{o}{\mathcal{H}}{}^{W}{}_{(n,m)} = \overline{Sp}\,\{w(s,t) | (s,t) \preceq (n,m)\}$.

Definition 3: A regular field $\{y(n,m)\}$ is called *purely indeterministic* if for all (n,m) $\overset{o}{\mathcal{H}}{}^{Y}{}_{(n,m)} = \overset{o}{\mathcal{H}}{}^{U}{}_{(n,m)}$, i.e., if its deterministic component $\{v(n,m)\}$ vanishes, so that $\{y(n,m)\}$ can be represented completely by the moving average term of (8):

$$y(n,m) = \sum_{(0,0) \preceq (k,l)} a_{(n,m)}(k,l) u(n-k, m-l) \; . \tag{17}$$

Theorem 3: Let $\{y(n,m)\}$ be a 2-D regular random field. Its component $\{w(n,m)\}$ is purely-indeterministic and regular.

Proof: Let us rewrite (8) as

$$w(n,m) = a_{(n,m)}(0,0) u(n,m) + \sum_{(0,0) \prec (k,l)} a_{(n,m)}(k,l) u(n-k, m-l). \tag{18}$$

If $E[u^2(n,m)] > 0$, then $a_{(n,m)}(0,0) = 1$. We have already proved that $w(n,m)$ and $u(n,m) \in \overset{o}{\mathcal{H}}{}^{Y}{}_{(n,m)}$, that $u(n,m) \perp \overset{o}{\mathcal{H}}{}^{Y}{}_{(n,m-1)}$, and that $\sum_{(0,0) \prec (k,l)} a_{(n,m)}(k,l) u(n-k, m-l) \in \overset{o}{\mathcal{H}}{}^{Y}{}_{(n,m-1)}$. Therefore, the orthogonal projection theorem and the uniqueness of both the projection and the residual, together with the above representation of $w(n,m)$, imply that $\hat{w}(n,m)$, which is the projection of $w(n,m)$ on $\overset{o}{\mathcal{H}}{}^{Y}{}_{(n,m-1)}$, is given by

$$\hat{w}(n,m) = \sum_{(0,0) \prec (k,l)} a_{(n,m)}(k,l) u(n-k, m-l). \tag{19}$$

Clearly (19) holds also if $E[u^2(n,m)] = 0$, since in that case $w(n,m) = \hat{w}(n,m)$, and both are elements of $\overset{o}{\mathcal{H}}{}^{Y}{}_{(n,m-1)}$.

In order to prove that $\{w(n,m)\}$ is a purely-indeterministic random field we show that $\overset{o}{\mathcal{H}}{}^{W}{}_{(n,m)} = \overset{o}{\mathcal{H}}{}^{U}{}_{(n,m)}$. Since $w(n,m)$ is a linear combination of the elements $u(k,l)$ where $(k,l) \preceq$

(n,m), $\overset{o}{\mathcal{H}}^W{}_{(n,m)} \subset \overset{o}{\mathcal{H}}^U{}_{(n,m)}$. On the other hand, $\hat{w}(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,m-1)}$. Hence, there exists a sequence of constants $\{c_{(n,m)}(k,l)\}$ such that $\hat{w}(n,m)$ is represented by

$$\hat{w}(n,m) = \sum_{(0,0)\prec(k,l)} c_{(n,m)}(k,l) y(n-k, m-l) \tag{20}$$

or by a limit of such expression. Using (7), we can rewrite (20):

$$\hat{w}(n,m) = \sum_{(0,0)\prec(k,l)} c_{(n,m)}(k,l) w(n-k, m-l) + \sum_{(0,0)\prec(k,l)} c_{(n,m)}(k,l) v(n-k, m-l)\ . \tag{21}$$

From (19), $\hat{w}(n,m) \in \overset{o}{\mathcal{H}}^U{}_{(n,m-1)}$. Also, since $\overset{o}{\mathcal{H}}^W{}_{(n,m-1)} \subset \overset{o}{\mathcal{H}}^U{}_{(n,m-1)}$, we have that $\sum_{(0,0)\prec(k,l)} c_{(n,m)}(k,l) w(n-k, m-l) \in \overset{o}{\mathcal{H}}^U{}_{(n,m-1)}$. However, Theorem 2 (d) implies that for all $(0,0) \prec (k,l)$, $v(n-k, m-l) \perp \overset{o}{\mathcal{H}}^U{}_{(n,m-1)}$. Hence (21) holds if and only if for all $(0,0) \prec (k,l)$, $v(n-k,m-l) \equiv 0$. Therefore, $\hat{w}(n,m) = \sum_{(0,0)\prec(k,l)} c_{(n,m)}(k,l) w(n-k, m-l)$. This implies that $\hat{w}(n,m) \in \overset{o}{\mathcal{H}}^W{}_{(n,m-1)} \subset \overset{o}{\mathcal{H}}^W{}_{(n,m)}$. From (18), $u(n,m) = w(n,m) - \hat{w}(n,m)$ and therefore $u(n,m) \in \overset{o}{\mathcal{H}}^W{}_{(n,m)}$ for all (n,m), so that $\overset{o}{\mathcal{H}}^U{}_{(n,m)} \subset \overset{o}{\mathcal{H}}^W{}_{(n,m)}$. We finally conclude that $\overset{o}{\mathcal{H}}^W{}_{(n,m)} = \overset{o}{\mathcal{H}}^U{}_{(n,m)}$.

Since $\hat{w}(n,m) \in \overset{o}{\mathcal{H}}^W{}_{(n,m-1)}$, we conclude that $\{u(n,m)\}$ is the innovation field of $\{w(n,m)\}$ as well. Therefore if $\{y(n,m)\}$ is a regular field, then $\{w(n,m)\}$ is also a regular field. □

Corollary: $\overset{o}{\mathcal{H}}^Y{}_{(n,m)}$ has a direct sum representation

$$\overset{o}{\mathcal{H}}^Y{}_{(n,m)} = \overset{o}{\mathcal{H}}^U{}_{(n,m)} \oplus \overset{o}{\mathcal{H}}^V{}_{(n,m)} \tag{22}$$

Proof: The definition of $\{w(n,m)\}$ (8), and Theorem 2 (d) imply that $w(n,m) \perp v(s,t)$ for all (n,m) and (s,t). By Theorem 2, for all (n,m), $y(n,m)$ can be represented uniquely as $y(n,m) = w(n,m) + v(n,m)$, where $w(n,m) \in \overset{o}{\mathcal{H}}^U{}_{(n,m)}$ and $v(n,m) \in \overset{o}{\mathcal{H}}^V{}_{(n,m)}$. Since the two subspaces $\overset{o}{\mathcal{H}}^U{}_{(n,m)}$ and $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$ are orthogonal, it follows that $\overset{o}{\mathcal{H}}^Y{}_{(n,m)} = \overset{o}{\mathcal{H}}^U{}_{(n,m)} \oplus \overset{o}{\mathcal{H}}^V{}_{(n,m)}$ for all (n,m). □

Theorem 4: Let $\{y(n,m)\}$ be a 2-D regular random field. Its component $\{v(n,m)\}$ is a deterministic random field.

Proof: The direct sum representation (22), implies in particular that $\overset{o}{\mathcal{H}}^Y{}_{(n,m-1)} = \overset{o}{\mathcal{H}}^U{}_{(n,m-1)} \oplus \overset{o}{\mathcal{H}}^V{}_{(n,m-1)}$. By Theorem 2 (f), $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)} \subset \overset{o}{\mathcal{H}}^Y{}_{(n,m-1)}$. Since $v(n,m) \perp u(s,t)$ for all

(n,m) and (s,t), it follows that $v(n,m) \perp \overset{o}{\mathcal{H}^U}_{(n,m-1)}$. Finally, because $v(n,m) \in \overset{o}{\mathcal{H}^Y}_{(n,m-1)}$ and $v(n,m) \perp \overset{o}{\mathcal{H}^U}_{(n,m-1)}$, we conclude that $v(n,m) \in \overset{o}{\mathcal{H}^V}_{(n,m-1)}$, i.e., $\{v(n,m)\}$ is a deterministic random field. □

Define $\overset{o}{\mathcal{H}^Y}_{(-\infty,-\infty)} = \bigcap_{(n,m)\in\mathcal{Z}^2} \overset{o}{\mathcal{H}^Y}_{(n,m)}$. The Hilbert space $\overset{o}{\mathcal{H}^Y}_{(-\infty,-\infty)}$ is called the *remote past space w.r.t. the NSHP total-order definition o*. It is spanned by the intersection of *all* the Hilbert spaces spanned by samples of the regular field $\{y(n,m)\}$ at all (n,m), with respect to the specific order definition denoted by o.

Before we proceed to prove a much stronger result concerning the properties of the deterministic component $\{v(n,m)\}$ of the regular field, we will elaborate on the meaning of "determinism" in the framework of 2-D random fields. The following example is illustrative. Let $\{\alpha(i) | -\infty < i < \infty\}$ be an infinite two sided sequence of i.i.d. Gaussian random variables with zero mean and unit variance. Define the 2-D random field $\{y(k,l)\}$ as $y(k,l) = \alpha(k)$. It is clear that $\hat{y}(k,l) = y(k,l-1) = \alpha(k) = y(k,l)$. Therefore $u(k,l) \equiv 0$ and the field $\{y(k,l)\}$ is deterministic. On the other hand it is obvious that $y(k,l)$ is not predictable from $\overset{o}{\mathcal{H}^Y}_{(k-1,m)}$ for any m, since the Hilbert space $\overset{o}{\mathcal{H}^Y}_{(k-1,m)}$ is spanned by $\{\alpha(i) | -\infty < i \le k-1\}$ which contains no information about $\alpha(k)$. Therefore although $\{y(k,l)\}$ is a deterministic 2-D field it is not in $\overset{o}{\mathcal{H}^Y}_{(-\infty,-\infty)}$.

Let $\overset{o}{\mathcal{H}^V}_{(n,-\infty)} = \bigcap_{m=-\infty}^{\infty} \overset{o}{\mathcal{H}^V}_{(n,m)}$. In order to prove the next theorem we shall first prove the following lemma.

Lemma 1: $\overset{o}{\mathcal{H}^Y}_{(n,-\infty)} = \overset{o}{\mathcal{H}^U}_{(n,-\infty)} \oplus \overset{o}{\mathcal{H}^V}_{(n,-\infty)}$.

Proof: Let z be any vector in $\overset{o}{\mathcal{H}^Y}_{(n,-\infty)}$. Hence, $z \in \overset{o}{\mathcal{H}^Y}_{(n,m)}$ for any $m \in \mathbb{Z}$. Because $\overset{o}{\mathcal{H}^Y}_{(n,m)} = \overset{o}{\mathcal{H}^U}_{(n,m)} \oplus \overset{o}{\mathcal{H}^V}_{(n,m)}$, z can be uniquely written, for all m, in the form $z = u_n(m) + v_n(m)$, where $u_n(m) \in \overset{o}{\mathcal{H}^U}_{(n,m)}$ and $v_n(m) \in \overset{o}{\mathcal{H}^V}_{(n,m)}$. In order to prove that this unique representation of z is the same for all m, we show that for all k and l, $u_n(l) = u_n(k)$ and $v_n(l) = v_n(k)$.

Assume that $u_n(l) \neq u_n(k)$, and without any loss of generality that $l < k$. The unique representation of z implies that for both k and l, z can be uniquely represented as

$$z = u_n(k) + v_n(k) \ ; \quad u_n(k) \in \overset{o}{\mathcal{H}^U}_{(n,k)} \quad \text{and} \quad v_n(k) \in \overset{o}{\mathcal{H}^V}_{(n,k)} \tag{23}$$

$$z = u_n(l) + v_n(l) \ ; \quad u_n(l) \in \overset{o}{\mathcal{H}^U}_{(n,l)} \quad \text{and} \quad v_n(l) \in \overset{o}{\mathcal{H}^V}_{(n,l)} \ . \tag{24}$$

Since $\overset{o}{\mathcal{H}}^U{}_{(n,l)} \subset \overset{o}{\mathcal{H}}^U{}_{(n,k)}$ it follows that $u_n(l) \in \overset{o}{\mathcal{H}}^U{}_{(n,k)}$. Because by the above assumption $u_n(l) \neq u_n(k)$, $u_n(k)$ can be written as $u_n(k) = u_n(l) + x$, where $x \neq 0$ and $x \in \overset{o}{\mathcal{H}}^U{}_{(n,k)}$. We can therefore rewrite equation (23): $z = u_n(l) + x + v_n(k)$. If we compare this with equation (24), it must be that $v_n(l) = x + v_n(k)$. But this cannot hold since $x \in \overset{o}{\mathcal{H}}^U{}_{(n,k)}$, while $v_n(l)$ and $v_n(k) \perp \overset{o}{\mathcal{H}}^U{}_{(n,k)}$. Therefore, $u_n(l) = u_n(k)$ and $v_n(l) = v_n(k)$ for any k and l. Hence, we can write $z = u_n + v_n$, where $u_n = u_n(m)$ and $v_n = v_n(m)$ for all m. Also, $u_n \in \overset{o}{\mathcal{H}}^U{}_{(n,m)}$ for all m implies that $u_n \in \bigcap_{m=-\infty}^{\infty} \overset{o}{\mathcal{H}}^U{}_{(n,m)} = \overset{o}{\mathcal{H}}^U{}_{(n,-\infty)}$ and similarly $v_n \in \overset{o}{\mathcal{H}}^V{}_{(n,m)}$ for all m implies that $v_n \in \bigcap_{m=-\infty}^{\infty} \overset{o}{\mathcal{H}}^V{}_{(n,m)} = \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. Because $\overset{o}{\mathcal{H}}^U{}_{(n,-\infty)} \subset \overset{o}{\mathcal{H}}^U{}_{(n,m)}$ and $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)} \subset \overset{o}{\mathcal{H}}^V{}_{(n,m)}$, the orthogonality of $\overset{o}{\mathcal{H}}^U{}_{(n,m)}$ and $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$ implies that $\overset{o}{\mathcal{H}}^U{}_{(n,-\infty)} \perp \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. Since the unique representation of z as $z = u_n + v_n$ holds for any $z \in \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$, we conclude using the orthogonality of $\overset{o}{\mathcal{H}}^U{}_{(n,-\infty)}$ and $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$ that $\overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)} \subset \overset{o}{\mathcal{H}}^U{}_{(n,-\infty)} \oplus \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$.

On the other hand, let u' be any vector in $\overset{o}{\mathcal{H}}^U{}_{(n,-\infty)}$ and let v' be any vector in $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. Therefore, for all m, $u' \in \overset{o}{\mathcal{H}}^U{}_{(n,m)}$ and $v' \in \overset{o}{\mathcal{H}}^V{}_{(n,m)}$. Since the subspaces $\overset{o}{\mathcal{H}}^U{}_{(n,m)}$ and $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$ are orthogonal it follows that for all m, $u' + v' \in \overset{o}{\mathcal{H}}^U{}_{(n,m)} \oplus \overset{o}{\mathcal{H}}^V{}_{(n,m)} = \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$. Hence, $u' + v' \in \bigcap_{m=-\infty}^{\infty} \overset{o}{\mathcal{H}}^Y{}_{(n,m)}$, where by definition $\bigcap_{m=-\infty}^{\infty} \overset{o}{\mathcal{H}}^Y{}_{(n,m)} = \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$. Therefore, $\overset{o}{\mathcal{H}}^U{}_{(n,-\infty)} \oplus \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)} \subset \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$. □

Theorem 5: Let $\{v(n,m)\}$ be the deterministic component of a regular field. Then, $\overset{o}{\mathcal{H}}^V{}_{(n,m)} = \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$ for all m.

Proof: By the definition of $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$, it follows that $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)} \subset \overset{o}{\mathcal{H}}^V{}_{(n,m)}$ for every m. We now have to show that for every m, $\overset{o}{\mathcal{H}}^V{}_{(n,m)} \subset \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. Using Theorem 2 (f) we have $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$. It follows from Lemma 1 that $\overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)} = \overset{o}{\mathcal{H}}^U{}_{(n,-\infty)} \oplus \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. Since $v(n,m) \perp u(s,t)$ for any (s,t) we get that $v(n,m) \perp \overset{o}{\mathcal{H}}^U{}_{(n,-\infty)}$. Because $v(n,m) \in \overset{o}{\mathcal{H}}^Y{}_{(n,-\infty)}$ and $v(n,m) \perp \overset{o}{\mathcal{H}}^U{}_{(n,-\infty)}$ it must be that $v(n,m) \in \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. The same argument holds for every $k \leq m$, so that we can conclude that $v(n,k) \in \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$ for every $k \leq m$. Recall that by definition $\overset{o}{\mathcal{H}}^V{}_{(n,m)} = \overline{Sp}\,\{v(k,l) | (k,l) \preceq (n,m)\}$, and that all the elements $v(s,t)$ such that $s < n$ are both in $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$ and $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. Thus all of the vectors that span $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$ are in $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$, so $\overset{o}{\mathcal{H}}^V{}_{(n,m)} \subset \overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$. □

From Theorem 5 we can now conclude that the knowledge of the values of the deterministic component $\{v(n,m)\}$ at all points of the columns preceding the present one, which is denoted by the index n, and the knowledge of its values up to a point which is as far in the "past" as we wish in the present column, are sufficient to achieve a perfect prediction of $v(n,m)$ for any m.

By reapplying Theorem 5 to each of the columns, we have that for every $s \leq n$ and for all t, $\overset{o}{\mathcal{H}}^{V}{}_{(s,t)} = \overset{o}{\mathcal{H}}^{V}{}_{(s,-\infty)}$. We can thus extend the above observation and conclude that a perfect prediction of the (n,m)-th sample of the deterministic component is guaranteed, given the complete knowledge of $\{v(s,t)\}$ for all $s \leq s_0$, where s_0 is an arbitrarily small integer, and the values of $\{v(s,t)\}$ for all $t < t_0$, where t_0 is also an arbitrarily small integer, for s such that $s_0 < s \leq n$.

Define $\overset{o}{\mathcal{H}}^{V}{}_{(-\infty,-\infty)} = \bigcap_{(n,m)\in\mathbb{Z}^2} \overset{o}{\mathcal{H}}^{V}{}_{(n,m)}$ and $\overset{o}{\mathcal{H}}^{U}{}_{(-\infty,-\infty)} = \bigcap_{(n,m)\in\mathbb{Z}^2} \overset{o}{\mathcal{H}}^{U}{}_{(n,m)}$.

Lemma 2: $\overset{o}{\mathcal{H}}^{Y}{}_{(-\infty,-\infty)} = \overset{o}{\mathcal{H}}^{V}{}_{(-\infty,-\infty)}$.

Proof: Let z be any vector in $\overset{o}{\mathcal{H}}^{Y}{}_{(-\infty,-\infty)}$. Hence, $z \in \overset{o}{\mathcal{H}}^{Y}{}_{(n,m)}$ for all $(n,m) \in \mathbb{Z}^2$. Because $\overset{o}{\mathcal{H}}^{Y}{}_{(n,m)} = \overset{o}{\mathcal{H}}^{U}{}_{(n,m)} \oplus \overset{o}{\mathcal{H}}^{V}{}_{(n,m)}$, z can be uniquely written, for all (n,m), in the form $z = u_{nm} + v_{nm}$, where $u_{nm} \in \overset{o}{\mathcal{H}}^{U}{}_{(n,m)}$ and $v_{nm} \in \overset{o}{\mathcal{H}}^{V}{}_{(n,m)}$. In order to prove that this unique representation of z is the same for all (n,m), we show that for all (i,j) and (k,l), $u_{kl} = u_{ij}$ and $v_{kl} = v_{ij}$.

Assume that $u_{kl} \neq u_{ij}$, and without any loss of generality that $(i,j) \prec (k,l)$. The unique representation of z implies that for both (i,j) and (k,l), z can be uniquely represented as

$$z = u_{ij} + v_{ij} \;; \quad u_{ij} \in \overset{o}{\mathcal{H}}^{U}{}_{(i,j)} \quad \text{and} \quad v_{ij} \in \overset{o}{\mathcal{H}}^{V}{}_{(i,j)} \tag{25}$$

$$z = u_{kl} + v_{kl} \;; \quad u_{kl} \in \overset{o}{\mathcal{H}}^{U}{}_{(k,l)} \quad \text{and} \quad v_{kl} \in \overset{o}{\mathcal{H}}^{V}{}_{(k,l)} \;. \tag{26}$$

Since $\overset{o}{\mathcal{H}}^{U}{}_{(i,j)} \subset \overset{o}{\mathcal{H}}^{U}{}_{(k,l)}$ it follows that $u_{ij} \in \overset{o}{\mathcal{H}}^{U}{}_{(k,l)}$. Because by the above assumption $u_{ij} \neq u_{kl}$, u_{kl} can be written as $u_{kl} = u_{ij} + x$, where $x \neq 0$ and $x \in \overset{o}{\mathcal{H}}^{U}{}_{(k,l)}$. We can therefore rewrite equation (26): $z = u_{ij} + x + v_{kl}$. If we compare this with equation (25), it must be that $v_{ij} = x + v_{kl}$. But this cannot hold since $x \in \overset{o}{\mathcal{H}}^{U}{}_{(k,l)}$, while v_{ij} and $v_{kl} \perp \overset{o}{\mathcal{H}}^{U}{}_{(k,l)}$. Therefore, $u_{ij} = u_{kl}$ and $v_{ij} = v_{kl}$ for any (i,j) and (k,l). Hence, we can write $z = u + v$, where $u = u_{nm}$ and $v = v_{nm}$ for all (n,m). Also, $v \in \overset{o}{\mathcal{H}}^{V}{}_{(n,m)}$ for all (n,m) implies that $v \in \bigcap_{(n,m)\in\mathbb{Z}^2} \overset{o}{\mathcal{H}}^{V}{}_{(n,m)} = \overset{o}{\mathcal{H}}^{V}{}_{(-\infty,-\infty)}$ and similarly $u \in \overset{o}{\mathcal{H}}^{U}{}_{(n,m)}$ for all (n,m) implies that $u \in \bigcap_{(n,m)\in\mathbb{Z}^2} \overset{o}{\mathcal{H}}^{U}{}_{(n,m)} = \overset{o}{\mathcal{H}}^{U}{}_{(-\infty,-\infty)}$. However, Theorem 2 implies that $\overset{o}{\mathcal{H}}^U{}_{(-\infty,-\infty)}=\{0\}$ and hence $u \equiv 0$. Therefore, for any $z \in\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)}$, $z = v \in\overset{o}{\mathcal{H}}^V{}_{(-\infty,-\infty)}$. Hence, $\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)}\subset\overset{o}{\mathcal{H}}^V{}_{(-\infty,-\infty)}$.

On the other hand, let v' be any vector in $\overset{o}{\mathcal{H}}^V{}_{(-\infty,-\infty)}$. Therefore, for all (n,m), $v' \in\overset{o}{\mathcal{H}}^V{}_{(n,m)}\subset\overset{o}{\mathcal{H}}^Y{}_{(n,m)}$. Hence, $v' \in \bigcap_{(n,m)\in\mathbb{Z}^2} \overset{o}{\mathcal{H}}^Y{}_{(n,m)}=\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)}$. Therefore, $\overset{o}{\mathcal{H}}^V{}_{(-\infty,-\infty)}\subset\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)}$. □

$\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$ and $\overset{o}{\mathcal{H}}^V{}_{(n-1,m)}$ are subspaces of $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$, the Hilbert space spanned by the deterministic component of the regular field. Define, $\overset{o}{\mathcal{H}}^V{}_n= \overline{Sp}\left\{v | v \in\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}, v\perp \overset{o}{\mathcal{H}}^V{}_{(n-1,-\infty)}\right\}$. We can thus write $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}=\overset{o}{\mathcal{H}}^V{}_{(n-1,-\infty)} \oplus \overset{o}{\mathcal{H}}^V{}_n$.

Theorem 6:

$$\overset{o}{\mathcal{H}}^V{}_{(n,m)}=\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}=\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)} \oplus \bigoplus_{l=-\infty}^{n} \overset{o}{\mathcal{H}}^V{}_l \tag{27}$$

Proof: We first show that for all $k \neq l$, $\overset{o}{\mathcal{H}}^V{}_l \perp \overset{o}{\mathcal{H}}^V{}_k$. Assume there is a vector z such that $z \in\overset{o}{\mathcal{H}}^V{}_l$, $z \in\overset{o}{\mathcal{H}}^V{}_k$ and assume $k < l$. Since $z \in\overset{o}{\mathcal{H}}^V{}_l$, we have that $z \in\overset{o}{\mathcal{H}}^V{}_{(l,-\infty)}$ and $z\perp \overset{o}{\mathcal{H}}^V{}_{(l-1,-\infty)}$. Since by assumption $z \in\overset{o}{\mathcal{H}}^V{}_{(k,-\infty)}\subset\overset{o}{\mathcal{H}}^V{}_{(l-1,-\infty)}$, $z \equiv 0$.

Let $x \in\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)}$. Hence for all (n,m) $x \in\overset{o}{\mathcal{H}}^V{}_{(n,m)}, x \in\overset{o}{\mathcal{H}}^V{}_{(n-1,m)}$. Assume that there exists some n for which $x \in\overset{o}{\mathcal{H}}^V{}_n$ as well. By the definition of $\overset{o}{\mathcal{H}}^V{}_n$, $x\perp \overset{o}{\mathcal{H}}^V{}_{(n-1,m)}$. Hence, $x \equiv 0$. Therefore, $\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)} \perp \bigoplus_{l=-\infty}^{n} \overset{o}{\mathcal{H}}^V{}_l$. Since each of the Hilbert spaces in the right hand side of (27) is a subspace of $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$, we conclude that $\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)} \oplus \bigoplus_{l=-\infty}^{n} \overset{o}{\mathcal{H}}^V{}_l\subset\overset{o}{\mathcal{H}}^V{}_{(n,m)}$.

On the other hand, let $y \in\overset{o}{\mathcal{H}}^V{}_{(n,m)}=\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$, and assume that $y\perp \overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)} \oplus \bigoplus_{l=-\infty}^{n} \overset{o}{\mathcal{H}}^V{}_l$. By definition, $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}=\overset{o}{\mathcal{H}}^V{}_{(n-1,-\infty)} \oplus \overset{o}{\mathcal{H}}^V{}_n$. Since $y \in\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$, and is orthogonal to $\overset{o}{\mathcal{H}}^V{}_n$ by assumption, we have that $y \in\overset{o}{\mathcal{H}}^V{}_{(n-1,-\infty)}$. Repeating the above argument, we conclude that for all $k \leq n$, $y \in\overset{o}{\mathcal{H}}^V{}_{(k,-\infty)}$. Since for all $k > n$, $\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}\subset\overset{o}{\mathcal{H}}^V{}_{(k,-\infty)}$ and since by assumption $y \in\overset{o}{\mathcal{H}}^V{}_{(n,-\infty)}$, we have that for all $k > n$, $y \in\overset{o}{\mathcal{H}}^V{}_{(k,-\infty)}$. Hence,

$$y \in \bigcap_{k=-\infty}^{\infty} \overset{o}{\mathcal{H}}^V{}_{(k,-\infty)}= \bigcap_{k=-\infty}^{\infty} \left(\bigcap_{l=-\infty}^{\infty} \overset{o}{\mathcal{H}}^V{}_{(k,l)}\right) = \bigcap_{(k,l)\in\mathbb{Z}^2} \overset{o}{\mathcal{H}}^V{}_{(k,l)}=\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)} \tag{28}$$

where the last equality follows from Lemma 2. We therefore have that $y \equiv 0$. □

We thus conclude that $\overset{o}{\mathcal{H}}^Y{}_{(-\infty,-\infty)}$ is the orthogonal complement of $\bigoplus_{l=-\infty}^{n} \overset{o}{\mathcal{H}}^V{}_l$ in the Hilbert space $\overset{o}{\mathcal{H}}^V{}_{(n,m)}$ spanned by the deterministic component of the regular field. The subspace

$\bigoplus_{l=-\infty}^{n} \overset{o}{\mathcal{H}^V}_l$ is spanned by the *column to column innovations* of the deterministic field. The field $\{y(n,m)\} = \alpha(n)$ of the previous example belongs to the subspace $\bigoplus_{l=-\infty}^{n} \overset{o}{\mathcal{H}^V}_l$ and for each l, $\dim \overset{o}{\mathcal{H}^V}_l = 1$.

We shall conclude this section with the following definition, after [HL2].

Definition 4: A 2-D deterministic random field $\{e_o(n,m)\}$ is called *evanescent w.r.t. the NSHP total-order o* if it spans a Hilbert space identical to the one spanned by its *column-to-column innovations* at each coordinate (n,m) (w.r.t. the total order o).

As mentioned in Section 2, all definitions and theorems are stated w.r.t. the NSHP total-ordering definition induced by (2). In the following section we shall generalize the previously obtained results for other NSHP total-ordering definitions.

5. The Total-Order Selection

The NSHP support definition which results from the total-order definition in (2) is not the only possible one of that type on the 2-D lattice. In [KL], Korezlioglu and Loubaton define "horizontal" and "vertical" total-orders and describe the horizontally and vertically evanescent components of homogeneous random fields. Kallianpur et al. [KMN], as well as Chiang [C1], employ similar techniques to obtain four-fold orthogonal decompositions of regular and homogeneous random fields. In the following we introduce a family of NSHP total-ordering definitions in which the boundary line of the NSHP has a rational slope. Note that it is only the total-order imposed on the random field that is changed, but not the 2-D discrete grid itself.

Definition 5: Let α, β be two coprime integers, such that $\alpha \neq 0$. Let us define a new NSHP total-ordering by rotating the NSHP support which was defined with respect to (2), through a counterclockwise angle θ about the origin of its coordinate system, such that $\tan\theta = \beta/\alpha$.

Let the coordinates (n^*, m^*) be defined by

$$\begin{pmatrix} n^* \\ m^* \end{pmatrix} = \begin{pmatrix} \sqrt{\alpha^2+\beta^2} & 0 \\ 0 & 1/\sqrt{\alpha^2+\beta^2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \tag{29}$$

where (n, m) are the original coordinates, and

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad ; \quad \begin{pmatrix} \sqrt{\alpha^2+\beta^2} & 0 \\ 0 & 1/\sqrt{\alpha^2+\beta^2} \end{pmatrix}$$

are the rotation transformation matrix and the normalization matrix, respectively. The normalization matrix is such that the indices n^* of the "columns" under the new total-order definition are consecutive integers and the distance between two neighboring samples on the same "column" is one. Thus, the new coordinates $(n^{(\alpha,\beta)}, m^{(\alpha,\beta)})$ of the original point (n, m) are given by

$$\begin{aligned} n^{(\alpha,\beta)} &= n^* \\ m^{(\alpha,\beta)} &= m^* - c(n^{(\alpha,\beta)}) \,. \end{aligned} \tag{30}$$

$c(n^{(\alpha,\beta)})$ is a correction term which guarantees that $m^{(\alpha,\beta)}$ be an integer as well. For each fixed column index $n^{(\alpha,\beta)}$ of the new total-order, $c(n^{(\alpha,\beta)})$ is determined by $c(n^{(\alpha,\beta)}) = \arg \min_{(n^*,m^*)} \{|m^*|\}$, i.e ., $c(n^{(\alpha,\beta)})$ is set equal to the m^* of the least absolute value in the $n^{(\alpha,\beta)}$ column. For $\theta = \pi/2$ the transformation is obtained by interchanging the roles of columns and rows. The total-order in the rotated system is defined similarly to (2), i.e.,

$$\begin{aligned} (i^{(\alpha,\beta)}, j^{(\alpha,\beta)}) &\prec (s^{(\alpha,\beta)}, t^{(\alpha,\beta)}) \text{ iff} \\ (i^{(\alpha,\beta)}, j^{(\alpha,\beta)}) &\in \left\{(k,\ell)|k = s^{(\alpha,\beta)}, l < t^{(\alpha,\beta)}\right\} \bigcup \left\{(k,l)|k < s^{(\alpha,\beta)}, -\infty < \ell < \infty\right\} \,. \end{aligned} \tag{31}$$

Let us denote by O the above defined set of all possible NSHP total-ordering definitions on the 2-D lattice, in which the boundary line of the NSHP has a rational slope, i.e., $O = \left\{(\alpha,\beta)|\alpha,\beta \text{ are coprime integers}\right\}$. We shall call such support *rational non-symmetrical half-plane (RNSHP)*. An example is illustrated in Fig. 1. Note the way the "column" is defined. (The NSHP total-ordering $o = (1, 0)$ used in the previous sections, corresponds to $\theta = 0$).

The results proved in the previous sections are valid for *any* RNSHP total-ordering definition, since the proofs require only such a total-order definition and finiteness of the second-order moments of the random field. Note however that contrary to the homogeneous case [FMP1], the regularity and determinism properties are total-order dependent and hence a field which is regular with respect to one NSHP total-ordering definition might be deterministic with respect to another definition, as the next example shows.

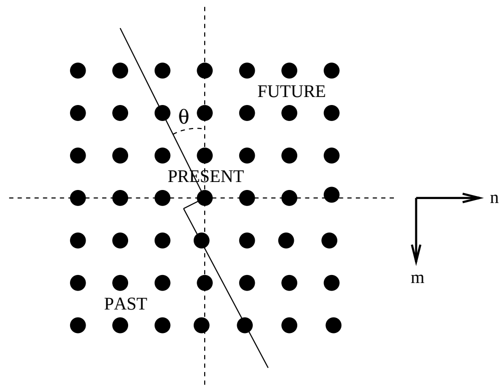

Figure 1: RNSHP total-order definition.

Let γ be a Gaussian random variable with zero mean and unit variance. Define the random field $\{y(k,l)\}$ such that $y(k,l) = 0$ for all $(k,l) \prec (0,0)$, and $y(k,l) = \gamma$ for all $(k,l) \succeq (0,0)$. Using the NSHP total-ordering definition $o = (1,0)$, we conclude that the field $\{y(k,l)\}$ is regular since for $(k,l) = (0,0)$, we have $E[u^2(0,0)] = 1 > 0$. On the other hand, if we rotate this order definition by $\theta = \pi$, i.e., we change the roles of "past" and "future", the field $\{y(k,l)\}$ is deterministic, since every vector $y(k,l)$ can be represented as a linear combination of "past" samples.

Nevertheless, Theorem 6 implies that under each total-order definition $o \in O$, at most one evanescent field can be resolved: The field that generates the column-to-column innovations of the deterministic component with respect to the order definition o. Hence, if one is interested in detecting an evanescent component of a 2-D nonhomogeneous random filed, where the evanescent component is not necessarily aligned with the "conventional" orientation of the NSHP support, an RNSHP total-ordering of the above type *must* be used.

6. Summary and Discussion

We have presented here a three-fold Wold-like decomposition for nonhomogeneous, regular 2-D discrete random fields. The presented decomposition can be obtained with respect to any RNSHP total-ordering definition. A construction in the spatial domain was used in order to prove and to discuss the properties of the regular field decomposition into purely-indeterministic, remote-past, and evanescent random fields. In [FMP1] it was shown that for regular and homogeneous random fields the decomposition into purely-indeterministic and deterministic components is NSHP-support invariant. This property results in a countably-infinite-fold decomposition of the regular field. However, since for regular nonhomogeneous random fields the decomposition into purely-indeterministic and deterministic components is not invariant to the choice of NSHP-support, the results in [FMP1], cannot be extended to the case of nonhomogeneous fields.

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A Proof of Theorem 1

Since $\overset{o}{\mathcal{H}}^Y{}_{(n,m);S_{N-1}} \subset \overset{o}{\mathcal{H}}^Y{}_{(n,m);S_N}$, and since $\overset{o}{\mathcal{H}}^Y{}_{(n,m-1)} = \overline{\bigcup_{N=0}^{\infty} \overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_N}}$, then, for every vector in $\overset{o}{\mathcal{H}}^Y{}_{(n,m-1)}$, and in particular for $\hat{y}(n,m)$, there is a sequence $\{x^{(N)}\}$ with $x^{(N)} \in \overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_N}$ such that $E[x^{(N)} - \hat{y}(n,m)]^2 \to 0$ as $N \to \infty$.

Let $E[y(n,m) - \hat{y}_{S_N}(n,m)]^2 = d^2_{(n,m);N}$ and let $E[y(n,m) - \hat{y}(n,m)]^2 = d^2_{(n,m)}$. Since $E[x^{(N)} - y(n,m)]^2 \geq d^2_{(n,m);N} \geq d^2_{(n,m)}$, we have by the triangle inequality

$$d_{(n,m)} \leq d_{(n,m);N} \leq \left(E[x^{(N)} - y(n,m)]^2\right)^{1/2} \leq \left(E[x^{(N)} - \hat{y}(n,m)]^2\right)^{1/2} + \left(E[\hat{y}(n,m) - y(n,m)]^2\right)^{1/2} \quad . \tag{32}$$

However, the right hand side of (32) tends to $d_{(n,m)}$ as $N \to \infty$, and therefore

$$\lim_{N\to\infty} E[x^{(N)} - y(n,m)]^2 = \lim_{N\to\infty} d^2_{(n,m);N} = d^2_{(n,m)} \quad . \tag{33}$$

Also,

$$E[x^{(N)} - y(n,m)]^2 = E[x^{(N)} - \hat{y}_{S_N}(n,m)]^2 + E[\hat{y}_{S_N}(n,m) - y(n,m)]^2 \tag{34}$$

since $[x^{(N)} - \hat{y}_{S_N}(n,m)] \in \overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_N}$, while $[y(n,m) - \hat{y}_{S_N}(n,m)]$ is orthogonal to every vector in $\overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_N}$. Because both $E[x^{(N)} - y(n,m)]^2$ and $E[y(n,m) - \hat{y}_{S_N}(n,m)]^2$ tend to $d^2_{(n,m)}$ as $N \to \infty$, we conclude that $E[x^{(N)} - \hat{y}_{S_N}(n,m)]^2 \to 0$ as $N \to \infty$.

However,

$$\left(E[\hat{y}(n,m)-\hat{y}_{S_N}(n,m)]^2\right)^{1/2} \leq \left(E[\hat{y}(n,m)-x^{(N)}]^2\right)^{1/2} + \left(E[x^{(N)}-\hat{y}_{S_N}(n,m)]^2\right)^{1/2} \quad . \tag{35}$$

Because the two terms on the right hand side of (35) tend to zero as $N \to \infty$, we conclude that $E[\hat{y}(n,m)-\hat{y}_{S_N}(n,m)]^2 \to 0$ as $N \to \infty$.

We now show by a similar technique that when we fix N

$$\lim_{M\to\infty} E[\hat{y}_{S_N}(n,m)-\hat{y}_{S_{N,M}}(n,m)]^2 = 0 \quad . \tag{36}$$

Let $E[y(n,m)-\hat{y}_{S_{N,M}}(n,m)]^2 = d^2_{(n,m);N}(M)$ and recall that $E[y(n,m)-\hat{y}_{S_N}(n,m)]^2 = d^2_{(n,m);N}$. Since $\overset{o}{\mathcal{H}}^Y{}_{(n,m);S_{N,M-1}} \subset \overset{o}{\mathcal{H}}^Y{}_{(n,m);S_{N,M}}$, and since $\overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_N} = \overline{\bigcup_{M=0}^{\infty} \overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_{N,M}}}$, there is a sequence $\{z^{(M)}\}$ with $z^{(M)} \in \overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_{N,M}}$ such that $E[z^{(M)}-\hat{y}_{S_N}(n,m)]^2 \to 0$ as $M \to \infty$.

By the triangle inequality

$$\left(E[z^{(M)}-y(n,m)]^2\right)^{1/2} \leq \left(E[z^{(M)}-\hat{y}_{S_N}(n,m)]^2\right)^{1/2} + \left(E[\hat{y}_{S_N}(n,m)-y(n,m)]^2\right)^{1/2} \quad . \tag{37}$$

However, $E[z^{(M)}-y(n,m)]^2 \geq d^2_{(n,m);N}(M) \geq d^2_{(n,m);N}$, while the right hand side of (37) tends to $d_{(n,m);N}$ as $M \to \infty$. Therefore

$$\lim_{M\to\infty} E[z^{(M)}-y(n,m)]^2 = \lim_{M\to\infty} d^2_{(n,m);N}(M) = d^2_{(n,m);N} \quad . \tag{38}$$

Also,

$$E[z^{(M)}-y(n,m)]^2 = E[z^{(M)}-\hat{y}_{S_{N,M}}(n,m)]^2 + E[\hat{y}_{S_{N,M}}(n,m)-y(n,m)]^2 \tag{39}$$

since $[z^{(M)}-\hat{y}_{S_{N,M}}(n,m)] \in \overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_{N,M}}$, while $[y(n,m)-\hat{y}_{S_{N,M}}(n,m)]$ is orthogonal to every vector in $\overset{o}{\mathcal{H}}^Y{}_{(n,m-1);S_{N,M}}$. Because both $E[z^{(M)}-y(n,m)]^2$ and $E[y(n,m)-\hat{y}_{S_{N,M}}(n,m)]^2$ tend to $d^2_{(n,m);N}$ as $M \to \infty$, we conclude that $E[z^{(M)}-\hat{y}_{S_{N,M}}(n,m)]^2 \to 0$ as $M \to \infty$.

However,

$$\left(E[\hat{y}_{S_N}(n,m)-\hat{y}_{S_{N,M}}(n,m)]^2\right)^{1/2} \leq \left(E[\hat{y}_{S_N}(n,m)-z^{(M)}]^2\right)^{1/2} + \left(E[z^{(M)}-\hat{y}_{S_{N,M}}(n,m)]^2\right)^{1/2} \quad . \tag{40}$$

Because the two terms on the right hand side of (40) tend to zero as $M \to \infty$, we conclude that $E[\hat{y}_{S_N}(n,m)-\hat{y}_{S_{N,M}}(n,m)]^2 \to 0$ as $M \to \infty$, for a fixed N.

Finally,

$$\left(E[\hat{y}(n,m) - \hat{y}_{S_{N,M}}(n,m)]^2\right)^{1/2} \leq \left(E[\hat{y}(n,m) - \hat{y}_{S_N}(n,m)]^2\right)^{1/2} + \left(E[\hat{y}_{S_N}(n,m) - \hat{y}_{S_{N,M}}(n,m)]^2\right)^{1/2} \quad , \tag{41}$$

and hence when we first let M tend to infinity for a fixed N, and then N tends to infinity, the theorem follows. □