Maximum Likelihood Parameter Estimation of Textures Using a Wold-Decomposition Based Model

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***Abstract*—We present a solution to the problem of modeling, parameter estimation, and synthesis of natural textures. The texture field is assumed to be a realization of a regular homogeneous random field, which can have a mixed spectral distribution. On the basis of a 2-D Wold-like decomposition, the field is represented as a sum of a purely indeterministic component, a harmonic component, and a countable number of evanescent fields. We present a maximum-likelihood solution to the joint parameter estimation problem of these components from a single observed realization of the texture field. The proposed solution is a two-stage algorithm. In the first stage, we obtain an estimate for the number of harmonic and evanescent components in the field, and a suboptimal initial estimate for the parameters of their spectral supports. In the second stage, we refine these initial estimates by iterative maximization of the likelihood function of the observed data. By introducing appropriate parameter transformations the highly nonlinear least-squares problem that results from the maximization of the likelihood function, is transformed into a separable least-squares problem. In this new problem, the solution for the unknown spectral supports of the harmonic and evanescent components reduces the problem of solving for the transformed parameters of the field to linear least squares. Solution of the transformation equations then provides a complete solution of the field-model parameter estimation problem. The Wold-based model and the resulting analysis and synthesis algorithms are seen applicable to a wide variety of texture types found in natural images. The support or shape of the analyzed texture patch may be arbitrary. Our model is very efficient in terms of the number of parameters required to represent and faithfully reconstruct the original texture.**

I. Introduction

MANY natural images can be described as a finite set of patches of uniform textures. Hence, a fundamental problem in image processing applications such as segmentation, synthesis, restoration, and coding of textured images (or of textured regions in larger images), is the modeling and estimation of textures. In parametric texture modeling the objective is to find a general model whose parameters have a meaningful interpretation in terms of the visual properties of the observed texture. Such a general model, accompanied by an appropriate parameter estimation method, can then serve as the basic building block in a variety of applications.

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In essence, there are two approaches to the analysis and synthesis of textures: structural and statistical. The structural methods describe the texture as a cellular, and to some extent, an ordered phenomenon. Therefore, texture is characterized by a description of its primitives and their placement rules [2], [3], [4], (see also [13] for the references therein). In the statistical methods, texture is described by a collection of statistics of the selected features, or by a stochastic model. Parametric stochastic modeling approaches assume that the observed texture is a finite sample from a single observed realization of a two-dimensional (2-D) random field. Chellappa and Kashyap [5], [6], suggested the use of a 2-D autoregressive, noncausal model for synthesizing textures that visually resemble natural textures. Due to its properties, this type of model is more suitable for parameterizing fine (purely random) textures, than for structured ones. Causal Gaussian AR texture models were successfully used in [7], as well as in [8], for unsupervised texture segmentation algorithms. Both works use Markov random fields (MRF) to model the texture labeling process. Cohen and Cooper [9], and Geman and Graffine [10], also used a two-tiered MRF model in order to perform supervised segmentation of natural textures, and obtained accurate segmentation results. However, even though the estimated parameters allowed for a successful segmentation, samples generated using the estimated model did not produce a close replica of the original texture, [10]. A similar conclusion was recently obtained in [11], where the SAR model of [5], [6], was used as the texture model in a two-tiered segmentation scheme. Gaussian MRF (GMRF) texture models were applied in an unsupervised texture segmentation procedure [12], and in a classification algorithm of rotated and scaled textured images [13]. Most of the above stochastic models perform reasonably well in synthesizing the original texture from the estimated parameters as long as the original texture is purely random (microtextures). However, they perform poorly when the texture becomes more structured.

In [14] we have presented a 2-D Wold-like decomposition for homogeneous random fields. The texture model suggested in the present paper is based on this decomposition. The texture

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field is decomposed into a sum of two mutually orthogonal components: a *deterministic* component which results in the structural attributes of the observed realization, and a *purely indeterministic* component which is the structureless, "random looking" component of the texture field. The deterministic field is further orthogonally decomposed into a harmonic component and a countable number of evanescent components. The harmonic field results in the periodic attributes of the texture, whereas the evanescent components result in directional ones. Using the decomposition results, the problem of estimating the texture model parameters becomes that of simultaneously estimating the parameters of the harmonic and evanescent components of the observed realization of the texture, in the presence of an unknown colored noise generated by the purely-indeterministic component, jointly with estimating the purely-indeterministic component parameters. Since the proposed texture model is based on the Wold decomposition results, it enables a rigorous mathematical treatment of the texture modeling and parameter estimation problems. Moreover, due to the generality of the decomposition, the model is not tailored to any specific type of texture.

In the proposed model, and in the derived analysis/synthesis method, the texture field is assumed to be a realization of a 2-D homogeneous random field. Hence, the problem of estimating the texture parameters is strongly related to the more general problem of fitting a parametric model to a finite observed sample from *a single realization* of a 2-D homogeneous random field. This homogeneous random field is characterized in general, by a mixed spectral distribution (the singular component of the spectral distribution is due to the deterministic component of the field, while the absolutely continuous component of the distribution is associated with the purely indeterministic component of the field). Until recently [16], this estimation problem has been largely unsolved. Existing methods either assume the field has an absolutely continuous spectral distribution and try to fit noise driven linear models to the observed field, or treat the special case of estimating the parameters of a sinusoidal signal in white noise. The existence of evanescent random fields has not received any attention in the estimation literature, although the evanescent components have major impact on the structure and properties of the random field, as they result in directional attributes in the observed realizations. Among the noise driven models we find the 2-D AR and ARMA models, as well as the Gauss-Markov random field model. As mentioned earlier, these types of models were also suggested for texture modeling applications. However, both white- and correlated-noise driven linear models can only produce fields with absolutely continuous spectral distribution functions. Hence, they cannot be applied to the general case in which the spectral distribution of the observed field contains singular components as well. Since noise-driven models cannot model the singular components of the spectral distribution, and since the singular components result in the structural features of the texture, nois-driven models can be successful only in modeling random looking textures, and fail to model structured ones, as was previously concluded based on experimental results, [10], [11].

In [16], we have presented a maximum likelihood solution to the problem of fitting a parametric model to a finite observed sample from a single realization of a 2-D homogeneous random field, with mixed spectral distribution. This solution is based on the theoretical results of the 2-D Wold decomposition. In the present paper, we employ the same theoretical framework to formulate the texture modeling problem, and extend the algorithm suggested in [16] so that it can be used for estimating the texture model parameters. The parametric modeling and estimation approach suggested in the present paper provides a unifying framework for many applications in which texture analysis is involved, together with the optimality properties of an ML estimation algorithm. Since different textures have different models, such a model can be used for image segmentation, for texture classification, for model based adaptive restoration of textured images, etc. In the present paper, the estimated parametric model is applied to synthesize textures which are statistically and visually similar to the observed ones. The synthesis procedure uses *only* the estimated parameters.

This paper is organized as follows: In Section II, we present the texture model. The model is based on the results of the 2-D Wold decomposition. Section III presents an overview on the maximum likelihood estimation algorithm for the parameters of the harmonic, evanescent and purely indeterministic components of the texture. In Section IV, we further elaborate on the estimation problem of the evanescent components, and present a complete estimation algorithm for these components. In Section V, we present and analyze the texture estimation and synthesis results obtained by applying the proposed method to a number of natural textures. We compare the results with those obtained by other models and methods.

II. The Wold-Based Texture Model

We derive our texture model based on the results of the Wold-type decomposition of 2-D regular and homogeneous random fields, [14]. Let $\{y(n,m),(n,m) \in \mathcal{Z}^2\}$ be a real valued, regular, and homogeneous random field. Then $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + v(n,m) \quad . \tag{1}$$

The field $\{w(n,m)\}$ is purely indeterministic and has a unique white innovations driven moving average representation. The field $\{v(n,m)\}$ is a deterministic random field. It can be shown that it is possible to define a family of nonsymmetric half-plane (NSHP) total-order definitions such that the boundary line of the NSHP is of rational slope. Let α, β be two coprime integers, such that $\alpha \neq 0$. The angle θ of the slope is given by $\tan\ \theta = \beta/\alpha$ (see, for example, Fig. 1). Each of these supports is called rational nonsymmetrical half-plane (RNSHP). We denote by O the set of all possible RNSHP definitions on the 2-D lattice, (i.e., the set of all NSHP definitions in which the boundary line of the NSHP is of rational slope). The introduction of the family of RNSHP

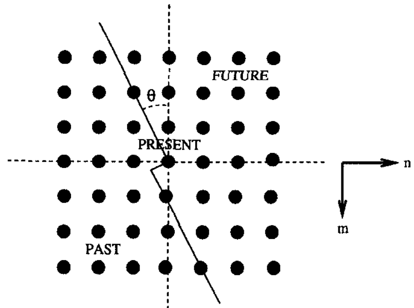

Fig. 1. RNSHP support.

total-ordering definitions results in the following countably infinite orthogonal decomposition of the field's deterministic component, [14]

$$v(n,m) = p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m)\ . \tag{2}$$

The random field $\{p(n,m)\}$ is a *half-plane deterministic* field, and $\{e_{(\alpha,\beta)}(n,m)\}$ is the evanescent field corresponding to the RNSHP total-ordering definition $(\alpha,\beta)\in O$.

Hence, if $\{y(n,m)\}$ is a 2-D regular and homogeneous random field, then $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m)\ . \tag{3}$$

In this paper, all spectral measures are defined on the square region $K = [-1/2, 1/2] \times [-1/2, 1/2]$. It is shown in [14] that the spectral measures of the decomposition components in (3) are mutually singular. The spectral distribution function of the purely-indeterministic component is absolutely continuous, while the spectral measures of the half-plane deterministic component and all the evanescent components are concentrated on a set L of Lebesgue measure zero in K. Since for practical applications we can exclude singular-continuous spectral distribution functions from the framework of our treatment, a model for the evanescent field which corresponds to the RNSHP defined by $(\alpha,\beta)\in O$ is given by

$$\begin{aligned} &e_{(\alpha,\beta)}(n,m) \\ &= \sum_{i=1}^{I^{(\alpha,\beta)}} s_i^{(\alpha,\beta)}(n\alpha - m\beta)\cos\left(2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta+m\alpha)\right) \\ &\quad + t_i^{(\alpha,\beta)}(n\alpha - m\beta)\sin\left(2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta+m\alpha)\right) \end{aligned} \tag{4}$$

where the 1-D purely-indeterministic processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$, $\{s_j^{(\alpha,\beta)}(n\alpha - m\beta)\}$, $\{t_k^{(\alpha,\beta)}(n\alpha - m\beta)\}$, $\{t_\ell^{(\alpha,\beta)}(n\alpha - m\beta)\}$ are mutually orthogonal for all $i,j,k,\ell, i\neq j, k\neq \ell$, and for all i the processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ and $\{t_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ have an identical autocorrelation function. Hence, the "spectral density function" of each evanescent field has the form of a countable sum of 1-D delta functions which are supported on lines of rational slope in the 2-D spectral domain.

Let $n^{(\alpha,\beta)} = n\alpha - m\beta$. In the following, we assume that the modulating 1-D processes $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ of each evanescent field can be modeled by a finite order AR model, i.e.,

$$\begin{aligned} s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = &-\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau) s_i^{(\alpha,\beta)}[n^{(\alpha,\beta)} - \tau] \\ &+ \xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) \end{aligned} \tag{5}$$

and

$$\begin{aligned} t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = &-\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau) t_i^{(\alpha,\beta)}[n^{(\alpha,\beta)} - \tau] \\ &+ \zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) \end{aligned} \tag{6}$$

where $\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})$, $\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})$ are independent 1-D white innovation processes of identical variance, $(\sigma_i^{(\alpha,\beta)})^2$.

One of the half-plane-deterministic field components, which is often found in natural textures, is the harmonic random field

$$\begin{aligned} &h(n,m) \\ &= \sum_{p=1}^{P}\Big(C_p\cos 2\pi(n\omega_p + m\nu_p) + D_p \sin 2\pi(n\omega_p + m\nu_p)\Big) \end{aligned} \tag{7}$$

where the C_p's and D_p's are mutually orthogonal random variables, $E[C_p]^2 = E[D_p]^2 = \sigma_p^2$, and (ω_p,ν_p) are the spatial frequencies of the pth harmonic. In general, P is infinite. This component generates the 2-D delta functions of the "spectral density" (the 2-D delta functions are singular functions supported on discrete points in the frequency plane). The parametric modeling of deterministic random fields whose spectral measures are concentrated on curves other than lines of rational slope, or discrete points in the frequency plane, is still an open question to the best of our knowledge. Since such components seem to be of very little practical importance for the texture modeling problem, we assume that the half-plane deterministic field consists only of the harmonic random field.

As stated earlier, the most general model for the purely indeterministic component $w(n,m)$ is the MA model. However, if its spectral density function is strictly positive on the unit bicircle and analytic in some neighborhood of it, a 2-D AR representation for the purely indeterministic field exists as well [15]. In the following, we assume that the above requirements are satisfied. Hence, the purely indeterministic component's *autoregressive model* is given by

$$w(n,m) = -\sum_{(0,0)\prec(k,\ell)} b(k,\ell)w(n-k,m-\ell) + u(n,m) \tag{8}$$

where $\{u(n,m)\}$ is the 2-D white innovations field, whose variance is σ^2. In the practical estimation problem, the model support is assumed finite.

Hence, the observed texture field $\{y(n,m)\}$ is uniquely represented by the orthogonal decomposition $y(n,m) =$

$w(n,m) + h(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m)$. Thus, the problem of estimating the texture model parameters, becomes one of estimating the parameters of the harmonic and evanescent components of the field in the presence of an unknown colored noise generated by the purely-indeterministic component, jointly with estimating the purely-indeterministic component parameters.

III. The ML Estimation Algorithm

The texture parameter estimation algorithm (which we now present) is a two-stage procedure for simultaneously estimating the parameters of all the texture components (purely indeterministic, harmonic, and evanescent) using a finite sample from a single observed realization of the texture field. In the first stage, we obtain an estimate for the number of harmonic and evanescent components in the observed field, and a suboptimal initial estimate for the parameters of their spectral supports. This first stage is implemented by solving the set of 2-D overdetermined normal equations for the parameters of an approximate high-order linear prediction model of the observed texture field. In the second stage of our joint-parameter estimation algorithm, we refine these initial estimates by iterative maximization of the likelihood function of the observed data. This proceeds via the minimization of an objective function which is expressed in terms of only the parameters of the spectral supports of the evanescent and harmonic components, rather than the entire set of parameters of the original problem. This significant reduction in the complexity of the original problem is made possible by a set of suitable parameter transformations. These parameter transformations transform the highly nonlinear least-squares problem that results from the maximization of the likelihood function into a separable least-squares problem. In this new problem, the solution for the unknown spectral supports of the harmonic and evanescent components reduces the problem of solving for the transformed parameters of the field to linear least squares. Solution of the transformation equations provides a complete solution of the field model parameter estimation problem.

Next, we briefly summarize the suggested algorithm and introduce some necessary notations and definitions. A detailed description of the algorithm can be found in [16].

When expressed in the general form (7), the coefficients $\{C_p, D_p\}$ of the harmonic component are real valued, mutually orthogonal random variables. However, since in general, only a single realization of the random field is observed we cannot infer anything about the variation of these coefficients over different realizations. The best we can do is to estimate the particular values which the C_p's and D_p's take for the given realization; in other words we might just as well treat the C_p's and D_p's as unknown constants. Note that in applications like image coding, the estimation of the values that the deterministic components assume in the given texture realization is essential, in order to preserve the exact structural properties of the observed image.

The question of estimating the number of harmonic and evanescent components is discussed later in this section. Let us assume, for now, that the number P of harmonic components, the number of the different evanescent fields, the orders $V_i^{(\alpha,\beta)}$ of the AR models of the 1-D purely-indeterministic processes $\{s_i^{(\alpha,\beta)}\}$, $\{t_i^{(\alpha,\beta)}\}$, and $S_{N,M}$, which is the support of the 2-D NSHP AR model, are all known and finite. Hence, $w(n,m)$ is given by (8) with $(k,\ell) \in S_{N,M}\backslash\{(0,0)\}$, where $S_{N,M} = \{(i,j)|i=0, 0\le j\le M\}\bigcup\{(i,j)|1\le i\le N, -M\le j\le M\}$.

Let $\{y(n,m),(n,m)\in D\}$ be the observed random field. The shape of D may be *arbitrary*. The parameters to be estimated are $\{C_p, D_p, \omega_p, \nu_p\}_{p=1}^P$, $\{b(k,\ell)\}_{(k,\ell)\in S_{N,M}}$, σ^2, and (α,β), $\nu_i^{(\alpha,\beta)}$, $\{a_i^{(\alpha,\beta)}(\tau)\}_{\tau=1}^{V_i^{(\alpha,\beta)}}$, $(\sigma_i^{(\alpha,\beta)})^2$ for all (α,β) pairs and for all $i=1,\ldots,I^{(\alpha,\beta)}$. We denote this vector of unknown parameters by $\boldsymbol{\theta}$.

In the proposed algorithm we take the approach of first estimating a *nonparametric* representation of the 1-D purely-indeterministic processes $\{s_i^{(\alpha,\beta)}\}$, $\{t_i^{(\alpha,\beta)}\}$, and only in a second stage the AR models of these processes are estimated. Hence, in the first stage we estimate the particular values which the processes take for the given realization, i.e., we treat these as unknown constants.

Assuming that the white innovations field $\{u(n,m)\}$ is Gaussian with unknown variance σ^2, we have,

$$p(Y;\boldsymbol{\theta},D\setminus D_1)=\frac{1}{(\sqrt{2\pi}\sigma)^{|D_1|}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{(n,m)\in D_1}u^2(n,m)\right\}. \tag{8}$$

The conditional maximum likelihood estimate of $\boldsymbol{\theta}$ is found by maximizing (9), or equivalently by minimizing

$$J=\sum_{(n,m)\in D_1}u^2(n,m) \tag{9}$$

where D_1 is the *interior* of D, and $D\backslash D_1$ is the set of required initial conditions. Thus, only actually occurring values of the observed field are used in the estimation procedure. Using this method we sum the squares of only $|D_1|$ values of $u(n,m)$, but this slight loss of information will be unimportant if the size of the observed field, $|D|$, is large enough.

Let $S'_{N,M} \triangleq S_{N,M}\backslash\{(0,0)\}$, $\gamma \triangleq \frac{\alpha}{\alpha^2+\beta^2}$, $\delta \triangleq \frac{\beta}{\alpha^2+\beta^2}$. Using (8), $u(n,m)$ is given by $u(n,m) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)w(n-k,m-\ell)$ with $b(0,0)=1$. Since $w = y - h - \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}$, we can express $u(n,m)$ in the form [16],

$$\begin{aligned} u(n,m) &= y(n,m) \\ &+ \sum_{(k,\ell)\in S'_{N,M}} b(k,\ell)y(n-k,m-\ell) \\ &- \sum_{p=1}^{P}\mu_p^1\cos 2\pi(n\omega_p+m\nu_p) - \sum_{p=1}^{P}\mu_p^2\sin 2\pi(n\omega_p+m\nu_p) \\ &- \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}}\Big\{\eta_i^1(n^{(\alpha,\beta)})\cos 2\pi\nu_i^{(\alpha,\beta)}(n\delta+m\gamma) \\ &+ \eta_i^2(n^{(\alpha,\beta)})\sin 2\pi\nu_i^{(\alpha,\beta)}(n\delta+m\gamma)\Big\}\quad (n,m)\in D_1 \end{aligned} \tag{10}$$

where we define the following systems of transformations (see bottom of the next page):

Note that $J = \sum_{(n,m)\in D_1} u^2(n,m) = \mathbf{u}^T\mathbf{u}$, where $\mathbf{u}$is a column vector whose elements are the samples $u(n,m)$, $(n,m) \in D_1$. By writing (10) for all $(n,m) \in D_1$, we obtain the following matrix representation for J:

$$J = \Bigg|\Bigg|\mathbf{y} - \mathbf{Y}\mathbf{b} - \mathbf{E}_h^R\boldsymbol{\mu}^1 - \mathbf{E}_h^I\boldsymbol{\mu}^2 - \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}}\left\{\mathbf{E}^R_{e_{(\alpha,\beta)_i}}\boldsymbol{\eta}^1_{(\alpha,\beta)_i} + \mathbf{E}^I_{e_{(\alpha,\beta)_i}}\boldsymbol{\eta}^2_{(\alpha,\beta)_i}\right\}\Bigg|\Bigg|^2. \tag{15}$$

The vector $\boldsymbol{\eta}^j_{(\alpha,\beta)_i}$, $j = 1,2$, comprises the set of $\eta_i^j(n^{(\alpha,\beta)})$'s for $n^{(\alpha,\beta)} = n\alpha - m\beta$ such that $(n,m) \in D_1$. Hence, the structure of the matrices $\mathbf{E}^R_{e_{(\alpha,\beta)_i}}$ and $\mathbf{E}^I_{e_{(\alpha,\beta)_i}}$ of the cosine and sine terms of the evanescent fields, respectively, is determined by the structure of $\boldsymbol{\eta}^j_{(\alpha,\beta)_i}$. The remaining parameter vectors are defined as follows: $\boldsymbol{\mu}^j = \left[\mu_1^j, \mu_2^j \dots \mu_P^j\right]^T$, $j = 1,2$ and

$$\mathbf{b} = -[b(0,1),\dots,b(0,M),b(1,-M),\dots,b(1,M),\dots, b(N,-M),\dots,b(N,M)]^T.$$

$\mathbf{y}$ is a column vector whose elements are the values of the observed field $y(n,m)$ for all $(n,m) \in D_1$. $\mathbf{Y}$ is a data matrix whose elements are the given observed values of the field. The matrices $\mathbf{E}_h^R$ and $\mathbf{E}_h^I$, contain the cosine and sine terms of the harmonic field.

Let $\mathbf{E}_e^R$ be the matrix obtained by concatenating all matrices $\mathbf{E}^R_{e_{(\alpha,\beta)_i}}$, for all $(\alpha,\beta) \in O$, and for all $i = 1,\dots,I^{(\alpha,\beta)}$. Correspondingly, let $\boldsymbol{\eta}^1$ be the column vector obtained by stacking all column vectors $\boldsymbol{\eta}^1_{(\alpha,\beta)_i}$ for all $(\alpha,\beta) \in O$, and for all $i = 1,\dots,I^{(\alpha,\beta)}$. The matrix $\mathbf{E}_e^I$, and the vector $\boldsymbol{\eta}^2$ are defined in a similar way.

Define $\mathbf{D} \triangleq [\mathbf{Y}\mathbf{E}_h^R\mathbf{E}_h^I\mathbf{E}_e^R\mathbf{E}_e^I]$ and

$$\boldsymbol{\theta}_1 \triangleq [\mathbf{b}^T(\boldsymbol{\mu}^1)^T(\boldsymbol{\mu}^2)^T(\boldsymbol{\eta}^1)^T(\boldsymbol{\eta}^2)^T]^T.$$

Then we can rewrite (15) as

$$J(\boldsymbol{\theta}) = ||\mathbf{y} - \mathbf{D}\boldsymbol{\theta}_1||^2. \tag{16}$$

Because the objective function is a quadratic function of $\boldsymbol{\theta}_1$, the minimization over $\boldsymbol{\theta}_1$ can be carried out analytically for any given value of $\mathbf{D}$. Using the well known solution to the least-squares problem we have that

$$\hat{\boldsymbol{\theta}_1} = \left(\mathbf{D}^T\mathbf{D}\right)^{-1}\mathbf{D}^T\mathbf{y} \tag{17}$$

will minimize $J(\boldsymbol{\theta})$ over $\boldsymbol{\theta}_1$. By inserting (17) into (16) we find that the minimum value of $J(\boldsymbol{\theta})$ is given by

$$J_{min} = \mathbf{y}^T(\mathbf{I} - \mathbf{D}(\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T)\mathbf{y}. \tag{18}$$

Here, $\mathbf{D}$ is assumed to be full rank so that $(\mathbf{D}^T\mathbf{D})^{-1}$ exists.

Thus, maximization of the likelihood function is achieved by minimizing the new objective function J_{min}, which is a function only of the deterministic component spectral support parameters. We have thus shown that the minimization problem (15) which is obtained after taking the transformations (12)–(14) is *separable* since its solution can be reduced to a minimization problem in the nonlinear deterministic component's spectral support parameters, $\{\omega_p,\nu_p\}_{p=1}^P$, the $(\alpha,\beta) \in O$ and the $\{\nu_i^{(\alpha,\beta)}\}_{i=1}^{I^{(\alpha,\beta)}}$ for each (α,β), *only*, while $\mathbf{b},\boldsymbol{\mu}^1,\boldsymbol{\mu}^2,\boldsymbol{\eta}^1,\boldsymbol{\eta}^2$ can then be determined by solving a *linear* least-squares problem. This new minimization problem is of a considerably lower complexity, as we end up minimizing a function of only the spectral support parameters of the harmonic and evanescent components, instead of minimizing an objective function over the high dimensional parameter

$$\mu_p^1 \triangleq C_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\cos 2\pi(k\omega_p + \ell\nu_p) - D_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\sin 2\pi(k\omega_p + \ell\nu_p) \tag{11}$$

$$\mu_p^2 \triangleq C_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\sin 2\pi(k\omega_p + \ell\nu_p) + D_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\cos 2\pi(k\omega_p + \ell\nu_p) \tag{12}$$

and

$$\eta_i^1(n^{(\alpha,\beta)}) \triangleq \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)s_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta]\cos 2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha) - \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)t_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta]\sin 2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha) \tag{13}$$

$$\eta_i^2(n^{(\alpha,\beta)}) \triangleq \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)s_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta]\sin 2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha) + \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)t_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta]\cos 2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha) \tag{14}$$

space of the original problem. Since J_{min} is a nonlinear function of the spectral support parameters of the harmonic and evanescent components, this optimization problem cannot be solved analytically and we must resort to numerical methods. In order to avoid the enormous computational burden of an exhaustive search, we use a two-step procedure.

The first step in solving the presented estimation problem is the minimization of J_{min} with respect to the spectral support parameters of the harmonic and evanescent components. However, we have first to determine the number of harmonic and evanescent components in the given texture field. Order selection criteria like minimum description length (MDL) require the minimization of the log-likelihood function for all possible combinations of the numbers of harmonic and evanescent components and orders of the 2-D AR model NSHP support. Since this approach is computationally very expensive, we have adopted a suboptimal approach which is based on solving the set of 2-D overdetermined normal equations for the parameters of an approximate high-order linear prediction model of the observed texture field. This is followed by a search for the peaks of the magnitude of the predictor transfer function inverse. The harmonic components result in isolated peaks, while the evanescent components result in peaks that form continuous lines. Hence, this method allows us to obtain an estimate for the number of harmonic and evanescent components, as well as initial estimates for the values of the deterministic components spectral support parameters.

These estimates are used to initialize the iterative numerical minimization of J_{min}. Our minimization approach is based on the conjugate gradient method of Fletcher and Reeves [18]. Note that in general, the procedure could result in local minima. As a result, the choice of an initial guess of the problem parameters is of prime importance. A good initial starting point can alleviate the problems associated with local minima, and considerably reduce the computational burden.

For our problem, the vector of the deterministic components' spectral support parameters comprises the set of the harmonic component frequencies given by $\{(\omega_p,\nu_p)\}_{p=1}^{P}$ along with the set of the evanescent components' spectral support parameters: The α,β 's (which are expressed as γ,δ 's) and the frequencies $\{\nu_i^{(\alpha,\beta)}\}_{i=1}^{I^{(\alpha,\beta)}}$, for all (α,β). We then seek the minimum of J_{min} with respect to these parameters. The algorithm is summarized in Table I. In the algorithm, C is some predetermined constant which guarantees that we consider only RNSHP definitions for which there is a "sufficiently" large number of samples per column (row), and ϵ is a small predetermined constant. The details of the algorithm derivation can be found in [16]. The solution for the unknown spectral supports of the harmonic and evanescent components reduces the problem of solving for the transformed parameters in (15) to linear least squares. Using the estimated parameters, we can now return to the parameter transformation (11)–(12) to obtain estimates for the amplitude parameters C_p, D_p of each harmonic component. The estimates are obtained by solving the simultaneous equations (11)–(12) for each p. The solution for the parameters of the 1-D modulating purely-indeterministic processes associated with each evanescent field is given next.

IV. Estimation of the Evanescent Field Parameters

The algorithm presented in the previous section provides estimates of the evanescent components' spectral support parameters and of the $\eta_i^1(n^{(\alpha,\beta)})$, $\eta_i^2(n^{(\alpha,\beta)})$, for $i=1,\ldots,I^{(\alpha,\beta)}$, $(\alpha,\beta)\in O$. In this section, we present an algorithm for estimating the parameters of the 1-D modulating sequences of each evanescent field, using the above set of estimated parameters.

Let $k^{(\alpha,\beta)}=k\alpha-\ell\beta$, $\ell^{(\alpha,\beta)}=\frac{k\beta+\ell\alpha}{\alpha^2+\beta^2}$. Let us also denote by $b^{(\alpha,\beta)}(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})$ the coefficient $b(k,\ell)$ for $(k,\ell)\in S_{N,M}$, under the total-order definition $(\alpha,\beta)\in O$. Define

$$G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)}) \triangleq \sum_{\ell^{(\alpha,\beta)}} b^{(\alpha,\beta)}(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})\ \cos 2\pi\nu_i^{(\alpha,\beta)}\ell^{(\alpha,\beta)} \tag{19}$$

$$H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)}) \triangleq \sum_{\ell^{(\alpha,\beta)}} b^{(\alpha,\beta)}(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})\ \sin 2\pi\nu_i^{(\alpha,\beta)}\ell^{(\alpha,\beta)} \tag{20}$$

where the summation w.r.t. $\ell^{(\alpha,\beta)}$ is taken over all pairs $(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})$ which result from the mapping of $(k,\ell)\in S_{N,M}$ by the above transformation, while holding $k^{(\alpha,\beta)}$ fixed. By substituting (19)–(20) into (13)–(14), and substituting the AR models (5), (6) of $\{s_i^{(\alpha,\beta)}[n^{(\alpha,\beta)}]\}$ and $\{t_i^{(\alpha,\beta)}[n^{(\alpha,\beta)}]\}$, respectively, into the resulting equations, we obtain

$$\begin{aligned}\eta_i^1(n^{(\alpha,\beta)}) &= -\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)\eta_i^1(n^{(\alpha,\beta)}-\tau)\\ &+\sum_{k^{(\alpha,\beta)}}\Big[G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})\\ &-H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})\Big]\end{aligned} \tag{21}$$

and

$$\begin{aligned}\eta_i^2(n^{(\alpha,\beta)}) &= -\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)\eta_i^2(n^{(\alpha,\beta)}-\tau)\\ &+\sum_{k^{(\alpha,\beta)}}\Big[H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})\\ &+G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})\Big]\end{aligned} \tag{22}$$

where the summations are taken over all $k^{(\alpha,\beta)}$ such that $(k,\ell)\in S_{N,M}$.

Hence, (21) and (22) imply that solving the problem of estimating the unknown parameters of the 1-D purely indeterministic processes associated with each evanescent component is equivalent to solving the foregoing one-dimensional two-channel ARMA problem, where the $\{G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\}$ and $\{H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\}$ have previously been estimated. The "observations" sequences are the $\{\eta_i^1(n^{(\alpha,\beta)})\}$, and $\{\eta_i^2(n^{(\alpha,\beta)})\}$.

This two-channel ARMA problem can be solved using any standard estimation procedure for vector ARMA processes,

TABLE I
THE ESTIMATION ALGORITHM.

0. Let Q be the total number of evanescent components in the field. Let $\mathbf{x} = \left\{ \{(\omega_p, \nu_p)\}_{p=1}^{P}, \{\gamma_q, \delta_q, \nu_q\}_{q=1}^{Q} \right\}$.
1. Find the minimum of J_{min} with respect to $\mathbf{x}$.
2. If for some evanescent component(s) $\hat{\delta}_q << \epsilon$ then for these component(s) $\hat{\beta}_q = 0, \hat{\alpha}_q = 1$.
3. If for some evanescent component(s) $\hat{\gamma}_q << \epsilon$ then for these component(s) $\hat{\beta}_q = 1, \hat{\alpha}_q = 0$.
4. For each one of the remaining evanescent components, find all coprime integer pairs (k_q, ℓ_q) such that $0 < |k_q|, |\ell_q| < min(S,T)/C$ for which $(\hat{\delta}_q/\hat{\gamma}_q) - \epsilon < \ell_q/k_q < (\hat{\delta}_q/\hat{\gamma}_q) + \epsilon$. For all q's for which only a single pair results, set $(\hat{\alpha}_q, \hat{\beta}_q) = (k_q, \ell_q)$.
5. If for one (or more) q's, more than one pair (k_q, ℓ_q) results from step 4, then
   a) For each resolved evanescent component, set $\gamma_q = \frac{\hat{\alpha}_q}{\hat{\alpha}_q^2 + \hat{\beta}_q^2}, \delta_q = \frac{\hat{\beta}_q}{\hat{\alpha}_q^2 + \hat{\beta}_q^2}$.
   b) For *each* possible combination of (k_q, ℓ_q)'s, where each (k_q, ℓ_q) is associated with a different unresolved evanescent component: Set for each unresolved evanescent component $(\gamma_q, \delta_q) = (\frac{k_q}{k_q^2+\ell_q^2}, \frac{\ell_q}{k_q^2+\ell_q^2})$. Minimize J_{min} w.r.t. the remaining unknown parameters.
6. For each unresolved q from step 5, set $(\hat{\alpha}_q, \hat{\beta}_q) = (k_q, \ell_q)$ where (k_q, ℓ_q) is the pair for which the minimal value of J_{min} was achieved.
7. Set the $\{(\hat{\omega}_p, \hat{\nu}_p)\}_{p=1}^{P}$ and $\{\hat{\nu}_q\}_{q=1}^{Q}$, to their values obtained by the minimization procedure for which the minimal value of J_{min} was achieved.

like the modified Yule-Walker method of [17], or the maximum likelihood estimator (if we further assume the innovation processes $\{\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ to be Gaussian). Note however that the AR parameters are identical in both channels, and the regression part of (21) is a function of $\eta_i^1(n^{(\alpha,\beta)})$ only, while the regression part of (22) is a function of $\eta_i^2(n^{(\alpha,\beta)})$ only. Hence, the estimation procedure can be significantly simplified. Applying the standard single-channel modified Yule-Walker method to the ARMA equation in (21), we obtain estimates of the $\{a_i^{(\alpha,\beta)}(\tau)\}_{\tau=1}^{V_i^{(\alpha,\beta)}}$. Let $\mathcal{A}(z) = \sum_{\tau=0}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau) z^{-\tau}$, $a_i^{(\alpha,\beta)}(0) = 1$. Once the AR parameters have been estimated, the sequences $\{\eta_i^1(n^{(\alpha,\beta)})\}$ and $\{\eta_i^2(n^{(\alpha,\beta)})\}$ are filtered by the estimated filter $\mathcal{A}(z)$ to produce approximate MA processes. We denote these sequences by $\{\epsilon_i^1(n^{(\alpha,\beta)})\}$ and $\{\epsilon_i^2(n^{(\alpha,\beta)})\}$, respectively. Thus, we have (23) and (24), shown at the bottom of this page, where all the parameters in the right hand side of (23) and in the right hand side of (24) have previously been estimated, except the variances of $\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})$ and $\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})$.

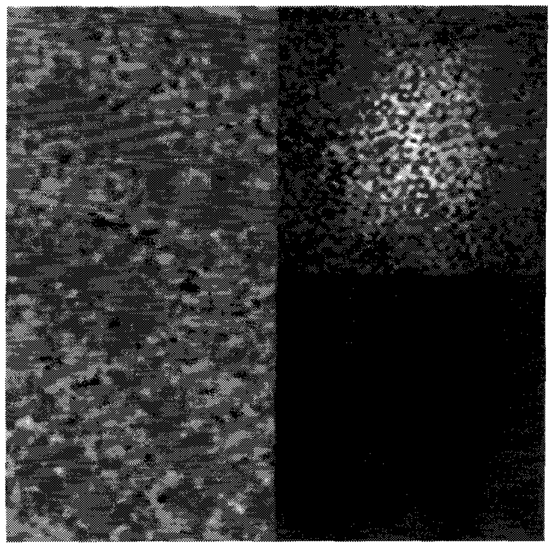

Fig. 2. Top left: Original texture. Top right: Periodogram. Bottom left: Reconstruction (purely indeterministic).

Since the sequences $\{\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$, $\{\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ are independent 1-D white innovation processes of identical variance, $(\sigma_i^{(\alpha,\beta)})^2$, we obtain by multiplying (23) by $\epsilon_i^1(n^{(\alpha,\beta)})$ and taking the expected value of both sides

$$\begin{aligned} & E\left[\epsilon_i^1(n^{(\alpha,\beta)})\epsilon_i^1(n^{(\alpha,\beta)})\right] \\ & \approx E\bigg\{ \sum_{k^{(\alpha,\beta)}} \Big[G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - k^{(\alpha,\beta)}) \\ & - H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - k^{(\alpha,\beta)}) \Big] \bigg\}^2 \\ & = \left(\sigma_i^{(\alpha,\beta)}\right)^2 \sum_{k^{(\alpha,\beta)}} \left(G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)}) \right)^2 + \left(H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)}) \right)^2. \end{aligned} \tag{25}$$

By replacing $\text{Var}\{\epsilon_i^1(n^{(\alpha,\beta)})\}$ with the sample variance, an estimate of $(\sigma_i^{(\alpha,\beta)})^2$ is obtained.

Finally, note that the two-channel ARMA model (21), (22), has in general, a noncausal MA part. Nevertheless, since $\xi_i^{(\alpha,\beta)}$, and $\zeta_i^{(\alpha,\beta)}$ are stationary white noise processes, the noncausal MA part can be replaced by its shifted, and hence causal, version (i.e., the white input sequence is replaced with its shifted version, which has the same statistics). Hence, estimates of $\{a_i^{(\alpha,\beta)}(\tau)\}_{\tau=1}^{V_i^{(\alpha,\beta)}}$, and $(\sigma_i^{(\alpha,\beta)})^2$ for $i = 1 \ldots I^{(\alpha,\beta)}$,

$$\epsilon_i^1(n^{(\alpha,\beta)}) \approx \sum_{k^{(\alpha,\beta)}} \left[G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - k^{(\alpha,\beta)}) - H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - k^{(\alpha,\beta)}) \right] \tag{23}$$

and

$$\epsilon_i^2(n^{(\alpha,\beta)}) \approx \sum_{k^{(\alpha,\beta)}} \left[H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - k^{(\alpha,\beta)}) + G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - k^{(\alpha,\beta)}) \right] \tag{24}$$

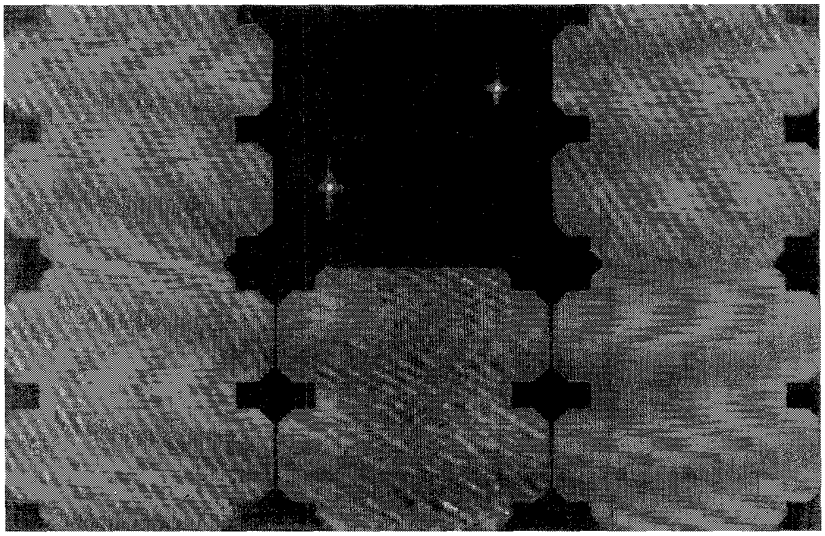

Fig. 3. Clockwise, from top left: Original texture, periodogram, harmonic component, purely indeterministic component, reconstruction using 2-D AR model, reconstruction by the proposed algorithm.

are obtained in this case, using the same procedures described earlier.

V. Estimation and Synthesis of Natural Textures

Experimental results are first presented below. This is followed by a section on discussion of the results.

Experimental Results

In this section, we present some examples to illustrate the performance of the suggested parametric texture model and the joint parameter estimation algorithm. All the textures presented here are natural textures, and hence, the true parameters are unknown. For the analysis of synthetic examples (where the true parameters are known) we refer the interested reader to [16]. The synthesis algorithm reconstructs the original texture using *only* the estimated parameters. In all six examples presented, the original image is such that it can be bounded by a 64 $\times$ 64 pixels box. For each example, the original texture, and the texture synthesized using the estimated parameters are shown. Also shown are the components (harmonic, purely indeterministic, evanescent) which exist in each texture. Through experiments on a few textures, it was determined that a $S_{(4,4)}$ NSHP AR model yields a sufficiently accurate reconstruction of the purely indeterministic component for the textures considered here. As indicated below, in some cases even a smaller support yields a sufficiently accurate reconstruction. For comparison with the quality of the estimation/synthesis results obtained by a widely used continuous spectrum model, we show for each texture the reconstruction obtained by "fitting" the texture with a high-order 2-D AR model.

Note that for the synthesis of the evanescent components, two different parametric representations can be used. The first uses a complete parametric model as described above. In the second, the synthesis is done using the estimated $\eta_i^1(n^{(\alpha,\beta)})$, $\eta_i^2(n^{(\alpha,\beta)})$ for all (α,β) and $i = 1,\ldots,I^{(\alpha,\beta)}$ by substituting them into the synthesis equation (10), without explicitly solving for the parameters of the modulating purely indeterministic processes of each evanescent field. In the following examples, the full parametric modeling approach was adopted for modeling and estimating the parameters of the evanescent components.

TABLE II
PARAMETERS $\{b(k,l)\}$ FOR THE TEXTURE OF FIG. 2, (e.g., read $b(3,-4) = -0.00871$).

	4	3	2	1	0
4	-0.00294	-0.01728	0.00617	-0.03012	0.01571
3	-0.00618	0.01758	-0.01515	0.04286	-0.08477
2	-0.01199	-0.00630	-0.02654	-0.00569	0.23206
1	-0.00159	0.10184	-0.00352	0.10134	-0.63309
0	0.00470	-0.13068	0.11484	-0.32291	-
-1	0.00921	0.00566	0.08055	-0.11271	-
-2	-0.00954	0.01707	0.00915	-0.02517	-
-3	-0.00105	0.03271	-0.02087	0.04089	-
-4	-0.01405	-0.00871	-0.00638	-0.00622	-

$\sigma^2 = 456.47$

In Fig. 2 we consider a carpet texture, which is modeled as a purely indeterministic field using a $S_{(4,4)}$ NSHP AR model. Note that for this example, our solution is identical to that obtained by fitting a 2-D AR model to the texture, since this texture contains no singular spectral components. For this example, the estimated parameters are listed in Table II.

Fig. 3 shows a corduroy fabric texture. It is modeled as the sum of a single harmonic component and a purely indeterministic component. The purely indeterministic component is modeled by a $S_{(1,1)}$ NSHP AR model. For this example, the estimated parameters are listed in Table III.

Fig. 4 presents an example of a fabric texture. This texture is composed of harmonic and purely indeterministic components. For this example, we used 25 harmonic components and a $S_{(4,4)}$ NSHP AR model for the purely indeterministic component.

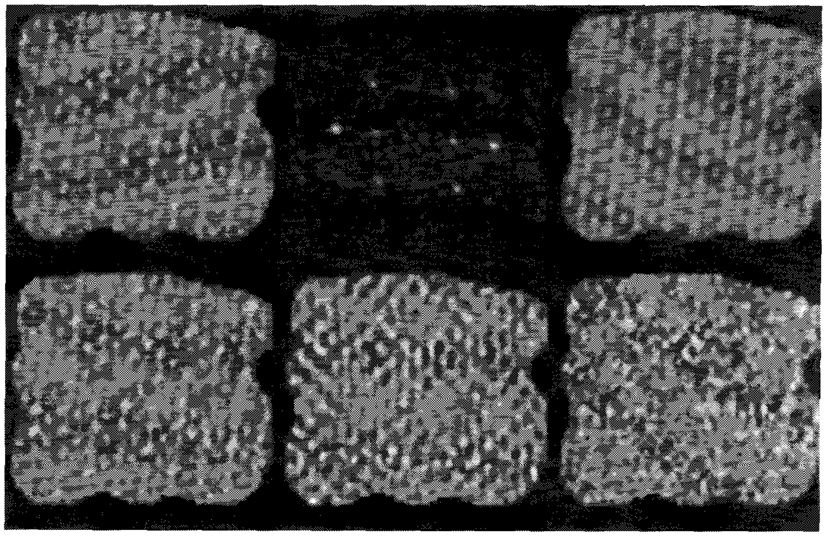

Fig. 4. Clockwise, from top left: Original texture, periodogram, harmonic component, purely indeterministic component, reconstruction using 2-D AR model, reconstruction by the proposed algorithm.

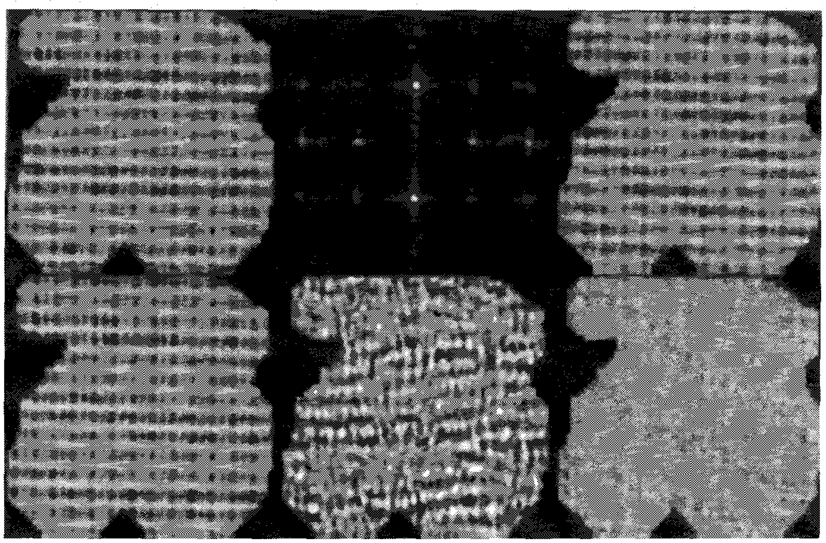

Fig. 5. Clockwise, from top left: Original texture, periodogram, harmonic component, purely indeterministic component, reconstruction using 2-D AR model, reconstruction by the proposed algorithm.

TABLE III
PARAMETERS $\{b(k,l)\}$, $\{\omega,\nu\}$, C, D FOR THE TEXTURE OF FIG. 3

	1	0
1	0.018005	0.100081
0	0.030010	-
-1	0.200301	-

$\sigma^2 = 5.65$

1 harmonic component at $(\omega,\nu) = (0.1825, -0.3095)$
with C = 2.001 and D = 2.120

Fig. 5 shows another fabric texture, composed of harmonic and purely indeterministic components. This texture is modeled using 20 harmonic components and a $S_{(4,4)}$ NSHP AR model for the purely indeterministic component.

In Fig. 6 we consider a wood texture. Observe from the periodogram in Fig. 6 that this texture has dominant evanescent components. For this example, we use five evanescent components with $(\alpha,\beta) = (1,0)$, two evanescent components with $(\alpha,\beta) = (0,1)$, and a $S_{(4,4)}$ NSHP AR model for the purely indeterministic component. The evanescent components image in Fig. 6 shows the image obtained by summing the synthesis results produced by the estimated models of all seven evanescent components.

Fig. 7 shows a brick texture. It contains evanescent and purely indeterministic components. For this example, we use

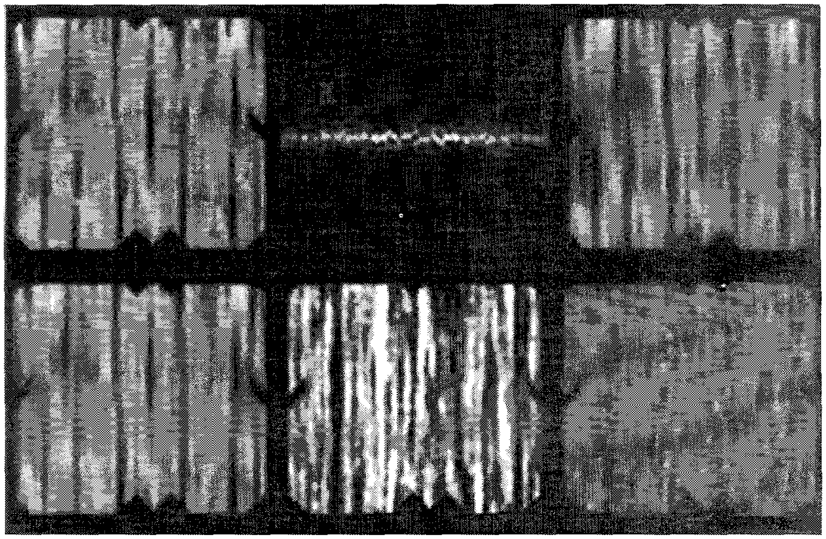

Fig. 6. Clockwise, from top left: Original texture, periodogram, evanescent components image, purely indeterministic component, reconstruction using 2-D AR model, reconstruction by the proposed algorithm.

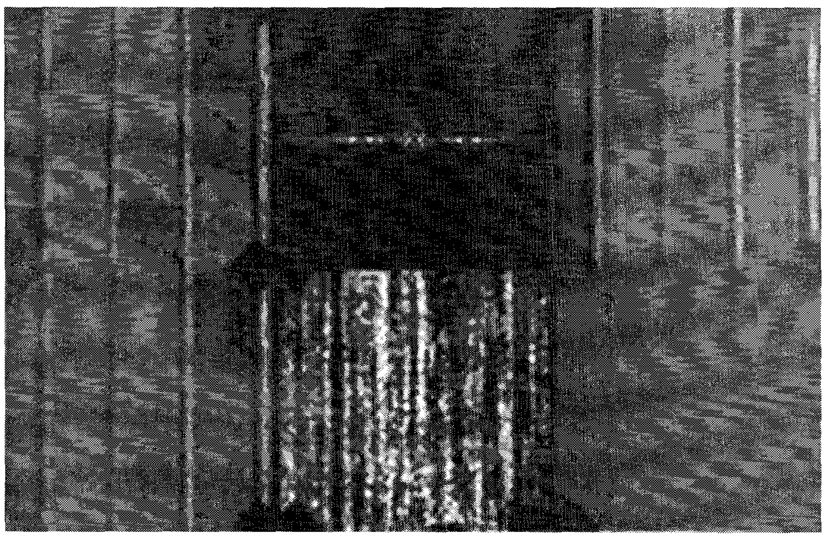

Fig. 7. Clockwise, from top left: Original texture, periodogram, evanescent components image, purely indeterministic component, reconstruction using 2-D AR model, reconstruction by the proposed algorithm.

three evanescent components with $(\alpha, \beta) = (1, 0)$, one evanescent component with $(\alpha, \beta) = (0, 1)$, and a $S_{(2,2)}$ NSHP AR model for the purely indeterministic component. The evanescent components image in Fig. 7 shows the image obtained by summing the synthesis results produced by the estimated models of all four evanescent components. For this example, the estimated parameters are listed in Table IV (a) and (b).

TABLE IV(a)
PARAMETERS OF THE EVANESCENT COMPONENTS FOR THE TEXTURE OF FIG. 7

α	β	$\sigma_i^{(\alpha,\beta)}$	a_1	a_2	a_3	a_4	ν
0	1	0.3920	0.0311	-	-	-	0.03
1	0	0.8102	0.0149	0.1570	0.221	0.344	0
1	0	0.3940	0.0310	0.1240	0.1721	0.3275	0.0158
1	0	0.0520	0.0292	0.0935	0.0152	0.2812	0.0301

Discussion

The synthesis results obtained by the suggested ML algorithm are both visually and statistically very similar to the originals, and in some cases, indistinguishable. On the other hand, it is clear that any attempt to estimate the texture parameters using a continuous spectral density estimator is both theoretically inappropriate, as well as practically unsuccessful as illustrated above. However, practically indistinguishable synthesis results are obtained when a 2-D AR model is

TABLE IV (b)
PARAMETERS $\{b(k,l)\}$ FOR THE TEXTURE OF FIG. 7

	2	1	0
2	0.00436	0.01113	0.04133
1	0.00986	0.02294	0.04370
0	0.04742	0.08592	-
-1	0.00865	0.01909	-
-2	0.00485	0.00782	-

$\sigma^2 = 126.16$

fitted to a purely-indeterministic texture since the purely-indeterministic field has no singular spectral components.

Since both white- and correlated-noise driven linear models can only produce fields with absolutely continuous spectral distribution functions, they cannot be applied to the general case, in which the spectral distribution of the observed field contains singular components as well. Because noise driven models cannot model the singular components of the spectral distribution, and since the singular components result in the structural features of the texture, noise driven models can be successful only in modeling random looking textures, and fail to model structured ones. Our theoretical and experimental analyses indicate that any estimation method of homogeneous texture parameters has to account for the existence of the various components of the texture; those of singular spectral distribution and those of continuous distribution.

In [1], we have introduced a simple to implement, but suboptimal, texture analysis and synthesis algorithm which is based on the same texture model that was described here, in Section II. The model and the derived estimation/synthesis algorithms presented in [1], enable the synthesis of purely random, as well as structured textures from the estimated parameters. Although the algorithm in [1] is suboptimal, the parameter estimation and synthesis results obtained by this method were far superior to those obtained by the frequently used AR and MRF models. The algorithm in [1] is a sequential, periodogram based estimation algorithm. In the first stage the parameters of the harmonic and evanescent components are estimated and their contribution to the observed realization is removed. Ideally, the obtained residual is the purely indeterministic component of the texture. In a second stage, a 2-D AR model is fitted to the residual. However, the algorithm in [1] suffers from some practical and theoretical limitations. Most importantly, since the algorithm in [1] uses the Discrete Fourier Transform to estimate the evanescent components, the evanescent components had to be approximated by a linear combination of harmonic components whose frequencies are along a "line" in the sampled frequency domain. Hence, [1] results in an approximate parameterization of the evanescent fields, which is different from the evanescent component model given by (4), (5), (6). Moreover, the asymptotic considerations which motivate the approach of [1] are problematic when the number of samples in the observed realization of the field is small. In addition, the algorithm itself is useful mainly for rectangular texture patches.

The new algorithm suggested here, is an ML estimation procedure for *jointly* estimating the parameters of the harmonic, evanescent, and purely indeterministic components of the texture, based on a small patch from a single observed realization of the field. As an ML algorithm it is asymptotically unbiased and efficient. Contrary to [1], the proposed algorithm results in a complete estimate of the model parameters for each of the evanescent components. In addition, in the suggested algorithm, the shape of the observed texture patch may be arbitrary. Note, however, that the new method overcomes the practical and theoretical limitations of the suboptimal estimation procedure in [1], at the cost of higher computational requirements.

VI. CONCLUSION

In this paper, we have presented a solution to the problem of modeling, parameter estimation, and synthesis of natural textures. The texture field was assumed to be a realization of a regular homogeneous random field possessing, in general, a discontinuous spectral distribution. Our suggested parametric modeling and estimation approach provides a unifying framework for many applications in which texture analysis is involved, together with the optimality properties of an ML estimation algorithm. The model and the resulting analysis and synthesis algorithms are seen to be applicable to a wide variety of texture types found in natural images. It also turns out that the model is very efficient in terms of the number of parameters required to represent, and faithfully reconstruct the original texture from the estimated parameters.

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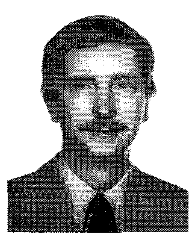

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