Bounds for Estimation of Multicomponent Signals with Random Amplitude and Deterministic Phase

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***Abstract*— We study a class of nonstationary multicomponent signals, where each component has the form $a(t)\exp j\phi(t)$, where $a(t)$ is a random amplitude function, and $\phi(t)$ is a deterministic phase function. The amplitude function consists of a stationary Gaussian process and a time varying mean. The phase and the amplitude mean are characterized by a linear parametric model, while the covariance of the amplitude function is parameterized in some general manner. This model encompasses signals that are commonly used in communications, radar, sonar, and other engineering systems. We derive the Cramér-Rao bound (CRB) for the estimates of the amplitude and phase parameters, and of functions of these parameters, such as the instantaneous frequencies of the signal components.**

I. Introduction

Many signals used in communications, sonar, radar and other engineering systems, as well as various natural signals, involve amplitude and/or frequency modulation of a carrier. The complex representation of such signals is given by $s(t) = a(t)e^{j\phi(t)}$, where $a(t)$ and $\phi(t)$ are the amplitude and phase functions. Signals with a more complicated structure can be represented by a combination of signals of this type. In the following, we refer to signals as being either single-component or multicomponent signals, where the word "component" refers to a term of the form $a(t)e^{j\phi(t)}$.

We are interested in developing parametric models for such signals, and in using these models to detect, estimate, and classify the signals. The amplitude and phase functions of these signals can be modeled as either deterministic functions with some unknown parameters or as stochastic processes whose statistics are specified parametrically. In a previous paper [2], we considered the case where both the amplitude and phase are deterministic functions of time. In this paper, we study the case where the phase is deterministic but the amplitude is random. More precisely, we assume that the phase function is a linear combination of some known functions of time (basis functions), where the coefficients associated with each function are unknown constants. The amplitude function is a stationary Gaussian process with a possibly time varying mean, which is also characterized by a linear parametric model. The covariance matrix of the amplitude is assumed to have some arbitrary parameterization. As an example, we may assume that it is the covariance matrix of a finite dimensional autoregressive (AR) process, parameterized by the AR coefficients.

Having specified a parametric model for the signals of interest, we want to estimate the model parameters given a finite number of possibly noise-corrupted measurements of the signal. The estimated parameters can then be used for signal estimation/reconstruction, classification, or another purpose. In this paper, we focus on the achievable accuracy of the estimated parameters, using the Cramér–Rao bound (CRB) as the principal tool for our investigation. We do not address here the question of how to estimate the model parameters, which is treated elsewhere [3].

While modulated signals of the type considered here are widely used in engineering applications, parametric modeling of such signals seems to have received relatively little attention in the literature, until very recently. The case of constant modulus components with a phase function that is a polynomial function of time has been studied in [9]–[13]. Signals with deterministic amplitude and phase functions have been studied in [2]. The case of signals with random *non-Gaussian* amplitudes and polynomial phase has been addressed in [14] and [15].

The structure of this paper is as follows. In Section II, we define the problem studied in this paper and introduce some necessary notations. In Section III, we derive the CRB for the general case where the signal is a multicomponent signal and the measurements are corrupted by additive complex Gaussian white noise. In Section IV, we specialize the general results of Section III for the case in which the signal is a monocomponent signal. We consider separately the case where the amplitude function has zero mean and when it has a nonzero time varying mean.

In the case of a single component signal we are able to make some interesting observations: i) The CRB for the phase and amplitude parameters are decoupled. ii) The CRB for the parameters of the random component of the amplitude is independent of the phase function and of the mean of the amplitude function. In fact, the CRB is the same as the CRB for the random component observed directly (without the modulating function or an additive mean). iii) The CRB for the phase parameters does not depend on the parametric model used for the amplitude process. iv) The CRB on the parameters of the amplitude mean depends on the general class of functions to which the time-varying mean function belongs and the covariance of the random component of the amplitude but is independent of the signal phase. Moreover, the bound on the mean parameters is decoupled from the bounds on the

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random component of the amplitude, the noise variance, and the phase. It is therefore identical to the bound that is obtained when the modulation is not present.

Section V presents some numerical examples illustrating the behavior of the bounds derived in this paper as function of signal-to-noise ratio (SNR) and various signal parameters. Section VI contains some concluding remarks.

II. Problem Formulation

In this section, we describe the problem to be treated in this paper and introduce the necessary notations.

We assume that the observed process $\{y(t)\}_{t=0}^{N-1}$ is given by

$$y(t) = \sum_{i=1}^{I} x_i(t) + n(t), \qquad t = 0, \Delta, \cdots, (N-1)\Delta \quad (1)$$

where I is the number of signal components. Each of the components has the form

$$x_i(t) = s_i(t) e^{j\phi_i(t)} \quad (2)$$

where $s_i(t)$ is the amplitude of the ith component and $\phi_i(t)$ is its phase. The amplitudes of the different components are assumed to be independent random processes. The amplitude of the ith component is the sum of a real, zero-mean, stationary Gaussian process and a real time-varying mean. The time-varying mean is assumed to obey the linear parametric model

$$m_i(t) = \sum_{k=0}^{P_i} c_{i,k}\, \rho_{i,k}(t) \quad (3)$$

where $\{\rho_{i,k}(t)\}$ is some arbitrary set of real basis functions that may be different from component to component, the $c_{i,k}$'s are the model parameters, and P_i is the model order. Hence, the mean of $x_i(t)$ is given by

$$\mu_i(t) = m_i(t) e^{j\phi_i(t)} \quad (4)$$

and the mean of $y(t)$ is given by

$$\mu(t) = \sum_{i=1}^{I} \mu_i(t). \quad (5)$$

Let $\boldsymbol{\rho}_i$ be an $N \times (P_i + 1)$ matrix whose columns contain the samples of the ith component basis functions over the observation interval. In other words

$$\boldsymbol{\rho}_i = [\rho_{i,0}(\boldsymbol{t})\rho_{i,1}(\boldsymbol{t}) \cdots \rho_{i,P_i}(\boldsymbol{t})] \quad (6)$$

where

$$\rho_{i,k}(\boldsymbol{t}) = [\rho_{i,k}(0), \rho_{i,k}(\Delta), \cdots, \rho_{i,k}((N-1)\Delta)]^T. \quad (7)$$

The vector $\boldsymbol{t}$ contains the sampling times, i.e.

$$\boldsymbol{t} = [0, \Delta, \cdots, (N-1)\Delta]^T. \quad (8)$$

The random part of the amplitude function $s_i(t)$ is characterized by its covariance matrix, which is denoted by $\boldsymbol{R}_{s_i}$. We assume that the covariance matrix has some known parametric form, where $\boldsymbol{a}_i$ is the parameter vector. At the moment, we will not specify the functional dependence of $\boldsymbol{R}_{s_i}$ on $\boldsymbol{a}_i$ but rather leave it implicit.

The phase of the ith signal is given by the linear model

$$\phi_i(t) = \sum_{\ell=0}^{Q_i} b_{i,\ell} \psi_{i,\ell}(t) \quad (9)$$

where $\{\psi_{i,\ell}(t)\}$ is some arbitrary set of real basis functions, which may be different for each component, the $b_{i,\ell}$'s are the model parameters, and Q_i is the model order. The vector of phase values of the ith component is denoted by $\boldsymbol{\phi}_i(\boldsymbol{t})$. Also, let

$$\boldsymbol{\psi}_{i,\ell} = [\psi_{i,\ell}(0), \psi_{i,\ell}(\Delta), \cdots, \psi_{i,\ell}((N-1)\Delta)]^T \quad (10)$$

and

$$\boldsymbol{T}_{i,k} = \operatorname{diag}\{\boldsymbol{\psi}_{i,k}\}. \quad (11)$$

The observation noise $n(t)$ is an additive, complex-valued, zero-mean, white Gaussian noise of unknown variance σ^2. It is assumed that the noise process can be written as $n(t) = n_1(t) + jn_2(t)$, with $n_1(t)$ and $n_2(t)$ being independent, identically distributed, real-valued white Gaussian noise processes, with variance $\sigma^2/2$ each. Both $n_1(t)$ and $n_2(t)$ are assumed to be independent of the amplitude functions $s_i(t)$.

Let the data, noise, and signal vectors be

$$\boldsymbol{y} = [y(0), y(\Delta), \cdots, y((N-1)\Delta)]^T \quad (12)$$
$$\boldsymbol{n} = [n(0), n(\Delta), \cdots, n((N-1)\Delta)]^T \quad (13)$$

and

$$\boldsymbol{s} = [\boldsymbol{s}_1^T, \boldsymbol{s}_2^T, \cdots, \boldsymbol{s}_I^T]^T \quad (14)$$

where

$$\boldsymbol{s}_i = [s_i(0), s_i(\Delta), \cdots, s_i((N-1)\Delta)]^T. \quad (15)$$

Also, let

$$\boldsymbol{b}_i = [b_{i,0}, b_{i,1}, \cdots, b_{i,Q_i}]^T \quad (16)$$
$$\boldsymbol{b} = [\boldsymbol{b}_1^T, \boldsymbol{b}_2^T, \cdots, \boldsymbol{b}_I^T]^T \quad (17)$$
$$\boldsymbol{c}_i = [c_{i,0}, c_{i,1}, \cdots, c_{i,P_i}]^T \quad (18)$$
$$\boldsymbol{c} = [\boldsymbol{c}_1^T, \boldsymbol{c}_2^T, \cdots, \boldsymbol{c}_I^T]^T \quad (19)$$

and

$$\boldsymbol{a} = [\boldsymbol{a}_1^T, \boldsymbol{a}_2^T, \cdots, \boldsymbol{a}_I^T]^T. \quad (20)$$

Finally, we collect all of the unknown parameters into a single vector $\boldsymbol{\theta}$, such that

$$\boldsymbol{\theta} = \{\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c}\}. \quad (21)$$

The problem considered in this paper can now be stated as follows. Given the measurements $\{y(t)\}_{t=0}^{N-1}$, how accurately can the parameter vector $\boldsymbol{\theta}$ be estimated?

III. THE CRB FOR SIGNAL PLUS NOISE

The CRB provides a lower bound for the covariance matrix of any unbiased estimator. As is well known, the bound is given by the inverse of the so-called Fisher information matrix (FIM) [1], [4], which will be denoted by $\boldsymbol{J}$. Let

$$\boldsymbol{A}_i = \mathrm{diag}\,\{\cos\phi_i(\boldsymbol{t})\} \tag{22}$$

$$\boldsymbol{B}_i = \mathrm{diag}\,\{\sin\phi_i(\boldsymbol{t})\} \tag{23}$$

be $N \times N$ diagonal matrices. Also, let

$$\boldsymbol{A} = [\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_I] \tag{24}$$

$$\boldsymbol{B} = [\boldsymbol{B}_1\boldsymbol{B}_2\cdots\boldsymbol{B}_I] \tag{25}$$

and

$$\boldsymbol{u} = \sum_{i=1}^{I}\boldsymbol{A}_i\boldsymbol{s}_i + \boldsymbol{n}_1 \tag{26}$$

$$\boldsymbol{v} = \sum_{i=1}^{I}\boldsymbol{B}_i\boldsymbol{s}_i + \boldsymbol{n}_2 \tag{27}$$

where $\boldsymbol{u}, \boldsymbol{v}$ are N-dimensional column vectors. Using this notation, we can now rewrite the measurement equation using real quantities only, as

$$\boldsymbol{z} = \begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{bmatrix}\boldsymbol{s} + \begin{bmatrix}\boldsymbol{n}_1\\ \boldsymbol{n}_2\end{bmatrix}. \tag{28}$$

Since $\boldsymbol{n}_1, \boldsymbol{n}_2$, and all $\boldsymbol{s}_i$'s are Gaussian and independent, $\boldsymbol{z}$ is Gaussian as well. Its covariance matrix Γ is of the form

$$\Gamma = \begin{bmatrix}\boldsymbol{R}_{uu} & \boldsymbol{R}_{uv}\\ \boldsymbol{R}_{vu} & \boldsymbol{R}_{vv}\end{bmatrix} \tag{29}$$

with

$$\boldsymbol{R}_{uu} = \sum_{i=1}^{I}\boldsymbol{A}_i\boldsymbol{R}_{s_i}\boldsymbol{A}_i + \frac{\sigma^2}{2}\boldsymbol{I}_N \tag{30}$$

$$\boldsymbol{R}_{vv} = \sum_{i=1}^{I}\boldsymbol{B}_i\boldsymbol{R}_{s_i}\boldsymbol{B}_i + \frac{\sigma^2}{2}\boldsymbol{I}_N \tag{31}$$

$$\boldsymbol{R}_{uv} = \sum_{i=1}^{I}\boldsymbol{A}_i\boldsymbol{R}_{s_i}\boldsymbol{B}_i, \tag{32}$$

$$\boldsymbol{R}_{vu} = \sum_{i=1}^{I}\boldsymbol{B}_i\boldsymbol{R}_{s_i}\boldsymbol{A}_i \tag{33}$$

where $\boldsymbol{R}_{s_i}$ is the covariance matrix of $\boldsymbol{s}_i$, and $\boldsymbol{I}_N$ is N-dimensional identity matrix. It follows that

$$\Gamma = \sum_{i=1}^{I}\begin{bmatrix}\boldsymbol{A}_i\\ \boldsymbol{B}_i\end{bmatrix}\boldsymbol{R}_{s_i}[\boldsymbol{A}_i\boldsymbol{B}_i] + \frac{\sigma^2}{2}\boldsymbol{I}_{2N}. \tag{34}$$

Let

$$\boldsymbol{X}_i = \begin{bmatrix}\boldsymbol{A}_i\\ \boldsymbol{B}_i\end{bmatrix}. \tag{35}$$

Hence, we can rewrite (34) in the form

$$\Gamma = \sum_{i=1}^{I}\boldsymbol{X}_i\boldsymbol{R}_{s_i}\boldsymbol{X}_i^T + \frac{\sigma^2}{2}\boldsymbol{I}_{2N}. \tag{36}$$

Rewriting the mean (5) using real quantities, we have

$$\boldsymbol{\mu} = \sum_{i=1}^{I}\begin{bmatrix}\boldsymbol{A}_i\\ \boldsymbol{B}_i\end{bmatrix}\boldsymbol{\rho}_i c_i = \sum_{i=1}^{I}\boldsymbol{X}_i\boldsymbol{\rho}_i c_i = \sum_{i=1}^{I}\boldsymbol{\mu}_i \tag{37}$$

where

$$\boldsymbol{\mu}_i = \begin{bmatrix}\boldsymbol{\mu}_i^R\\ \boldsymbol{\mu}_i^I\end{bmatrix} \tag{38}$$

and $\boldsymbol{\mu}_i^R = \boldsymbol{A}_i\boldsymbol{\rho}_i c_i, \boldsymbol{\mu}_i^I = \boldsymbol{B}_i\boldsymbol{\rho}_i c_i$. The FIM of the process $y(t)$ is given by

$$\boldsymbol{J}_{k,\ell}(\boldsymbol{\theta}) = \frac{\partial\boldsymbol{\mu}^T}{\partial\theta_k}\boldsymbol{\Gamma}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\theta_\ell} + \frac{1}{2}\mathrm{tr}\left\{\boldsymbol{\Gamma}^{-1}\frac{\partial\boldsymbol{\Gamma}}{\partial\theta_k}\boldsymbol{\Gamma}^{-1}\frac{\partial\boldsymbol{\Gamma}}{\partial\theta_\ell}\right\} \tag{39}$$

where $\boldsymbol{\theta}$ is a real valued parameter vector [1].

To evaluate (39), we need to compute the derivatives of $\boldsymbol{\Gamma}$ with respect to the various parameters of interest. Taking the partial derivatives of $\boldsymbol{\Gamma}$ we get

$$\frac{\partial\boldsymbol{\Gamma}}{\partial b_{i,k}} = \begin{bmatrix}\frac{\partial\boldsymbol{A}_i}{\partial b_{i,k}}\boldsymbol{R}_{s_i}\boldsymbol{A}_i + \boldsymbol{A}_i\boldsymbol{R}_{s_i}\frac{\partial\boldsymbol{A}_i}{\partial b_{i,k}} & \frac{\partial\boldsymbol{A}_i}{\partial b_{i,k}}\boldsymbol{R}_{s_i}\boldsymbol{B}_i + \boldsymbol{A}_i\boldsymbol{R}_{s_i}\frac{\partial\boldsymbol{B}_i}{\partial b_{i,k}}\\ \frac{\partial\boldsymbol{B}_i}{\partial b_{i,k}}\boldsymbol{R}_{s_i}\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{R}_{s_i}\frac{\partial\boldsymbol{A}_i}{\partial b_{i,k}} & \frac{\partial\boldsymbol{B}_i}{\partial b_{i,k}}\boldsymbol{R}_{s_i}\boldsymbol{B}_i + \boldsymbol{B}_i\boldsymbol{R}_{s_i}\frac{\partial\boldsymbol{B}_i}{\partial b_{i,k}}\end{bmatrix} \tag{40}$$

$$\frac{\partial\boldsymbol{\Gamma}}{\partial a_{i,k}} = \begin{bmatrix}\boldsymbol{A}_i\frac{\partial\boldsymbol{R}_{s_i}}{\partial a_{i,k}}\boldsymbol{A}_i & \boldsymbol{A}_i\frac{\partial\boldsymbol{R}_{s_i}}{\partial a_{i,k}}\boldsymbol{B}_i\\ \boldsymbol{B}_i\frac{\partial\boldsymbol{R}_{s_i}}{\partial a_{i,k}}\boldsymbol{A}_i & \boldsymbol{B}_i\frac{\partial\boldsymbol{R}_{s_i}}{\partial a_{i,k}}\boldsymbol{B}_i\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_i\\ \boldsymbol{B}_i\end{bmatrix}\frac{\partial\boldsymbol{R}_{s_i}}{\partial a_{i,k}}[\boldsymbol{A}_i\boldsymbol{B}_i] = \boldsymbol{X}_i\frac{\partial\boldsymbol{R}_{s_i}}{\partial a_{i,k}}\boldsymbol{X}_i^T \tag{41}$$

$$\frac{\partial\boldsymbol{\Gamma}}{\partial\sigma^2} = \frac{1}{2}\boldsymbol{I}_{2N}. \tag{42}$$

Note also that since the process covariance function $\boldsymbol{\Gamma}$ is independent of the components' means

$$\frac{\partial\boldsymbol{\Gamma}}{\partial c_{i,k}} = 0. \tag{43}$$

Hence, the $\frac{1}{2}\mathrm{tr}\{\cdot\}$ term in (39) vanishes for all the FIM entries that correspond to parameters of the components' means. Taking the partial derivatives w.r.t. the phase parameters yields

$$\frac{\partial\boldsymbol{A}_i}{\partial b_{i,k}} = -\mathrm{diag}\,\{\boldsymbol{\psi}_{i,k}\}\boldsymbol{B}_i = -\boldsymbol{T}_{i,k}\boldsymbol{B}_i = -\boldsymbol{B}_i\boldsymbol{T}_{i,k} \tag{44}$$

$$\frac{\partial\boldsymbol{B}_i}{\partial b_{i,k}} = \mathrm{diag}\,\{\boldsymbol{\psi}_{i,k}\}\boldsymbol{A}_i = \boldsymbol{T}_{i,k}\boldsymbol{A}_i = \boldsymbol{A}_i\boldsymbol{T}_{i,k}. \tag{45}$$

Substituting (44), (45) into (40) we have

$$\begin{aligned}\frac{\partial\boldsymbol{\Gamma}}{\partial b_{i,k}} &= \begin{bmatrix}-\boldsymbol{T}_{i,k}\boldsymbol{B}_i\boldsymbol{R}_{s_i}\boldsymbol{A}_i & -\boldsymbol{T}_{i,k}\boldsymbol{B}_i\boldsymbol{R}_{s_i}\boldsymbol{B}_i\\ \boldsymbol{T}_{i,k}\boldsymbol{A}_i\boldsymbol{R}_{s_i}\boldsymbol{A}_i & \boldsymbol{T}_{i,k}\boldsymbol{A}_i\boldsymbol{R}_{s_i}\boldsymbol{B}_i\end{bmatrix} + \begin{bmatrix}-\boldsymbol{A}_i\boldsymbol{R}_{s_i}\boldsymbol{B}_i\boldsymbol{T}_{i,k} & \boldsymbol{A}_i\boldsymbol{R}_{s_i}\boldsymbol{A}_i\boldsymbol{T}_{i,k}\\ -\boldsymbol{B}_i\boldsymbol{R}_{s_i}\boldsymbol{B}_i\boldsymbol{T}_{i,k} & \boldsymbol{B}_i\boldsymbol{R}_{s_i}\boldsymbol{A}_i\boldsymbol{T}_{i,k}\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{T}_{i,k} & 0\\ 0 & \boldsymbol{T}_{i,k}\end{bmatrix}\begin{bmatrix}-\boldsymbol{B}_i\\ \boldsymbol{A}_i\end{bmatrix}\boldsymbol{R}_{s_i}[\boldsymbol{A}_i\boldsymbol{B}_i] + \begin{bmatrix}\boldsymbol{A}_i\\ \boldsymbol{B}_i\end{bmatrix}\boldsymbol{R}_{s_i}[-\boldsymbol{B}_i\boldsymbol{A}_i]\begin{bmatrix}\boldsymbol{T}_{i,k} & 0\\ 0 & \boldsymbol{T}_{i,k}\end{bmatrix}\\ &= \boldsymbol{H}_{i,k}\boldsymbol{V}_i\boldsymbol{R}_{s_i}\boldsymbol{X}_i^T + \boldsymbol{X}_i\boldsymbol{R}_{s_i}\boldsymbol{V}_i^T\boldsymbol{H}_{i,k}\end{aligned} \tag{46}$$

where we define the notations

$$H_{i,k} = \begin{bmatrix} T_{i,k} & 0 \\ 0 & T_{i,k} \end{bmatrix} \quad (47)$$

$$V_i = \begin{bmatrix} -B_i \\ A_i \end{bmatrix}. \quad (48)$$

Taking the partial derivatives of μ yields

$$\frac{\partial \mu}{\partial b_{i,k}} = \begin{bmatrix} T_{i,k} & 0 \\ 0 & T_{i,k} \end{bmatrix} \begin{bmatrix} -B_i \\ A_i \end{bmatrix} \rho_i c_i = H_{i,k} V_i \rho_i c_i \quad (49)$$

and

$$\frac{\partial \mu}{\partial c_{i,\ell}} = \begin{bmatrix} A_i \\ B_i \end{bmatrix} \rho_i e_\ell = X_i \rho_{i,\ell} \quad (50)$$

where e_ℓ is a column vector whose ℓth element is one, and all its other elements are zero. $\rho_{i,\ell}$ is the ℓth column of ρ_i. Since μ is independent of the white noise process

$$\frac{\partial \mu}{\partial \sigma^2} = 0. \quad (51)$$

The mean μ does not depend on the parameters characterizing the covariance matrix and, therefore

$$\frac{\partial \mu}{\partial a_{i,k}} = 0. \quad (52)$$

Substituting (43) and (52) into (39), we conclude that

$$J^{a_i,c_n} = 0. \quad (53)$$

Similarly, substituting (43) and (51) yields

$$J^{c_i,\sigma^2} = 0. \quad (54)$$

In the following, we use the notation $J_{k,\ell}^{b_i,b_n}$ for the FIM matrix entry that corresponds to the kth element of b_i and the ℓth element of b_n. Thus

$$J_{k,\ell}^{b_i,b_n}(\theta) = \frac{\partial \mu^T}{\partial b_{i,k}} \Gamma^{-1} \frac{\partial \mu}{\partial b_{n,\ell}} + \frac{1}{2}\mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial b_{i,k}}\Gamma^{-1}\frac{\partial \Gamma}{\partial b_{n,\ell}}\right\} \quad (55)$$

$$J_{k,\ell}^{a_i,a_n}(\theta) = \frac{1}{2}\mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial a_{i,k}}\Gamma^{-1}\frac{\partial \Gamma}{\partial a_{n,\ell}}\right\} \quad (56)$$

$$J_{k,\ell}^{b_i,a_n}(\theta) = \frac{1}{2}\mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial b_{i,k}}\Gamma^{-1}\frac{\partial \Gamma}{\partial a_{n,\ell}}\right\} \quad (57)$$

$$J_{k,\ell}^{b_i,c_n}(\theta) = \frac{\partial \mu^T}{\partial b_{i,k}} \Gamma^{-1} \frac{\partial \mu}{\partial c_{n,\ell}} \quad (58)$$

$$J_{k,\ell}^{c_i,c_n}(\theta) = \frac{\partial \mu^T}{\partial c_{i,k}} \Gamma^{-1} \frac{\partial \mu}{\partial c_{n,\ell}} \quad (59)$$

$$J_{k,1}^{b_i,\sigma^2}(\theta) = \frac{1}{4}\mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial b_{i,k}}\Gamma^{-1}\right\} \quad (60)$$

$$J_{k,1}^{a_i,\sigma^2}(\theta) = \frac{1}{4}\mathrm{tr}\left\{\Gamma^{-1}\frac{\partial \Gamma}{\partial a_{i,k}}\Gamma^{-1}\right\} \quad (61)$$

$$J^{\sigma^2,\sigma^2}(\theta) = \frac{1}{8}\mathrm{tr}\{\Gamma^{-1}\Gamma^{-1}\}. \quad (62)$$

Note that bounds for the special case in which the amplitudes have only a deterministic mean component, i.e., $R_s = 0$, were studied in [2].

IV. The CRB for a Monocomponent Signal

In this section, we specialize the general results that were derived in the previous section for the case where the signal is a monocomponent signal. Hence, in the following, we omit the subindex notation i that refers to the component index. In this case, we have from (36)

$$\Gamma = X R_s X^T + \frac{\sigma^2}{2} I_{2N}. \quad (63)$$

In the following, we use the following identities:

$$X^T V = V^T X = 0 \quad (64)$$

$$X^T X = V^T V = A^2 + B^2 = I_N \quad (65)$$

$$I_{2N} - X X^T = V V^T \quad (66)$$

$$X^T H_k V = V^T H_k X = 0 \quad (67)$$

$$V^T H_k V = T_k \quad (68)$$

$$V^T H_k H_\ell V = T_k T_\ell. \quad (69)$$

Now, using the matrix inversion lemma (e.g., [5]), we find that

$$\begin{aligned} \Gamma^{-1} &= \frac{2}{\sigma^2} I_{2N} - \frac{2}{\sigma^2} X \left(\frac{2}{\sigma^2} X^T X + R_s^{-1}\right)^{-1} X^T \frac{2}{\sigma^2} \\ &= \frac{2}{\sigma^2} I_{2N} - \frac{2}{\sigma^2} X \left(I_N + \frac{\sigma^2}{2} R_s^{-1}\right)^{-1} X^T \\ &= \frac{2}{\sigma^2} I_{2N} - \frac{2}{\sigma^2} X D^{-1} X^T \end{aligned} \quad (70)$$

where the second equality results from (65), and we define

$$D = I_N + \frac{\sigma^2}{2} R_s^{-1}. \quad (71)$$

A. The Zero Mean Case

Note that the first term in (39) is a function of both the mean and the covariance function of the observed vector z, where as the last term is a function of only the covariance function and is independent of the mean of z. Hence, for a zero mean process, only this last term is nonzero.

Using (70) and (46), we have that

$$\begin{aligned} \Gamma^{-1}\frac{\partial \Gamma}{\partial b_k} &= \left(\frac{2}{\sigma^2} I_{2N} - \frac{2}{\sigma^2} X D^{-1} X^T\right) \\ &\quad \cdot (H_k V R_s X^T + X R_s V^T H_k) \\ &= \frac{2}{\sigma^2} H_k V R_s X^T + \frac{2}{\sigma^2} X R_s V^T H_k \\ &\quad - \frac{2}{\sigma^2} X D^{-1} X^T H_k V R_s X^T \\ &\quad - \frac{2}{\sigma^2} X D^{-1} X^T X R_s V^T H_k \\ &= \frac{2}{\sigma^2} H_k V R_s X^T + \frac{2}{\sigma^2} X R_s V^T H_k \\ &\quad - \frac{2}{\sigma^2} X D^{-1} R_s V^T H_k \end{aligned} \quad (72)$$

where the last equality results from (67) and (65). Hence

$$\begin{aligned} J^{b,b}_{k,\ell} &= \frac{1}{2}\mathrm{tr}\left\{\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial b_k}\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial b_\ell}\right\} \\ &= \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &= \frac{2}{\sigma^4}\mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad - \frac{2}{\sigma^4}\mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad + \frac{2}{\sigma^4}\mathrm{tr}\{\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{R}_s\} \\ &\quad - \frac{2}{\sigma^4}\mathrm{tr}\{\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{R}_s\} \\ &= \frac{2}{\sigma^4}\mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s(I_N - \boldsymbol{D}^{-1})\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \\ &\quad + \frac{2}{\sigma^4}\mathrm{tr}\{\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{R}_s(I_N - \boldsymbol{D}^{-1})\} \\ &= \frac{4}{\sigma^4}\mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s(I_N - \boldsymbol{D}^{-1})\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\} \end{aligned} \tag{73}$$

where we have used (65) and (67), the commutative property of the trace operator, the fact that the trace operator is invariant to transposition, and the symmetric property of $\boldsymbol{H}_k, \boldsymbol{H}_\ell$, and $\boldsymbol{R}_s$.

Using (41) and (70), we also have that

$$\begin{aligned} \boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial a_k} &= \left(\frac{2}{\sigma^2}\boldsymbol{I}_{2N} - \frac{2}{\sigma^2}\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\right)\left(\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\right) \\ &= \frac{2}{\sigma^2}\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T - \frac{2}{\sigma^2}\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T \end{aligned} \tag{74}$$

where the last equality follows from (65). Hence, using (56), (65), the trace operator commutative property, and the symmetric property of $\partial \boldsymbol{R}_s/\partial a_k, \boldsymbol{D}^{-1}$, we obtain

$$\begin{aligned} J^{a,a}_{k,\ell} &= \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &= \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\frac{\partial \boldsymbol{R}_s}{\partial a_k}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\right\} \\ &\quad - \left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\right\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\right\}. \end{aligned} \tag{75}$$

Using (72), (74), and (57) we have

$$\begin{aligned} J^{b,a}_{k,\ell} &= \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad - \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &\quad + \frac{1}{2}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &= 0 \end{aligned} \tag{76}$$

where we have again made use of (67) and the trace operator commutative property.

Finally, substituting (70) and (72) into (60) while using (67) and the commutativity of the trace operator, gives

$$\begin{aligned} J^{b,\sigma^2}_{k,1} &= \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\} \\ &\quad - \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\} \\ &\quad + \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\} \\ &\quad - \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\} \\ &\quad - \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\} \\ &\quad + \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \mathrm{tr}\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\} \\ &= 0. \end{aligned} \tag{77}$$

Substituting (70) and (74) into (61) while using (65), the commutativity of the trace operator, and the symmetric property of $\boldsymbol{R}_s$, gives

$$\begin{aligned} \boldsymbol{J}_{k,1}^{a,\sigma^2} &= \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \operatorname{tr}\left\{\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\right\} \\ &\quad - \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \operatorname{tr}\left\{\boldsymbol{X}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\right\} \\ &\quad - \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \operatorname{tr}\left\{\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\right\} \\ &\quad + \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \operatorname{tr}\left\{\boldsymbol{X}\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\right\} \\ &= \frac{1}{\sigma^4}\operatorname{tr}\left\{\frac{\partial \boldsymbol{R}_s}{\partial a_k}\right\} - \frac{2}{\sigma^4}\operatorname{tr}\left\{\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{D}^{-1}\right\} \\ &\quad + \frac{1}{\sigma^4}\operatorname{tr}\left\{\boldsymbol{D}^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{D}^{-1}\right\}. \end{aligned} \tag{78}$$

The FIM entry that corresponds to the noise parameter is given by

$$\begin{aligned} \boldsymbol{J}^{\sigma^2,\sigma^2} &= \frac{1}{8}\left(\frac{2}{\sigma^2}\right)^2 \operatorname{tr}\{\boldsymbol{I}_{2N}\} \\ &\quad - \frac{1}{4}\left(\frac{2}{\sigma^2}\right)^2 \operatorname{tr}\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\} \\ &\quad + \frac{1}{8}\left(\frac{2}{\sigma^2}\right)^2 \operatorname{tr}\{\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\} \\ &= \frac{N}{\sigma^4} - \frac{1}{\sigma^4}\operatorname{tr}\{\boldsymbol{D}^{-1}\} + \frac{1}{2\sigma^4}\operatorname{tr}\{\boldsymbol{D}^{-1}\boldsymbol{D}^{-1}\}. \end{aligned} \tag{79}$$

From (75)–(79), we conclude that for a monocomponent signal, the bounds on the amplitude parameters and the noise variance are both independent and decoupled of the phase, while the bound on the phase parameters is decoupled from the bounds on the amplitude and noise parameters. Hence, the CRB for the phase is obtained by inverting (73). Note that the FIM block that corresponds to parameters of the random component of the amplitude and the observation noise is *identical* to the block we would have obtained if this component was not modulated by $e^{j\phi(t)}$, (i.e., the case that is obtained by repeating the above derivation for $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{I}_N$).

We can now summarize our observations regarding the CRB for a zero-mean single component signal. The CRB for the phase and amplitude parameters are decoupled. Furthermore, the CRB for the amplitude parameters is the same as if the modulation by $e^{j\phi(t)}$ was not present, and it is a function only of the amplitude covariances and the noise variance. The CRB for the phase parameters is a function of the basis functions $\{\psi_n(t)\}$, the phase waveform, the covariances of the amplitude process $s(t)$, and the observation noise variance σ^2, but it does not depend on the parametric model used for the amplitude process. Similarly, the bound on the noise variance is decoupled from the bound on the phase and it is identical to the bound that is obtained when the modulation by $e^{j\phi(t)}$ is not present.

B. Nonzero Mean Case

We now extend the results of the previous section to the case in which the observed signal has a nonzero mean, whose parametric representation is given by (37). Using (49) and (70), we have that

$$\begin{aligned} &\frac{\partial \boldsymbol{\mu}^T}{\partial b_k}\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\mu}}{\partial b_\ell} \\ &\quad = (\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{\rho}\boldsymbol{c})^T\left(\frac{2}{\sigma^2}\boldsymbol{I}_{2N} - \frac{2}{\sigma^2}\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\right)\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{\rho}\boldsymbol{c} \\ &\quad = \frac{2}{\sigma^2}\boldsymbol{c}^T\boldsymbol{\rho}^T\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{\rho}\boldsymbol{c} \\ &\qquad - \frac{2}{\sigma^2}\boldsymbol{c}^T\boldsymbol{\rho}^T\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\boldsymbol{H}_\ell\boldsymbol{V}\boldsymbol{\rho}\boldsymbol{c} \\ &\quad = \frac{2}{\sigma^2}\boldsymbol{c}^T\boldsymbol{\rho}^T\boldsymbol{T}_k\boldsymbol{T}_\ell\boldsymbol{\rho}\boldsymbol{c} \end{aligned} \tag{80}$$

where the last equality is due to (69) and (67).

Substituting (73) and (80) into (55) yields

$$\begin{aligned} \boldsymbol{J}_{k,\ell}^{b,b} &= \frac{2}{\sigma^2}\boldsymbol{c}^T\boldsymbol{\rho}^T\boldsymbol{T}_k\boldsymbol{T}_\ell\boldsymbol{\rho}\boldsymbol{c} \\ &\quad + \frac{4}{\sigma^4}\operatorname{tr}\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s(\boldsymbol{I}_N - \boldsymbol{D}^{-1})\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\}. \end{aligned} \tag{81}$$

Substituting (49), (50), and (70) into (58), we have

$$\begin{aligned} \boldsymbol{J}_{k,\ell}^{b,c} &= (\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{\rho}\boldsymbol{c})^T\left(\frac{2}{\sigma^2}\boldsymbol{I}_{2N} - \frac{2}{\sigma^2}\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\right)\boldsymbol{X}\boldsymbol{\rho}_\ell \\ &= \frac{2}{\sigma^2}\boldsymbol{c}^T\boldsymbol{\rho}^T\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{\rho}_\ell \\ &\quad - \frac{2}{\sigma^2}\boldsymbol{c}^T\boldsymbol{\rho}^T\boldsymbol{V}^T\boldsymbol{H}_k\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\rho}_\ell \\ &= 0 \end{aligned} \tag{82}$$

where we have used (67).

Using (50), (70), and (59) we have

$$\begin{aligned} \boldsymbol{J}_{k,\ell}^{c,c} &= (\boldsymbol{X}\boldsymbol{\rho}_k)^T\left(\frac{2}{\sigma^2}\boldsymbol{I}_{2N} - \frac{2}{\sigma^2}\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\right)\boldsymbol{X}\boldsymbol{\rho}_\ell \\ &= \frac{2}{\sigma^2}\boldsymbol{\rho}_k^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\rho}_\ell - \frac{2}{\sigma^2}\boldsymbol{\rho}_k^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\rho}_\ell \\ &= \frac{2}{\sigma^2}\boldsymbol{\rho}_k^T(\boldsymbol{I}_N - \boldsymbol{D}^{-1})\boldsymbol{\rho}_\ell \end{aligned} \tag{83}$$

where we have used (65).

An important special case is the case where the mean is parameterized by sinusoidal basis functions, i.e.

$$\mu(t) = \left(A_0 + \sum_{\ell=1}^{L} A_\ell \cos\omega_\ell t + \sum_{\ell=1}^{L} B_\ell \sin\omega_\ell t\right)e^{j\phi(t)} \tag{84}$$

where the frequencies ω_ℓ are known. This parameterization arises, for example, if we expand the mean in a trigonometric Fourier series. In this case

$$\boldsymbol{\rho} = [1 \cos\omega_1 \boldsymbol{t} \cdots \cos\omega_L \boldsymbol{t} \sin\omega_1 \boldsymbol{t} \cdots \sin\omega_L \boldsymbol{t}] \tag{85}$$

and $\boldsymbol{c} = [A_0, A_1, \cdots, A_L, B_1, \cdots, B_L]^T$.

Finally, combining the results of Sections IV-A and B with (53) and (54), we conclude that for a single component signal, the FIM has the block diagonal form

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}^{b,b} & 0 & 0 & 0 \\ 0 & \boldsymbol{J}^{c,c} & 0 & 0 \\ 0 & 0 & \boldsymbol{J}^{a,a} & \boldsymbol{J}^{a,\sigma} \\ 0 & 0 & (\boldsymbol{J}^{a,\sigma})^T & \boldsymbol{J}^{\sigma,\sigma} \end{bmatrix}. \tag{86}$$

Using (86), we conclude that the bounds on the parameter estimates of the phase, the random component of the amplitude, and the mean of the amplitude are mutually decoupled.

We can now summarize our observations regarding the CRB for the nonzero mean single component case.

Similarly to the zero-mean case, we conclude that the elements of the FIM are independent of the specific model of the random component of the amplitude since those of them that depend on the random component are functions of its covariance matrix $\boldsymbol{R}_s$ only. Thus, the random part of the amplitude can be any stationary zero-mean Gaussian process whose covariance matrix is $\boldsymbol{R}_s$. The CRB for the parameters of the random component of the amplitude is independent of the phase function and of the mean of the amplitude function, but it is a function of the observation noise variance. In fact, the bound on the random component of the amplitude signal is the same as the CRB for the random component when it is observed directly (without the modulating function or an additive mean).

The CRB for the phase parameters does not depend on the parametric model used for the amplitude process, but rather on the amplitude mean waveform, the phase waveform, the covariance of the random component of the amplitude, the observation noise variance, and the phase basis functions. It is independent of the phase parameters.

The CRB on the parameters of the mean of the amplitude component is a function of the mean basis functions $\{\rho_n(t)\}$ (i.e., of the general class of functions to which the time varying mean function belongs), and the covariance of the random component of the amplitude, but is independent of the phase. Moreover, the bound on the mean parameters is decoupled from the bounds on the random component of the amplitude, the noise variance, and the phase. It is therefore identical to the bound that is obtained when the modulation by $e^{j\phi(t)}$ is not present.

C. The CRB for a Monocomponent Signal at High SNR

In this section, we specialize the general results that were derived in the previous sections for the case where the signal is a monocomponent signal and the measurements of the signal are known to be at a high SNR. In other words, we assume here that $\sigma^2 \to 0$. Hence, a first-order approximation of $\boldsymbol{D}^{-1}$ yields

$$\boldsymbol{D}^{-1} \approx \boldsymbol{I}_N - \frac{\sigma^2}{2}\boldsymbol{R}_s^{-1}. \tag{87}$$

Thus, (70) can be approximated by

$$\begin{aligned} \boldsymbol{\Gamma}^{-1} &\approx \frac{2}{\sigma^2}\boldsymbol{I}_{2N} - \frac{2}{\sigma^2}\boldsymbol{X}\left(\boldsymbol{I}_N - \frac{\sigma^2}{2}\boldsymbol{R}_s^{-1}\right)\boldsymbol{X}^T \\ &= \frac{2}{\sigma^2}(\boldsymbol{I}_{2N} - \boldsymbol{X}\boldsymbol{X}^T) + \boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T \\ &= \frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T + \boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T. \end{aligned} \tag{88}$$

Substituting (87) and (88) into the equations of the nonzero elements of the FIM (86) yields the monocomponent signal FIM for the case in which the measurements of the signal are known to be at a high SNR. In particular, substituting (87) into (73) we have that for the zero-mean case

$$\begin{aligned} J_{k,\ell}^{b,b} &= \frac{4}{\sigma^4}\mathrm{tr}\left\{\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\frac{\sigma^2}{2}\boldsymbol{R}_s^{-1}\boldsymbol{R}_s\boldsymbol{V}^T\boldsymbol{H}_\ell\right\} \\ &= \frac{2}{\sigma^2}\mathrm{tr}\{\boldsymbol{V}^T\boldsymbol{H}_\ell\boldsymbol{H}_k\boldsymbol{V}\boldsymbol{R}_s\} \\ &= \frac{2}{\sigma^2}\mathrm{tr}\{\boldsymbol{T}_k\boldsymbol{R}_s\boldsymbol{T}_\ell\} \end{aligned} \tag{89}$$

where we have used (69) and the commutativity property of the trace operator. For the nonzero mean case we have, using (80) and (89)

$$J_{k,\ell}^{b,b} = \frac{2}{\sigma^2}\boldsymbol{c}^T\boldsymbol{\rho}^T\boldsymbol{T}_k\boldsymbol{T}_\ell\boldsymbol{\rho}\boldsymbol{c} + \frac{2}{\sigma^2}\mathrm{tr}\{\boldsymbol{T}_k\boldsymbol{R}_s\boldsymbol{T}_\ell\}. \tag{90}$$

We therefore conclude that in contrast with the general case in which the bound on the phase parameters is a function of both the phase waveform and basis functions, when the SNR is high this bound is independent of the phase waveform. It depends on the general class of functions to which the phase function belongs (through the phase basis functions $\{\psi_n(t)\}$), but not on the specific values of the phase parameters. For example, if we represent the phase as a polynomial function of time, all signals whose phase is a Q_1th order polynomial of time, and whose amplitude have the same covariance function and time varying mean, will have the same values for the CRB of the phase parameters.

Using (41) and (88), we also have that

$$\begin{aligned} \boldsymbol{\Gamma}^{-1}\frac{\partial\boldsymbol{\Gamma}}{\partial a_k} &= \left(\frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T + \boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T\right)\left(\boldsymbol{X}\frac{\partial\boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\right) \\ &= \frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T\boldsymbol{X}\frac{\partial\boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T + \boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T\boldsymbol{X}\frac{\partial\boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T \\ &= \boldsymbol{X}\boldsymbol{R}_s^{-1}\frac{\partial\boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T \end{aligned} \tag{91}$$

where the last equality follows from (64) and (65). Hence, using (65) and the trace operator commutative property, we obtain

$$\begin{aligned} J_{k,\ell}^{a,a} &= \frac{1}{2}\mathrm{tr}\left\{\boldsymbol{X}\boldsymbol{R}_s^{-1}\frac{\partial\boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{R}_s^{-1}\frac{\partial\boldsymbol{R}_s}{\partial a_\ell}\boldsymbol{X}^T\right\} \\ &= \frac{1}{2}\mathrm{tr}\left\{\boldsymbol{R}_s^{-1}\frac{\partial\boldsymbol{R}_s}{\partial a_k}\boldsymbol{R}_s^{-1}\frac{\partial\boldsymbol{R}_s}{\partial a_\ell}\right\}. \end{aligned} \tag{92}$$

Note that the elements of the FIM that correspond to parameters of the random component of the amplitude are *identical* to the expressions we would have obtained if this component was measured directly (i.e., if the modulation by $e^{j\phi(t)}$ did not take place and the observations were noise free). Thus, we can use here any available expressions for the FIM of real stationary Gaussian processes. For example, if the amplitude was known to be a zero-mean AR or ARMA process, we could use the expressions presented in [7] and [6].

Substituting (88) and (91) into (61) while using (64), (65), and the commutativity of the trace operator gives

$$\begin{aligned} J_{k,1}^{a,\sigma^2} &= \frac{1}{4}\mathrm{tr}\left\{ \boldsymbol{X}\boldsymbol{R}_s^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T \right\} \\ &\quad + \frac{1}{4}\mathrm{tr}\left\{ \boldsymbol{X}\boldsymbol{R}_s^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T \right\} \\ &= \frac{1}{4}\mathrm{tr}\left\{ \boldsymbol{R}_s^{-1}\frac{\partial \boldsymbol{R}_s}{\partial a_k}\boldsymbol{R}_s^{-1} \right\} \\ &= -\frac{1}{4}\mathrm{tr}\left\{ \frac{\partial \boldsymbol{R}_s^{-1}}{\partial a_k} \right\}. \end{aligned} \tag{93}$$

The FIM entry that corresponds to the noise parameter is given by

$$\begin{aligned} J^{\sigma^2,\sigma^2} &= \frac{1}{8}\mathrm{tr}\left\{ \frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T\frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T \right\} \\ &\quad + \frac{1}{8}\mathrm{tr}\left\{ \frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T\boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T \right\} \\ &\quad + \frac{1}{8}\mathrm{tr}\left\{ \boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T\frac{2}{\sigma^2}\boldsymbol{V}\boldsymbol{V}^T \right\} \\ &\quad + \frac{1}{8}\mathrm{tr}\{ \boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{R}_s^{-1}\boldsymbol{X}^T \} \\ &= \frac{1}{2\sigma^4}\mathrm{tr}\{(\boldsymbol{V}\boldsymbol{V}^T)^2\} + \frac{1}{8}\mathrm{tr}\{\boldsymbol{R}_s^{-1}\boldsymbol{R}_s^{-1}\} \\ &= \frac{N}{2\sigma^4} + \frac{1}{8}\mathrm{tr}\{\boldsymbol{R}_s^{-1}\boldsymbol{R}_s^{-1}\}. \end{aligned} \tag{94}$$

Finally, substituting (87) into (83), we have

$$J_{k,\ell}^{c,c} = \boldsymbol{\rho}_k^T\boldsymbol{R}_s^{-1}\boldsymbol{\rho}_\ell. \tag{95}$$

Note that as $\sigma^2 \to 0, \boldsymbol{J}^{b,b}$ becomes singular, and hence, the phase of the signal can be perfectly estimated, regardless of the structure of amplitude covariance matrix $\boldsymbol{R}_s$ or the amplitude mean waveform. This result is due to the fact that in the absence of observation noise, the phase of the measured monocomponent signal $y(t)$ can be easily obtained by dividing the imaginary part of the measured signal by its real part.

D. Bounds for Functions of the Parameters

In most cases, we are interested not in the phase or amplitude parameters themselves but in estimating some function of these parameters. For example, we may be interested in estimating the signal or its individual components. Having estimated the model parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, the signal mean, spectral density, and phase functions can be computed using their known functional dependence on the (estimated) parameters. Other quantities of considerable practical interest are the instantaneous frequencies of the signal components, which can be computed in a straightforward manner by evaluating the derivative of the phase function.

In this section, we derive the CRB for the signal mean, spectral density, phase, and instantaneous frequency. Since all of these quantities are functions of the estimated parameters, we will use the following generalized form of the CRB.

Let β be a continuous and differentiable function of the parameter vector $\boldsymbol{\theta}$, i.e., $\beta = f(\boldsymbol{\theta})$. Then, the CRB for β is related to the CRB of $\boldsymbol{\theta}$ by

$$\mathrm{CRB}(\beta) = (\boldsymbol{f}')^T\mathrm{CRB}(\boldsymbol{\theta})\boldsymbol{f}' \tag{96}$$

where the column vector $\boldsymbol{f}'$ is defined by

$$\boldsymbol{f}' = \frac{\partial f}{\partial \boldsymbol{\theta}} \tag{97}$$

(see, *e.g.*, [1]). Using this formula, the CRB for the desired functions can be computed in a straightforward manner.

The instantaneous phase is defined in (9) as a linear vector function of $\boldsymbol{b}$. The CRB on the phase of the ith component is, therefore, given by

$$\mathrm{CRB}(\phi_i(t)) = \boldsymbol{g}_i^T\ \mathrm{CRB}(\boldsymbol{b}_i)\ \boldsymbol{g}_i \tag{98}$$

where

$$\begin{aligned} \boldsymbol{g}_i^T &= \left[\frac{\partial \phi_i}{\partial b_{i,0}}, \cdots, \frac{\partial \phi_i}{\partial b_{i,Q_i}} \right] \\ &= [\psi_{i,0}(t), \psi_{i,1}(t), \cdots, \psi_{i,Q_i}(t)]. \end{aligned} \tag{99}$$

The instantaneous frequency is the time derivative of the instantaneous phase

$$\omega_i(t) = \frac{1}{2\pi}\phi_i'(t) = \frac{1}{2\pi}\sum_{\ell=0}^{Q_i} b_{i,\ell}\psi_{i,\ell}'(t) \tag{100}$$

and therefore

$$\mathrm{CRB}(\omega_i(t)) = \boldsymbol{h}_i^T\ \mathrm{CRB}(\boldsymbol{b}_i)\ \boldsymbol{h}_i \tag{101}$$

where

$$\begin{aligned} \boldsymbol{h}_i^T &= \left[\frac{\partial \omega_i}{\partial b_{i,0}}, \cdots, \frac{\partial \omega_i}{\partial b_{i,Q_i}} \right] \\ &= \frac{1}{2\pi}[\psi_{i,0}'(t), \psi_{i,1}'(t), \cdots, \psi_{i,Q_i}'(t)]. \end{aligned} \tag{102}$$

Similarly, the CRB on the mean of the ith signal amplitude is

$$\mathrm{CRB}(m_i(t)) = \boldsymbol{f}_i^T\ \mathrm{CRB}(\boldsymbol{c}_i)\ \boldsymbol{f}_i \tag{103}$$

where

$$\begin{aligned} \boldsymbol{f}_i^T &= \left[\frac{\partial m_i}{\partial c_{i,0}}, \cdots, \frac{\partial m_i}{\partial c_{i,P_i}} \right] \\ &= [\rho_{i,0}(t), \rho_{i,1}(t), \cdots, \rho_{i,P_i}(t)]. \end{aligned} \tag{104}$$

Finally, the CRB for the spectral density function $S_i(e^{j\omega})$ of the amplitude of the ith component, is given by

$$\mathrm{CRB}(S_i(e^{j\omega})) = \boldsymbol{W}_i^T\ \mathrm{CRB}(\boldsymbol{a}_i)\ \boldsymbol{W}_i \tag{105}$$

where

$$\boldsymbol{W}_i^T = \left[\frac{\partial S_i(e^{j\omega})}{\partial a_{i,0}}, \cdots, \frac{\partial S_i(e^{j\omega})}{\partial a_{i,O_i}}\right] \tag{106}$$

and O_i is the number of parameters of the ith signal random component model.

E. Example: Single AR Component

So far, we have not specified the functional dependence of the covariance matrix $\boldsymbol{R}_s$ on the parameters $\boldsymbol{a}$. In this section, we consider a special case in which the amplitude function is a zero-mean Kth-order autoregressive process. In this case, the covariance matrix $\boldsymbol{R}_s$ is given as an explicit and relatively simple function of the AR parameters. Thus, we can derive closed-form formulas for the CRB. These formulas will be used in Section V to illustrate by means of numerical examples the results derived in this paper.

A real Gaussian AR process $s(t)$ obeys the recursion

$$s(t) = -\sum_{k=1}^{K} \alpha_k s(t-k) + u(t) \tag{107}$$

where $\{u(t)\}$ is a stationary, zero-mean Gaussian white noise with variance σ_{AR}^2.

It can be shown [8] that the inverse covariance matrix $\boldsymbol{R}_s^{-1}$ of a Kth-order AR process $(N \geq K)$ is given by

$$\boldsymbol{R}_s^{-1} = \frac{1}{\sigma_{AR}^2}(\boldsymbol{A}_1\boldsymbol{A}_1^T - \boldsymbol{A}_2\boldsymbol{A}_2^T) \tag{108}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are lower triangular Toeplitz matrices such that

$$(\boldsymbol{A}_1)_{i,j} = \begin{cases} 1, & i=j \\ \alpha_{i-j}, & i>j \; ; \\ 0, & i<j \end{cases} \tag{109}$$

$$(\boldsymbol{A}_2)_{i,j} = \begin{cases} \alpha_{N-i+j} & i \geq j \\ 0, & i<j \end{cases} \tag{110}$$

and $\alpha_k = 0$ for $k< 0$ and $k>K$. Hence

$$\boldsymbol{R}_s = \sigma_{AR}^2(\boldsymbol{A}_1\boldsymbol{A}_1^T - \boldsymbol{A}_2\boldsymbol{A}_2^T)^{-1}. \tag{111}$$

Substituting (111) into (81) (or (90) for the high SNR case), we obtain a closed-form expression for $\boldsymbol{J}_{k,\ell}^{b,b}(\boldsymbol{\theta})$ in terms of the observed signal parameters. Similar substitution into (79) (or (94)) results in a closed-form expression for $\boldsymbol{J}^{\sigma^2,\sigma^2}$.

In the present case, the parameter vector defining the covariance matrix of the amplitude is the $K+1$ dimensional vector $\boldsymbol{a} = [\sigma_{AR}^2, \alpha_1, \alpha_2, \cdots, \alpha_K]^T$. Taking the partial derivatives of $\boldsymbol{R}_s^{-1}$ using (108), we have

$$\frac{\partial \boldsymbol{R}_s^{-1}}{\partial \sigma_{AR}^2} = -\frac{1}{\sigma_{AR}^4}(\boldsymbol{A}_1\boldsymbol{A}_1^T - \boldsymbol{A}_2\boldsymbol{A}_2^T) = -\frac{1}{\sigma_{AR}^2}\boldsymbol{R}_s^{-1} \tag{112}$$

$$\frac{\partial \boldsymbol{R}_s^{-1}}{\partial \alpha_n} = \frac{1}{\sigma_{AR}^2}(\boldsymbol{Z}_n\boldsymbol{A}_1^T + \boldsymbol{A}_1\boldsymbol{Z}_n^T - \boldsymbol{Z}_{N-n}\boldsymbol{A}_2^T - \boldsymbol{A}_2\boldsymbol{Z}_{N-n}^T) \quad n = 1, \cdots, K \tag{113}$$

where $\boldsymbol{Z}_n$ is the down shift matrix

$$(\boldsymbol{Z}_n)_{i,j} = \begin{cases} 1, & i-j=n \\ 0, & \text{otherwise.} \end{cases} \tag{114}$$

Since

$$\frac{\partial \boldsymbol{R}_s}{\partial a_k} = -\boldsymbol{R}_s \frac{\partial \boldsymbol{R}_s^{-1}}{\partial a_k} \boldsymbol{R}_s \tag{115}$$

we obtain by substituting (115), (108), (112), and (113) into the general expressions of $\boldsymbol{J}^{a,a}(\boldsymbol{\theta})$ in (75) and $\boldsymbol{J}^{a,\sigma^2}(\boldsymbol{\theta})$ in (78) closed-form exact expressions for $\boldsymbol{J}^{a,a}(\boldsymbol{\theta})$ and $\boldsymbol{J}^{a,\sigma^2}(\boldsymbol{\theta})$.

In particular, for the case in which the observations are known to be at a high SNR, substituting (108), (112), (113), and (115) into (92) yields

$$\boldsymbol{J}_{1,1}^{a,a} = \frac{N}{2\sigma_{AR}^4} \tag{116}$$

and

$$\boldsymbol{J}_{k,1}^{a,a} = -\frac{1}{2\sigma_{AR}^2}\text{tr}\left\{\frac{\partial \boldsymbol{R}_s^{-1}}{\partial \alpha_{k-1}}\boldsymbol{R}_s\right\} \tag{117}$$

for $k \geq 2$. Note that $\boldsymbol{J}_{k,\ell}^{a,a}(\boldsymbol{\theta})$ is given by the expression of the exact FIM of a stationary Gaussian autoregressive process, as if the modulation by $\exp j\phi(t)$ did not take place. This expression was previously derived in [7]. Substituting (108), (112), and (113) into (93) yields

$$\boldsymbol{J}_{1,1}^{a,\sigma^2} = \frac{1}{4\sigma_{AR}^4}\text{tr}\{\boldsymbol{R}_s^{-1}\} \tag{118}$$

and

$$\boldsymbol{J}_{k,1}^{a,\sigma^2} = -\frac{1}{4}\text{tr}\left\{\frac{\partial \boldsymbol{R}_s^{-1}}{\partial \alpha_{k-1}}\right\} \tag{119}$$

for $k \geq 2$.

The CRB on the spectral density function of the amplitude is given by (105), where

$$\boldsymbol{W}_i^T = -2S(e^{j\omega})\,\text{Re}\left[-\frac{1}{2\sigma_{AR}^2}, \frac{e^{j\omega(P-1)}}{\alpha(e^{j\omega})}, \frac{e^{j\omega(P-2)}}{\alpha(e^{j\omega})}, \cdots, \frac{1}{\alpha(e^{j\omega})}\right] \tag{120}$$

and

$$\alpha(e^{j\omega}) = e^{j\omega P} + \alpha_1 e^{j\omega(P-1)} + \cdots + \alpha_P. \tag{121}$$

V. Numerical Examples

To gain more insight into the behavior of the bound, we resort to numerical evaluation of specific examples. In these examples, we restrict our attention to the case of a single component signal whose measurements are known to be at a high SNR. In all of the examples presented in this section, the white Gaussian observation noise is of variance $\sigma^2 = 0.01$, the time axis shows the sampling time, and the samples are equispaced.

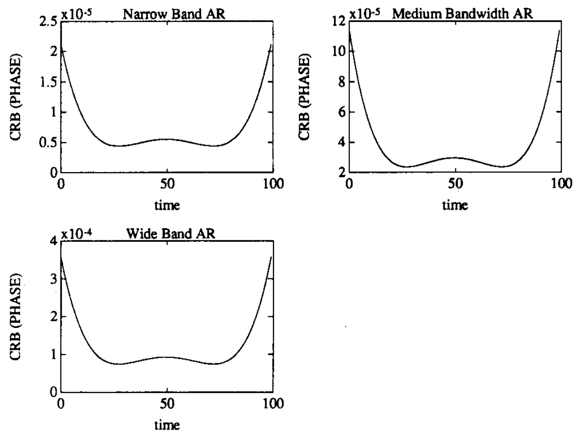


Fig. 1. CRB on the instantaneous phase of a chirp signal whose amplitude is a zero-mean Gaussian AR process. We consider narrowband, medium bandwidth, and wideband AR processes.

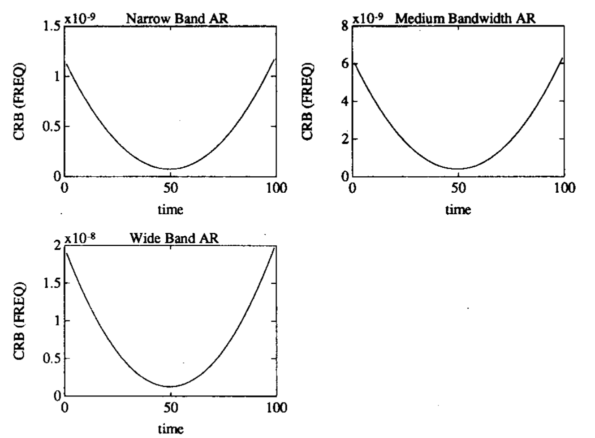


Fig. 2. CRB on the instantaneous frequency of a chirp signal whose amplitude is a zero-mean Gaussian AR process. We consider narrowband, medium bandwidth, and wideband AR processes.

TABLE I
THREE TEST CASES OF ZERO MEAN AR COMPONENTS

Test Case	σ^2_{AR}	α_1	α_2
Narrow Band AR	1	-1.378	0.95
Medium Bandwidth AR	1	-1.183	0.7
Wideband AR	1	-0.447	0.1

Example 1: Consider the case of three chirp signals, such that for all three, the signal amplitude is a zero-mean second-order Gaussian AR processes and the phase is a second-order time polynomial, whose parameter vector is $\boldsymbol{b} = [\pi/3, 0.04\pi, 0.001\pi]$. The phase is the same for all three signals. The first amplitude signal is a narrowband AR process, the second is of medium bandwidth, and the third is a wideband AR process. The parameters of the three amplitude components are given in Table I.

As can be seen from Figs. 1 and 2, the bounds on the error variance for estimating the instantaneous phase and frequency of the signal are inversely proportional to the bandwidth of the signal amplitude.

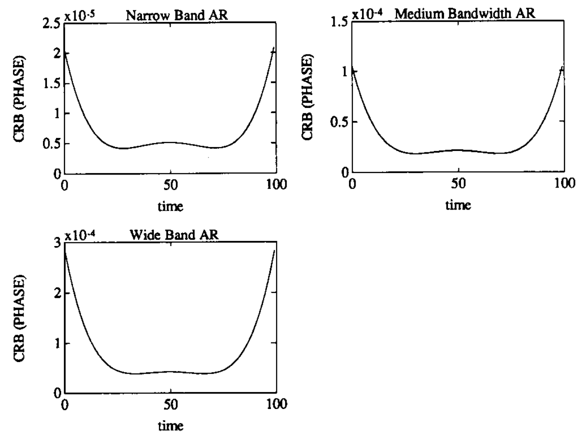


Fig. 3. CRB on the instantaneous phase of a chirp signal whose amplitude function is the sum of a time-varying mean and a zero-mean Gaussian AR process (Example 2)—narrowband, medium bandwidth, and wideband AR processes.

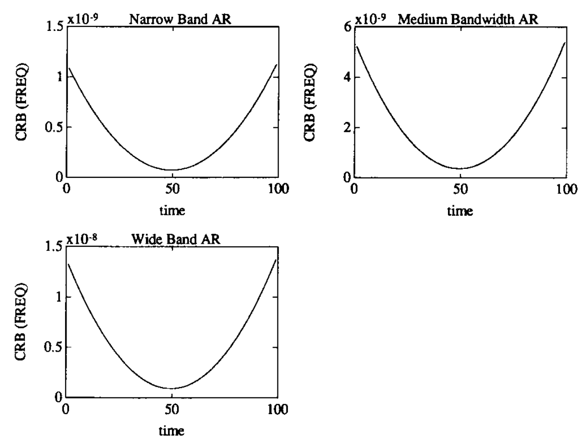


Fig. 4. CRB on the instantaneous frequency of a chirp signal whose amplitude function is the sum of a time-varying mean and a zero-mean Gaussian AR process (Example 2)—narrowband, medium bandwidth, and wideband AR processes.

Example 2: The second set of examples involves three nonzero-mean chirp signals. For all three, the mean of the signal amplitude is a second-order time polynomial whose parameter vector is $\boldsymbol{c} = [0.02672, 0.05343, -0.0005343]$.The amplitude parameters were chosen such that the amplitude energy in the observed time interval is equal to one. The amplitude random components are the zero-mean second-order Gaussian AR processes of Example 1. The time-varying phase is the same as in Example 1, i.e., $\boldsymbol{b} = [\pi/3, 0.04\pi, 0.001\pi]$.

Note that for the case in which the amplitude function is the sum of a time-varying deterministic mean and a wideband AR process, the CRB on the instantaneous frequency and phase of the signal is slightly lower than in the zero-mean case. However, the presence of the low-energy deterministic mean had little or no effect at all on the bounds for estimating the phase and frequency in the case of spectrally narrower random components.

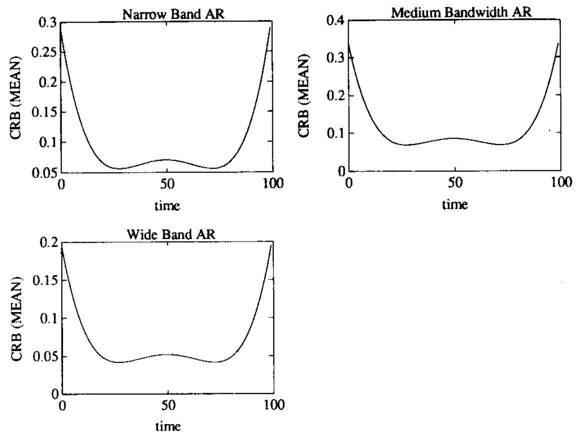


Fig. 5. CRB on the mean of the instantaneous amplitude. The signal is a chirp whose amplitude function is the sum of a time-varying mean and a zero-mean Gaussian AR process (Example 2)—narrowband, medium bandwidth, and wideband AR processes.

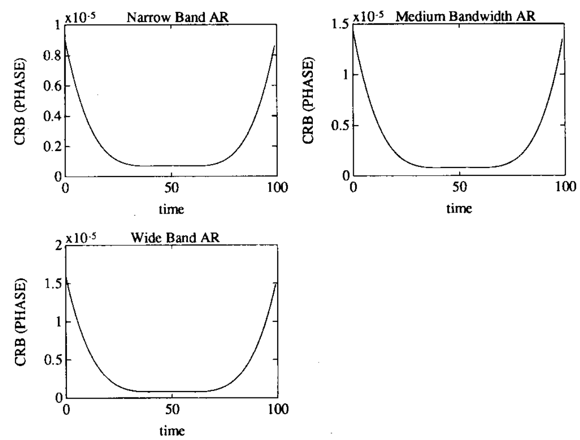


Fig. 6. CRB on the instantaneous phase of a chirp signal whose amplitude function is the sum of a time-varying mean and a zero-mean Gaussian AR process (Example 3)—narrowband, medium bandwidth, and wideband AR processes.

Example 3: Consider the same scenario as that of Example 2. However, in the present case the energy of the time-varying mean in the observed interval is 100 instead of one. In this case, the bounds on both the phase and frequency functions (Figs. 6 and 7), are considerably lower than in the case of a low energy time-varying mean, while the bound on the time-varying mean of the signal amplitude is unchanged since it is independent of the mean parameters (see (83) and (86)).

For examples that involve deterministic amplitude and phase function, i.e., with no random component for the amplitude function, we refer the interested reader to [2].

VI. Conclusions

In this paper, we presented a study of the achievable accuracy in estimating the phase and amplitude parameters of a class of nonstationary multicomponent signals. In the case of a single component signal, we were able to make some observations regarding the decoupling of the estimation

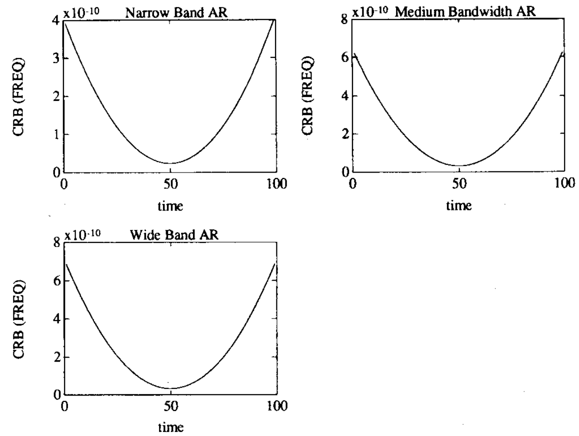


Fig. 7. CRB on the instantaneous frequency of a chirp signal whose amplitude function is the sum of a time-varying mean and a zero-mean Gaussian AR process (Example 3)—narrowband, medium bandwidth, and wideband AR processes.

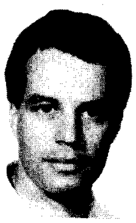
of the phase and amplitude parameters. In general, the phase and amplitude estimation are coupled, and the expressions are difficult to interpret without numerical evaluation.

Parametric modeling appears to be a promising approach to the analysis of nonstationary signals. The results presented here and in [2] and [3] represent some preliminary steps in the development of new techniques for estimation, detection, and classification of multicomponent amplitude and frequency modulated signals.

References

[1] C. R. Rao, *Linear Statistical Inference and Its Applications.* New York: Wiley, 1965.
[2] B. Friedlander and J. M. Francos, "Estimation of amplitude and phase parameters of multicomponent signals," IEEE *Trans. Signal Processing*, vol. 43, no. 4, pp. 917–926, Apr. 1995.
[3] ——, "Algorithms for estimation of multicomponent signals with random amplitude and deterministic phase," in preparation.
[4] S. Zacks, *The Theory of Statistical Inference.* New York: Wiley, 1971.
[5] S. M. Kay, *Modern Spectral Estimation.* Englewood Cliffs, NJ: Prentice-Hall, 1988.
[6] B. Porat and B. Friedlander, "Computation of the exact information matrix of Gaussian time series with stationary random components," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, no. 1, pp. 118–130, Feb. 1986.
[7] ——, "The exact Cramér-Rao bound for Gaussian autoregressive processes," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-23, no. 4, pp. 537–542, July 1987.
[8] T. Kailath, A. Vieira, and M. Morf, "Inverses of Toeplitz operators, innovations, and orthogonal polynomials," *SIAM Rev.*, vol. 20, no. 1, pp. 118–130, Jan. 1978.
[9] S. Peleg and B. Friedlander, "A technique for estimation the parameters of multiple polynomial phase signals," presented at 1992 Symp. Time-Frequency Time-Scale Anal., Victoria, Canada, Oct. 4–6, 1992.
[10] S. Peleg, "Estimation and detection with the discrete polynomial transform," Ph.D. dissertation, Dep. Elec. Comput. Eng., Univ. of California, Davis, CA, 1993.
[11] B. Friedlander, "Parametric signal analysis using the polynomial phase transform," *Proc. IEEE Signal Processing Workshop Higher Order Statistics*, Stanford Sierra Camp, South Lake Tahoe, June 7–9, 1993.
[12] R. Kumaresan and S. Verma, "On estimating the parameters of chirp signals using rank reduction techniques," in *Proc. Twenty-First Asilomar Conf. Signals, Syst., Comput.*, Nov. 1987, pp. 555–558.
[13] R. M. Liang and K. S. Arun, "Parameter estimation for superimposed chirp signals," *Proc. 1992 ICASSP*, San Francisco, CA, Mar. 23–27, 1992.

[14] S. Shamsunder and G. B. Giannakis, "Ambiguity functions, polynomial phase signals and higher-order cyclostationarity," in *Proc. IEEE Signal Processing Workshop Higher-Order Statistics*, June 7-9, 1993, pp. 173–177.
[15] ——, "Estimation of random amplitude polynomial phase signals using higher-order cyclic cumulants," in *Proc. Conf. Inform. Sciences Syst.*, Baltimore, MD, Mar. 24–26, 1993.



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