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Linear Least Squares Estimation of Regression Models for Two-Dimensional Random Fields[1]

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We consider the problem of estimating regression models of two-dimensional random fields. Asymptotic properties of the least squares estimator of the linear regression coefficients are studied for the case where the disturbance is a homogeneous random field with an absolutely continuous spectral distribution and a positive and piecewise continuous spectral density. We obtain necessary and sufficient conditions on the regression sequences such that a linear estimator of the regression coefficients is asymptotically unbiased and mean square consistent. For such regression sequences the asymptotic covariance matrix of the linear least squares estimator of the regression coefficients is derived.

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1. INTRODUCTION

In this paper we consider the problem of estimating the coefficients of a regression model of a two-dimensional random field. The disturbance field is a homogeneous random field with an absolutely continuous spectral distribution and a positive and piecewise continuous spectral density. In its simplest form, the problem is reduced to that of estimating the regression coefficient γ where the observed field is given by

$$y_{u,v} = \gamma \varphi_{u,v} + \epsilon_{u,v} \tag{1}$$

such that $\varphi_{u,v}$ is a given regression sequence and $\epsilon_{u,v}$ is a zero mean homogeneous random field. We shall restrict ourselves to estimates of the regression coefficients that are linear in the observations $\{y_{u,v}: 0 \leqslant u \leqslant M-1, 0 \leqslant v \leqslant N-1\}$. In the case where the noise $\epsilon_{u,v}$ is Gaussian, linear estimates

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are optimal. However, in many cases $\epsilon_{u,v}$ will not be normal and there will be nonlinear estimates of γ better than the linear estimates. Nonetheless, evaluation of the nonlinear estimates are often computationally prohibitive, especially in the case of multidimensional processes. In such cases the computational simplicity in evaluating the linear estimates makes them particularly appealing. The problems of one-dimensional linear and nonlinear regression have been extensively investigated, *e.g.*, [1, 3, 6, 8, 12]. More recently, a necessary condition for the consistency of l_1 estimates in linear regression models was derived in [2]. Estimation of multivariate regression functions of stationary random processes with errors-in-variables is addressed by Elias, [4], where the problems of strong consistency of the estimates and their uniform convergence rate are explored. Nonlinear regression models and the conditions for the consistency of l_α estimators are considered in [11].

The problem of estimating the parameters of multidimensional processes has received considerably less attention. Moreover, most of the of literature is concerned with estimating the regression models when the observation noise is uncorrelated. Leonenko [13], and more recently Ivanov and Leonenko [14], studied the properties of least-squares estimates of the linear regression coefficients for the case where a random field is observable on a system of subsets whose dimensions tend to infinity, and the disturbance field is homogeneous. An expression for the asymptotic normalized covariance of the estimates was developed, assuming the regressors are square-integrable functions on each of the subsets. In [15], Leonenko has introduced conditions under which distributions of functionals of least squares estimators for linear regression coefficients converge to distributions of the functionals of some Gaussian random field.

In this paper we derive necessary and sufficient conditions on the regression sequences such that a linear estimator of the regression coefficients is asymptotically unbiased and mean square consistent. As shown in the next section, for a linear estimator of the regression coefficients to be mean square consistent, the regressors should not belong to l_2. For such regression sequences the asymptotic covariance matrix of the linear least squares estimator of the regression coefficients is derived. In [13, 14], on the other hand, by concentrating on regression functions that are square-integrable on the above system of subsets an expression for the asymptotic normalized covariance of the regression coefficients estimates, is derived. Thus the resulting linear least-squares estimates of the regression coefficients are not necessarily mean square consistent. Therefore, the results of [13–16], though related, are not directly applicable to the framework of mean square consistent linear estimators for two-dimensional random fields, addressed in this paper. Moreover, the approach adopted in this paper is different from that of [13, 14], as we first establish conditions on the regressors such that the linear estimator of the regression coefficients is asymptotically unbiased and mean square consistent. Then, for this class of regressors we derive the asymptotic normalized covariance matrix of the linear least squares estimator of the regression coefficients. The result is established for the case where the spectral density function of the disturbance field is piecewise continuous and has no common discontinuities with the discontinuities of the spectral distribution function of the regression vectors, while the results in [13, 14] are given for the more restricted case where this spectral density is continuous.

Let $\mathbf{y}$, $\boldsymbol{\varphi}$, $\boldsymbol{\varepsilon}$ denote the observation, regression, and disturbance column vectors, respectively, where

$$\mathbf{y} = [y(0,0), ..., y(M-1,0), y(0,1), ..., y(M-1,1), ..., ..., y(0,N-1), ..., y(M-1,N-1)]^T,$$

and $\boldsymbol{\varphi}$, $\boldsymbol{\varepsilon}$ are similarly defined. Let $\boldsymbol{\Gamma}$ denote the covariance matrix of $\boldsymbol{\varepsilon}$ and hence of $\mathbf{y}$ as well. Thus,

$$\boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}^{(0)} & \boldsymbol{\Gamma}^{(-1)} & \cdots & \boldsymbol{\Gamma}^{(1-N)} \\ \boldsymbol{\Gamma}^{(1)} & \boldsymbol{\Gamma}^{(0)} & \cdots & \boldsymbol{\Gamma}^{(2-N)} \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{\Gamma}^{(N-1)} & \boldsymbol{\Gamma}^{(N-2)} & \cdots & \boldsymbol{\Gamma}^{(0)} \end{pmatrix}, \tag{2}$$

where

$$\boldsymbol{\Gamma}^{(k)} = \begin{pmatrix} r_{0,k} & r_{-1,k} & \cdots & r_{1-M,k} \\ r_{1,k} & r_{0,k} & \cdots & r_{2-M,k} \\ \vdots & \vdots & \cdots & \vdots \\ r_{M-1,k} & r_{M-2,k} & \cdots & r_{0,k} \end{pmatrix}. \tag{3}$$

In the following, the meaning of the notation $M, N \to \infty$, used throughout, is that $\min(M, N) \to \infty$.

THEOREM 1. *Suppose there exists a sequence of linear estimates $c_{M,N}$ of γ where*

$$c_{M,N} = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} a_{u,v}^{(M,N)} y_{u,v}. \tag{4}$$

Assume also that the spectral density $f(\omega, v)$ of the disturbance field is bounded away from zero. A necessary condition for $c_{M,N}$ to be both mean square consistent and asymptotically unbiased estimate of γ is that

$$\lim_{M,N\to\infty} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\varphi_{u,v}|^2 = \infty. \tag{5}$$

Proof. By the next identity (see Appendix A for the proof) we have that

$$E\,|c_{M,N}-\gamma|^2 = E\,|c_{M,N}-E(c_{M,N})|^2 + |E(c_{M,N})-\gamma|^2$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} a_{u,v}^{(M,N)} e^{i2\pi(\omega u+\nu v)} \right|^2 f(\omega, \nu)\, d\omega\, d\nu$$
$$+ |\gamma|^2 \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} a_{u,v}^{(M,N)} \varphi_{u,v} - 1 \right|^2. \tag{6}$$

For $c_{M,N}$ to be a consistent estimate of γ, the R.H.S. of (6) has to tend to zero. Hence,

$$\lim_{M,N\to\infty} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} a_{u,v}^{(M,N)} \varphi_{u,v} = 1. \tag{7}$$

Using (4), (7), and (1) we conclude that $c_{M,N}$ is an asymptotically unbiased estimate of γ. By Parseval's equality

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} a_{u,v}^{(M,N)} e^{i2\pi(\omega u+\nu v)} \right|^2 f(\omega, \nu)\, d\omega\, d\nu$$
$$\geqslant \min_{\omega,\nu} f(\omega, \nu) \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |a_{u,v}^{(M,N)}|^2.$$

Hence, assuming $f(\omega, \nu)$ is bounded away from zero, for $c_{M,N}$ to be a consistent estimate of γ we must have

$$\lim_{M,N\to\infty} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |a_{u,v}^{(M,N)}|^2 = 0. \tag{8}$$

Since by the triangle inequality

$$\left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} a_{u,v}^{(M,N)} \varphi_{u,v} \right| \leqslant \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |a_{u,v}^{(M,N)} \varphi_{u,v}|, \tag{9}$$

we have from (7) that

$$\lim_{M,N\to\infty} \left(\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |a_{u,v}^{(M,N)} \varphi_{u,v}| \right)^2 \geqslant 1. \tag{10}$$

However, from the Schwartz inequality we have that

$$\left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|a_{u,v}^{(M,N)}\varphi_{u,v}|\right)^2 \leqslant \left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|a_{u,v}^{(M,N)}|^2\right)\left(\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|\varphi_{u,v}|^2\right). \tag{11}$$

Thus, using (8)–(11) we conclude that

$$\lim_{M,N\to\infty}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|\varphi_{u,v}|^2=\infty \tag{12}$$

is a necessary condition for $c_{M,N}$ to be a mean square consistent estimate of γ. ∎

THEOREM 2. *Assume the disturbance spectral density $f(\omega, v)$ is bounded and strictly positive. Let*

$$c_{M,N}=\frac{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\bar{\varphi}_{u,v}\,y_{u,v}}{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|\varphi_{u,v}|^2}. \tag{13}$$

Then $c_{M,N}$ is an unbiased estimate of γ and (5) is a necessary and sufficient condition for $c_{M,N}$ to be a mean square consistent estimate of γ.

Proof. See Appendix B.

In the foregoing discussion we have concentrated on the case where the observed field is expressed in terms of a single regressor and an additive disturbance. In the more general case, the observation is a linear combination of some number $P << \min(M, N)$ of regressors and an additive disturbance such that

$$\mathbf{y}=\mathbf{\Phi}\boldsymbol{\gamma}+\boldsymbol{\varepsilon},$$

where $\boldsymbol{\gamma}=[\gamma_1,\gamma_2,\dots,\gamma_P]^T$, $\mathbf{\Phi}=[\boldsymbol{\varphi}^{(1)},\boldsymbol{\varphi}^{(2)},\dots,\boldsymbol{\varphi}^{(P)}]$ is $MN\times P$ matrix of known constants such that

$$\boldsymbol{\varphi}^{(j)}=[\varphi_{0,0}^{(j)},\varphi_{1,0}^{(j)},\dots,\varphi_{M-1,0}^{(j)},\dots,\varphi_{M-1,N-1}^{(j)}]^T.$$

Thus $E\mathbf{y}=\mathbf{\Phi}\boldsymbol{\gamma}$.

Let $\mathbf{c}=[c_1,\dots,c_P]^T$ denote the linear least squares estimate of $\boldsymbol{\gamma}$. Provided that $\mathbf{\Phi}^H\mathbf{\Phi}$ is non-singular the least squares estimate

$$\mathbf{c}_L=(\mathbf{\Phi}^H\mathbf{\Phi})^{-1}\mathbf{\Phi}^H\mathbf{y} \tag{14}$$

is an unbiased estimator of γ. Furthermore, if $\mathbf{\Gamma}$ is non-singular then among all the linear estimators, there exists an optimal one

$$c_o = (\mathbf{\Phi}^H \mathbf{\Gamma}^{-1} \mathbf{\Phi})^{-1} \, \mathbf{\Phi}^H \mathbf{\Gamma}^{-1} \mathbf{y}. \tag{15}$$

2. THE REGRESSION SPECTRUM

Consider for a fixed r the sequence

$$\Phi^{(r)}_{M,N} = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\varphi^{(r)}_{u,v}|^2 \tag{16}$$

From (5) we have that $\Phi^{(r)}_{M,N}$ approaches infinity with M and N since otherwise no consistent estimate of γ, exists. We further assume that $\Phi^{(r)}_{MN}$ is a *slowly increasing* sequence of M and N, *i.e.*, for every fixed h, k

$$\lim_{M,N \to \infty} \frac{\Phi^{(r)}_{M+h,N+k}}{\Phi^{(r)}_{M,N}} = 1. \tag{17}$$

Using the Schwartz inequality we have that

$$\begin{aligned} \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \varphi^{(r)}_{u+h,v+k} \bar{\varphi}^{(s)}_{u,v} \right| &\leqslant \left(\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\varphi^{(r)}_{u+h,v+k}|^2 \right)^{\frac{1}{2}} \left(\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\bar{\varphi}^{(s)}_{u,v}|^2 \right)^{\frac{1}{2}} \\ &\leqslant (\Phi^{(r)}_{M+h,N+k})^{\frac{1}{2}} \, (\Phi^{(s)}_{M,N})^{\frac{1}{2}} \end{aligned} \tag{18}$$

From the slowly increasing property of $\Phi^{(r)}_{MN}$ we have that for $1 \leqslant r, s \leqslant P$ and all nonnegative integers h, k the sequence $\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \varphi^{(r)}_{u+h,v+k} \bar{\varphi}^{(s)}_{u,v} / (\Phi^{(r)}_{M,N} \Phi^{(s)}_{M,N})^{1/2}$ is bounded. We assume that for all $1 \leqslant r, s \leqslant P$ this sequence also converges to a limit denoted by $R^{(r,s)}_{h,k}$, *i.e.*,

$$R^{(r,s)}_{h,k} = \lim_{M,N \to \infty} \frac{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \varphi^{(r)}_{u+h,v+k} \bar{\varphi}^{(s)}_{u,v}}{(\Phi^{(r)}_{M,N} \Phi^{(s)}_{M,N})^{\frac{1}{2}}}. \tag{19}$$

In order to deal with negative values of h, k we define $\varphi^{(r)}_{u,v} = 0$ for $u < 0$ or $v < 0$. Then for $h, k \geqslant 0$ we have

$$\begin{aligned} \frac{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \varphi^{(r)}_{u-h,v-k} \bar{\varphi}^{(s)}_{u,v}}{(\Phi^{(r)}_{M,N} \Phi^{(s)}_{M,N})^{\frac{1}{2}}} &= \frac{\sum_{u=0}^{M-1-h} \sum_{v=0}^{N-1-k} \bar{\varphi}^{(s)}_{u+h,v+k} \varphi^{(r)}_{u,v}}{(\Phi^{(r)}_{M-h,N-k} \Phi^{(s)}_{M-h,N-k})^{\frac{1}{2}}} \\ &\quad \times \left(\frac{\Phi^{(r)}_{M-h,N-k} \Phi^{(s)}_{M-h,N-k}}{\Phi^{(r)}_{M,N} \Phi^{(s)}_{M,N}} \right)^{\frac{1}{2}}. \end{aligned}$$

On taking the limits of both sides and using the slowly increasing property of the sequence $\Phi^{(r)}_{MN}$ we have using (17), (19)

$$R^{(r,s)}_{-h,-k} = \bar{R}^{(s,r)}_{h,k}. \tag{20}$$

Using similar arguments we also have that

$$R^{(r,s)}_{-h,k} = \bar{R}^{(s,r)}_{h,-k}. \tag{21}$$

Using matrix notations we have $\mathbf{R}_{-h,-k} = \mathbf{R}^H_{h,k}$ and $\mathbf{R}_{-h,k} = \mathbf{R}^H_{h,-k}$. We therefore conclude that $\mathbf{R}_{h,k}$ is an Hermitian series on $\mathscr{Z}^2$.

Using the definition of $\mathbf{R}_{h,k}$ we conclude that $\mathbf{R}_{h,k}$ is a double index positive semi-definite sequence: Indeed let $\boldsymbol{\alpha}$ be an arbitrary P dimensional vector and consider the quadratic form

$$\begin{aligned}\sigma_{\delta-\mu,\epsilon-\eta} &= \boldsymbol{\alpha}^H \mathbf{R}_{\delta-\mu,\epsilon-\eta} \boldsymbol{\alpha} \\ &= \lim_{M,N\to\infty} \sum_{r,s=1}^{P} \frac{\overline{\boldsymbol{\alpha}_r}}{(\Phi^{(r)}_{M,N})^{\frac{1}{2}}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \varphi^{(r)}_{u+\delta,v+\epsilon} \bar{\varphi}^{(s)}_{u+\mu,v+\eta} \frac{\boldsymbol{\alpha}_s}{(\Phi^{(s)}_{M,N})^{\frac{1}{2}}}.\end{aligned}$$

Now let $\mathbf{k}$ be any m-vector where m is arbitrary and let $r_1, r_2, \ldots, r_m$ and $s_1, s_2, \ldots, s_m$ be some integers. Then

$$\sum_{i,j=1}^{m} \bar{\mathbf{k}}_i \sigma_{r_i-r_j,s_i-s_j} \mathbf{k}_j = \lim_{M,N\to\infty} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \left| \sum_{i=1}^{m} \sum_{r=1}^{P} \frac{\bar{\mathbf{k}}_i \bar{\boldsymbol{\alpha}}_r \varphi^{(r)}_{u+r_i,v+s_i}}{(\Phi^{(r)}_{M,N})^{\frac{1}{2}}} \right|^2 \geqslant 0.$$

Hence, $\boldsymbol{\alpha}^H \mathbf{R}_{h,k} \boldsymbol{\alpha}$ is a 2-D positive semi-definite sequence. Thus, employing the spectral representation theorem, (see, *e.g.*, [9]), there exists a non-negative 2-D function $M_\alpha(\omega, v)$ such that

$$\boldsymbol{\alpha}^H \mathbf{R}_{h,k} \boldsymbol{\alpha} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi(h\omega+kv)}\, dM_\alpha(\omega, v). \tag{22}$$

However, $M_\alpha(\omega, v) = \boldsymbol{\alpha}^H \mathbf{M}(\omega, v)\, \boldsymbol{\alpha}$, where $\mathbf{M}(\omega, v)$ is a matrix valued function of ω and v taking as values Hermitian $P \times P$ positive semi-definite matrices whose elements are functions of bounded variation, while the functions on the diagonal are non-decreasing. It can be shown following similar arguments to those in [7, p. 45], that $\mathbf{R}_{h,k}$ has a spectral representation of the form

$$\mathbf{R}_{h,k} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi(h\omega+kv)}\, d\mathbf{M}(\omega, v).$$

$\mathbf{M}(\omega, \nu)$ is called the spectral distribution of the regression vectors $\boldsymbol{\varphi}^{(1)}, \boldsymbol{\varphi}^{(2)}, \ldots, \boldsymbol{\varphi}^{(P)}$. In the following we assume that $\mathbf{R}_{0,0} = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} d\mathbf{M}(\omega, \nu)$ is a nonsingular matrix.

3. ASYMPTOTIC EXPRESSION FOR THE COVARIANCE MATRIX

When estimating the regression coefficients vector γ one should theoretically use the best linear unbiased estimator (15). In general $\boldsymbol{\Gamma}$ is unknown. Hence, in the following we will be interested in the properties of the estimator (14) whose form does not depend on $\boldsymbol{\Gamma}$.

The covariance matrix of $\mathbf{c}_L$ in (14) is given by

$$E\{(\mathbf{c}_L - \gamma)(\mathbf{c}_L - \gamma)^H\} = (\boldsymbol{\Phi}^H \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^H \boldsymbol{\Gamma} \boldsymbol{\Phi} (\boldsymbol{\Phi}^H \boldsymbol{\Phi})^{-1}. \tag{23}$$

Define

$$\mathbf{D}_{M,N} = \begin{pmatrix} (\Phi_{M,N}^{(1)})^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & (\Phi_{M,N}^{(2)})^{\frac{1}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\Phi_{M,N}^{(P)})^{\frac{1}{2}} \end{pmatrix} \tag{24}$$

Next, we will investigate the asymptotic behavior of the normalized covariance of $\mathbf{c}_L$.

THEOREM 3. *Assume the disturbance field is a homogeneous random field with an absolutely continuous spectral distribution and a positive and piecewise continuous spectral density, $f(\omega, \nu)$, such that the set of discontinuity points of $f(\omega, \nu)$ and the set of discontinuity points of $\mathbf{M}$ are disjoint. Then*

$$\lim_{M,N \to \infty} \mathbf{D}_{M,N} E\{(\mathbf{c}_L - \gamma)(\mathbf{c}_L - \gamma)^H\} \mathbf{D}_{M,N} = \mathbf{R}_{0,0}^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega, \nu)\, d\bar{\mathbf{M}}(\omega, \nu)\, \mathbf{R}_{0,0}^{-1} \tag{25}$$

Proof. For generality, we assume the disturbance field $\{\epsilon_{u,v}\}$ is a non-causal moving average field such that

$$\epsilon_{u,v} = \sum_{\nu=-\alpha_1}^{\alpha_2} \sum_{\xi=-\beta_1}^{\beta_2} a_{\nu,\xi} \eta_{u-\nu, v-\xi}, \tag{26}$$

where $\{\eta_{u,v}\}$ is white noise field with zero mean and unit variance, while $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive integers. (Clearly the case of a nonsymmetrical half-plane (NSHP) moving average field is obtained when $\alpha_1 = 0$). Using

(26) we conclude that there exist two positive constants α and β such that the disturbance spectral density has the following 2-D trigonometric polynomial representation

$$f(\omega, v) = \sum_{\tau=-\alpha}^{\alpha} \sum_{\varsigma=-\beta}^{\beta} r_{\tau,\varsigma} e^{-i2\pi(\tau\omega+\varsigma v)}.$$

This is since the covariances of the disturbance field are nonzero at most for $|\tau| \leqslant \alpha$, $|\varsigma| \leqslant \beta$.

To evaluate the normalized asymptotic covariance on the L.H.S. of (25), we first evaluate $\mathbf{D}_{M,N}^{-1}\mathbf{\Phi}^H\mathbf{\Gamma\Phi D}_{M,N}^{-1}$. Let

$$f_1(\mu) = \mu \bmod M \tag{27}$$

$$f_2(\mu) = \left\lfloor \frac{\mu}{M} \right\rfloor. \tag{28}$$

We thus have

$$\{\mathbf{D}_{M,N}^{-1}\mathbf{\Phi}^H\mathbf{\Gamma}\}_{\xi,\mu} = \sum_{v=0}^{N-1} \sum_{u=0}^{M-1} \frac{\bar{\varphi}_{u,v}^{(\xi)} r_{u-f_1(\mu),v-f_2(\mu)}}{(\Phi_{M,N}^{(\xi)})^{\frac{1}{2}}} \tag{29}$$

and

$$\begin{aligned}
&\{\mathbf{D}_{M,N}^{-1}\mathbf{\Phi}^H\mathbf{\Gamma\Phi D}_{M,N}^{-1}\}_{\xi,\mu} \\
&= \sum_{\lambda=0}^{MN-1} \left(\sum_{v=0}^{N-1} \sum_{u=0}^{M-1} \frac{\bar{\varphi}_{u,v}^{(\xi)} r_{u-f_1(\lambda),v-f_2(\lambda)} \varphi_{f_1(\lambda),f_2(\lambda)}^{(\mu)}}{(\Phi_{M,N}^{(\xi)}\Phi_{M,N}^{(\mu)})^{\frac{1}{2}}} \right) \\
&= \sum_{\varepsilon=0}^{N-1} \sum_{\eta=0}^{M-1} \sum_{v=0}^{N-1} \sum_{u=0}^{M-1} \frac{\bar{\varphi}_{u,v}^{(\xi)} r_{u-f_1(M\varepsilon+\eta),v-f_2(M\varepsilon+\eta)} \varphi_{f_1(M\varepsilon+\eta),f_2(M\varepsilon+\eta)}^{(\mu)}}{(\Phi_{M,N}^{(\xi)}\Phi_{M,N}^{(\mu)})^{\frac{1}{2}}} \\
&= \sum_{\varepsilon=0}^{N-1} \sum_{\eta=0}^{M-1} \sum_{v=0}^{N-1} \sum_{u=0}^{M-1} \frac{\bar{\varphi}_{u,v}^{(\xi)} r_{u-\eta,v-\varepsilon} \varphi_{\eta,\varepsilon}^{(\mu)}}{(\Phi_{M,N}^{(\xi)}\Phi_{M,N}^{(\mu)})^{\frac{1}{2}}} \\
&= \sum_{p=0}^{\alpha} \sum_{q=0}^{\beta} r_{p,q} \sum_{\varepsilon=0}^{N-1-q} \sum_{\eta=0}^{M-1-p} \frac{\bar{\varphi}_{\eta+p,\varepsilon+q}^{(\xi)} \varphi_{\eta,\varepsilon}^{(\mu)}}{(\Phi_{M,N}^{(\xi)}\Phi_{M,N}^{(\mu)})^{\frac{1}{2}}} \\
&\quad + \sum_{p=-\alpha}^{-1} \sum_{q=-\beta}^{-1} r_{p,q} \sum_{v=0}^{N-1+q} \sum_{u=0}^{M-1+p} \frac{\bar{\varphi}_{u,v}^{(\xi)} \varphi_{u-p,v-q}^{(\mu)}}{(\Phi_{M,N}^{(\xi)}\Phi_{M,N}^{(\mu)})^{\frac{1}{2}}} \\
&\quad + \sum_{p=0}^{\alpha} \sum_{q=-\beta}^{-1} r_{p,q} \sum_{v=0}^{N-1+q} \sum_{\eta=0}^{M-1-p} \frac{\bar{\varphi}_{\eta+p,v}^{(\xi)} \varphi_{\eta,v-q}^{(\mu)}}{(\Phi_{M,N}^{(\xi)}\Phi_{M,N}^{(\mu)})^{\frac{1}{2}}} \\
&\quad + \sum_{p=-\alpha}^{-1} \sum_{q=0}^{\beta} r_{p,q} \sum_{\varepsilon=0}^{N-1-q} \sum_{u=0}^{M-1+p} \frac{\bar{\varphi}_{u,\varepsilon+q}^{(\xi)} \varphi_{u-p,\varepsilon}^{(\mu)}}{(\Phi_{M,N}^{(\xi)}\Phi_{M,N}^{(\mu)})^{\frac{1}{2}}}.
\end{aligned} \tag{30}$$

Using (19) and the slowly increasing property of the sequence $\Phi^{(r)}_{MN}$ we have that when $M, N \to \infty$ each of the four sum terms in (30) tends to a finite limit of the same form. Collecting the four terms into a single summation we have

$$\begin{aligned} \{\mathbf{D}^{-1}_{M,N}\mathbf{\Phi}^H\mathbf{\Gamma}\mathbf{\Phi}\mathbf{D}^{-1}_{M,N}\}_{\xi,\mu} &= \sum_{u=-\alpha}^{\alpha}\sum_{v=-\beta}^{\beta} r_{u,v}R^{(\mu,\xi)}_{-u,-v} \\ &= \sum_{u=-\alpha}^{\alpha}\sum_{v=-\beta}^{\beta} r_{u,v}\bar{R}^{(\xi,\mu)}_{u,v}, \end{aligned} \tag{31}$$

where the last equality is due to (20). Hence, using Parseval's equality we have

$$\lim_{M,N\to\infty} \mathbf{D}^{-1}_{M,N}\mathbf{\Phi}^H\mathbf{\Gamma}\mathbf{\Phi}\mathbf{D}^{-1}_{M,N} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega,\nu)\, d\bar{\mathbf{M}}(\omega,\nu) \tag{32}$$

where $\bar{\mathbf{M}} = (\mathbf{M}^H)^T$.

Rewriting (19) in matrix form for $h=k=0$ we obtain

$$\mathbf{R}_{0,0} = \lim_{M,N\to\infty} \mathbf{D}^{-1}_{M,N}\mathbf{\Phi}^H\mathbf{\Phi}\mathbf{D}^{-1}_{M,N}. \tag{33}$$

Finally,

$$\begin{aligned} &\lim_{M,N\to\infty} \mathbf{D}_{M,N}E\{(\mathbf{c}_L-\gamma)(\mathbf{c}_L-\gamma)^H\}\,\mathbf{D}_{M,N} \\ &\quad= \lim_{M,N\to\infty} \mathbf{D}_{M,N}(\mathbf{\Phi}^H\mathbf{\Phi})^{-1}\,\mathbf{\Phi}^H\mathbf{\Gamma}\mathbf{\Phi}(\mathbf{\Phi}^H\mathbf{\Phi})^{-1}\,\mathbf{D}_{M,N} \\ &\quad= \lim_{M,N\to\infty} \{\mathbf{D}_{M,N}(\mathbf{\Phi}^H\mathbf{\Phi})^{-1}\,\mathbf{D}_{M,N}\}\{\mathbf{D}^{-1}_{M,N}\mathbf{\Phi}^H\mathbf{\Gamma}\mathbf{\Phi}\mathbf{D}^{-1}_{M,N}\} \\ &\quad\quad\times\{\mathbf{D}_{M,N}(\mathbf{\Phi}^H\mathbf{\Phi})^{-1}\,\mathbf{D}_{M,N}\} \\ &\quad= \mathbf{R}^{-1}_{0,0}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega,\nu)\, d\bar{\mathbf{M}}(\omega,\nu)\,\mathbf{R}^{-1}_{0,0} \end{aligned} \tag{34}$$

as $\mathbf{R}_{0,0}$ is nonsingular. The first equality in (34) is due to (23), while the last equality is due to the existence of the limits in (32) and (33).

From now on we shall assume that the spectral density $f(\omega,\nu)$ is piecewise continuous and positive. Similarly to the 1-D case, see *e.g.*, [5, 8], it is assumed that the discontinuities of $f(\omega,\nu)$ do not coincide with those of $\mathbf{M}(\omega,\nu)$. Let $f_1(\omega,\nu)$ and $f_2(\omega,\nu)$ be finite trigonometric polynomials such that

$$f_1(\omega,\nu) \leqslant f(\omega,\nu) \leqslant f_2(\omega,\nu). \tag{35}$$

Let us denote the corresponding covariance matrices by $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}$, and $\mathbf{\Gamma}_2$, respectively.

From the derivation in Appendix C, we have that for any non-zero $NM \times 1$ arbitrary vector $\mathbf{Z}$

$$\mathbf{Z}^H \mathbf{\Gamma}_j \mathbf{Z} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} Z_{u,v} e^{i2\pi(\omega u + \nu v)} \right|^2 f_j(\omega, \nu)\, d\omega\, d\nu, \qquad j = 1, 2 \tag{36}$$

$$\mathbf{Z}^H \mathbf{\Gamma} \mathbf{Z} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} Z_{u,v} e^{i2\pi(\omega u + \nu v)} \right|^2 f(\omega, \nu)\, d\omega\, d\nu. \tag{37}$$

Using (35), (36) and (37) we have

$$\mathbf{\Gamma}_1 \leqslant \mathbf{\Gamma} \leqslant \mathbf{\Gamma}_2. \tag{38}$$

Note that the R.H.S. of (23) (and hence (34) as well) is a congruence. Since congruence preserves ordering in the sense of (38), applying this transformation to (38) while using (34) we conclude that for any non-zero $NM \times 1$ arbitrary vector $\mathbf{Z}$,

$$\begin{aligned} &\mathbf{Z}^H \mathbf{R}_{0,0}^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_1(\omega, \nu)\, d\bar{\mathbf{M}}(\omega, \nu)\, \mathbf{R}_{0,0}^{-1} \mathbf{Z} \\ &\quad \leqslant \lim_{M, N \to \infty} \mathbf{Z}^H \mathbf{D}_{M,N} E\{(\mathbf{c}_L - \boldsymbol{\gamma})(\mathbf{c}_L - \boldsymbol{\gamma})^H\}\, \mathbf{D}_{M,N} \mathbf{Z} \\ &\quad \leqslant \mathbf{Z}^H \mathbf{R}_{0,0}^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_2(\omega, \nu)\, d\bar{\mathbf{M}}(\omega, \nu)\, \mathbf{R}_{0,0}^{-1} \mathbf{Z}. \end{aligned} \tag{39}$$

Since it is assumed that $\mathbf{M}$ has no atoms at the discontinuity points of $f(\omega, \nu)$, $\max\{f_2(\omega, \nu) - f_1(\omega, \nu)\}$ can be made arbitrarily small (except in small neighborhoods with arbitrarily small total measure $\mathbf{M}$ of the discontinuity points of $f(\omega, \nu)$) and hence

$$\begin{aligned} &\lim_{M, N \to \infty} \mathbf{D}_{M,N} E\{(\mathbf{c}_L - \boldsymbol{\gamma})(\mathbf{c}_L - \boldsymbol{\gamma})^H\}\, \mathbf{D}_{M,N} \\ &\quad = \mathbf{R}_{0,0}^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega, \nu)\, d\bar{\mathbf{M}}(\omega, \nu)\, \mathbf{R}_{0,0}^{-1}. \quad \blacksquare \end{aligned} \tag{40}$$

Thus for two-dimensional random fields, (40) provides the asymptotic covariance matrix of the unbiased and mean square consistent linear least squares estimator of the regression coefficients. The disturbance is a homogeneous random field with an absolutely continuous spectral distribution and a positive and piecewise continuous spectral density.

APPENDIX A

We now prove the identity (6). We first prove the first equality in (6). A straightforward calculation shows that $E\,|c_{M,N}-\gamma|^2 = E\,|c_{M,N}|^2 - 2\,\mathrm{Re}\{\bar{\gamma}E(c_{M,N})\} + |\gamma|^2$. Evaluating the R.H.S. of the first equality in (6) in a similar way, this equality follows.

Evaluating $E(c_{M,N})$ we have

$$E(c_{M,N}) = E\left\{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} a_{u,v}^{(M,N)} y_{u,v}\right\} = \gamma \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} a_{u,v}^{(M,N)} \varphi_{u,v}. \tag{41}$$

The spectral representation of the disturbance field $\{\epsilon_{u,v}\}$ is given by

$$\epsilon_{u,v} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi(\omega u+\nu v)}\, dz(\omega,\nu),$$

where $z(\omega,\nu)$ is a doubly orthogonal increments process. Let $f(\omega,\nu)$ denote the spectral density function of $\{\epsilon_{u,v}\}$. Then

$$\begin{aligned} c_{M,N} - E(c_{M,N}) &= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} a_{u,v}^{(M,N)} \epsilon_{u,v} \\ &= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} a_{u,v}^{(M,N)} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi(\omega u+\nu v)}\, dz(\omega,\nu). \end{aligned} \tag{42}$$

Therefore

$$\begin{aligned} &E\{\,|c_{M,N} - E(c_{M,N})\,|^2\} \\ &\quad = E\left\{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} a_{u,v}^{(M,N)} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi(\omega u+\nu v)}\, dz(\omega,\nu)\right. \\ &\qquad \left.\cdot \sum_{u'=0}^{M-1}\sum_{v'=0}^{N-1} \overline{a_{u',v'}^{(M,N)}} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i2\pi(\omega' u'+\nu' v')}\, \overline{dz(\omega',\nu')}\right\} \\ &\quad = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} a_{u,v}^{(M,N)} e^{i2\pi(\omega u+\nu v)} \\ &\qquad \times \sum_{u'=0}^{M-1}\sum_{v'=0}^{N-1} \overline{a_{u',v'}^{(M,N)}}\, e^{-i2\pi(\omega u'+\nu v')} f(\omega,\nu)\, d\omega\, d\nu \\ &\quad = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} a_{u,v}^{(M,N)} e^{i2\pi(\omega u+\nu v)}\right|^2 f(\omega,\nu)\, d\omega\, d\nu. \end{aligned} \tag{43}$$

Thus the second equality in (6) follows from (41) and (43).

APPENDIX B

To show that $c_{M,N}$ defined in (13) is a mean square consistent estimate of γ, define

$$a_{u,v}^{(M,N)} = \frac{\bar{\varphi}_{u,v}}{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\varphi_{u,v}|^2}.$$

Using the same considerations as in (43) and Parseval's equality we have

$$\begin{aligned}
E(|c_{M,N} - \gamma|^2) &\\
&= E\{|c_{M,N} - E(c_{M,N})|^2\} \\
&= \frac{1}{(\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\varphi_{u,v}|^2)^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \bar{\varphi}_{u,v} e^{i2\pi(\omega u + \nu v)} \right|^2 f(\omega, \nu)\, d\omega\, d\nu \\
&\leqslant \frac{\max_{\omega,\nu} f(\omega,\nu) \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\bar{\varphi}_{u,v}|^2}{(\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\varphi_{u,v}|^2)^2} \\
&= \max_{\omega,\nu} f(\omega,\nu) \frac{1}{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} |\varphi_{u,v}|^2}.
\end{aligned} \tag{44}$$

Thus the condition (5) is sufficient for the R.H.S. of (44) to tend to zero as $N, M \to \infty$.

APPENDIX C

Let $\mathbf{\Gamma}$ be an $MN \times NM$ Toeplitz-block-Toeplitz covariance matrix of the form (2). Let $\mathbf{Z}$ be an arbitrary non-zero $MN \times 1$ vector, whose elements are indexed in the following way

$$\mathbf{Z} = [Z_{0,0}, Z_{1,0}, \ldots, Z_{M-1,0}, Z_{0,1}, \ldots, Z_{M-1,N-1}]^T \tag{45}$$

We wish to evaluate the quadratic form $\mathbf{Z}^H \mathbf{\Gamma} \mathbf{Z}$. Thus

$$\begin{aligned}
\mathbf{Z}^H \mathbf{\Gamma} \mathbf{Z} &= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \bar{Z}_{u,v} r_{p-u,q-v} Z_{p,q} \\
&= \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \bar{Z}_{u,v} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi[(p-u)\omega + (q-v)\nu]} f(\omega,\nu)\, d\omega\, d\nu\, Z_{p,q} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \bar{Z}_{u,v} e^{i2\pi[(p-u)\omega + (q-v)\nu]} Z_{p,q} f(\omega,\nu)\, d\omega\, d\nu \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} Z_{u,v} e^{i2\pi(\omega u + \nu v)} \right|^2 f(\omega,\nu)\, d\omega\, d\nu \geqslant 0.
\end{aligned} \tag{46}$$

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