Maximum-Likelihood Parameter Estimation of the Harmonic, Evanescent, and Purely Indeterministic Components of Discrete Homogeneous Random Fields

Joseph M. Francos, *Member, IEEE*, Anand Narasimhan, *Member, IEEE*, and John W. Woods, *Fellow, IEEE*

***Abstract*—This paper presents a maximum-likelihood solution to the general problem of fitting a parametric model to observations from a single realization of a two-dimensional (2-D) homogeneous random field with mixed spectral distribution. On the basis of a 2-D Wold-like decomposition, the field is represented as a sum of mutually orthogonal components of three types: purely indeterministic, harmonic, and evanescent. The suggested algorithm involves a two-stage procedure. In the first stage, we obtain a suboptimal initial estimate for the parameters of the spectral support of the evanescent and harmonic components. In the second stage, we refine these initial estimates by iterative maximization of the conditional likelihood of the observed data, which is expressed as a function of only the parameters of the spectral supports of the evanescent and harmonic components. The solution for the unknown spectral supports of the harmonic and evanescent components reduces the problem of solving for the other unknown parameters of the field to linear least squares. The Cramer–Rao lower bound on the accuracy of jointly estimating the parameters of the different components is derived, and it is shown that the bounds on the purely indeterministic and deterministic components are decoupled. Numerical evaluation of the bounds provides some insight into the effects of various parameters on the achievable estimation accuracy. The performance of the maximum-likelihood algorithm is illustrated by Monte Carlo simulations and is compared with the Cramer–Rao bound.**

***Index Terms*—ML estimation of 2-D random fields, 2-D Wold decomposition, 2-D mixed spectral distributions, purely indeterministic fields, harmonic fields, evanescent fields, Cramer–Rao bound.**

I. Introduction

IN THIS PAPER, we consider the problem of fitting a parametric model to observations from a single realization of a two-dimensional (2-D) complex-valued discrete and homogeneous random field with mixed spectral distribution. This fundamental problem is of great theoretical and practical importance. It arises in several areas of radar and sonar processing, and the special case of real-valued 2-D random fields arises quite naturally in terms of the texture estimation of images [9].

The general problem of random fields' parameter estimation has received considerable attention. Most approaches reported to date fall into one of two categories. They either try to fit noise-driven linear models (2-D autoregressive (AR), moving average (MA), or autoregressive moving average (ARMA)) to the observed field, *or* they treat the special case of estimation of the parameters of sinusoidal signals in white noise. Noise-driven linear models have absolutely continuous spectral distribution functions, and hence, are inappropriate for the general problem considered here. Parameter estimation techniques of sinusoidal signals in additive white noise include the periodogram-based approximation (applicable for widely spaced sinusoids) to the maximum-likelihood (ML) solution [2], extensions to the Pisarenko harmonic decomposition [3], or the singular value decomposition [5]. These methods rely heavily on the white noise assumption, and are therefore not applicable here, since in our more general setting, the noise is colored, and *a priori* unknown. Note that covariance-based estimation procedures must assume knowledge of the true covariances. If these are unknown, substituting them with the sample covariances is incorrect, since it is well known [6] that even under the Gaussian assumption, the sample covariances are not consistent estimates of the covariance function if the spectral distribution function has discontinuities.

The 2-D Wold-like decomposition [1] implies that any 2-D regular discrete and homogeneous random field can be represented as a sum of two mutually orthogonal components: A *purely indeterministic* field and a *deterministic* one. The deterministic component is further orthogonally decomposed into a *half-plane deterministic* field and a countable number of mutually orthogonal *evanescent* fields. This decomposition results in a corresponding decomposition of the spectral measure of the regular random field into a countable sum of mutually singular spectral measures. The spectral distribution function of the purely indeterministic component is absolutely continuous, while the spectral measure of the deterministic component is singular with respect to the Lebesgue measure, and therefore it is concentrated on a set of Lebesgue measure

Manuscript received November 17, 1994; revised January 5, 1995. This work was supported in part by the NSF under Grant MIP-9120377. The material in this paper was presented in part at the IEEE International Symposium on Information Theory, San Antonio, TX, 1993.

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Publisher Item Identifier S 0018-9448(96)02907-0.

zero in the frequency plane. For practical applications, the "spectral density function" of the regular field's deterministic component can be assumed to have the form of a countable sum of one–dimensional (1-D) and two–dimensional (2-D) delta functions. The 1-D delta functions are singular functions supported on curves in the 2-D spectral domain. The 2-D delta functions are singular functions supported on discrete points in the spectral domain.

In this paper, we consider the problem of estimating the parameters of the different components of the decomposition from a single realization of the field. In general, an unbiased estimator of the field parameters will require joint estimation of the parameters of the harmonic, evanescent, and purely indeterministic components. We present a conditional maximum-likelihood solution to this simultaneous parameter estimation problem, for the case in which the purely indeterministic component is a complex-valued Gaussian random field. The algorithm is a two-stage procedure. In the first stage, we obtain suboptimal initial estimates for the parameters of the spectral support of the evanescent and harmonic components. The initial estimates are obtained by solving the set of overdetermined 2-D normal equations for the parameters of a high-order linear predictor of the observed data. In the second stage, we refine these initial estimates by iterative maximization of the conditional likelihood of the observed data. This maximization requires the solution of a highly nonlinear least squares problem. By introducing appropriate parameter transformations the nonlinear least squares problem is transformed into a *separable* least squares problem [11], [12]. In this new problem, the solution for the unknown spectral supports of the harmonic and evanescent components reduces the problem of solving for the other unknown parameters of the field to linear least squares. Hence, the solution of the original least squares problem becomes much simpler. The proposed method is useful even when the separation between the spectral supports of any two deterministic components is less than $1/N$ in each dimension (for an $N \times N$ observed field). We also present the Cramer–Rao lower bound (CRB) for this estimation problem. We show that the bounds on the deterministic and purely indeterministic components are decoupled, and derive closed-form CR bounds on the accuracy of estimating the parameters of the harmonic and evanescent components of the field.

An early discussion on the problem of analyzing 2-D homogeneous random fields with discontinuous spectral distribution functions can be found in [7]. There, harmonic analysis is employed to analyze the long-lag sample covariances, since for such lags the contribution of the purely indeterministic component is assumed to be insignificant. In this framework, the detection problem for a special case of evanescent fields is also discussed. The idea in [7] is to first test for the existence of the deterministic components. If such components are detected, their parameters are estimated and their contribution to the sample covariances is removed. Next, the spectral density function of the purely indeterministic component can be estimated from the "corrected" sample covariances. In [9], a similar periodogram-based approach was used.

The paper is organized as follows. In Section II we briefly summarize the results of the 2-D Wold-like decomposition, which establish the theoretical basis for the suggested solution. Then, in Section III, assuming that the purely indeterministic component is a complex-valued Gaussian AR field, we formalize the parameter estimation problem and derive the conditional maximum-likelihood estimator in the presence of a single evanescent field. In Section IV we elaborate on the problem of estimating the parameters of the evanescent random field. Section V describes an iterative solution for the parameters of the spectral support of the evanescent and harmonic components and its initialization algorithm. In Section VI the Cramer–Rao bound is derived, and in Section VII we present some numerical examples to illustrate the performance of the suggested algorithm, and the behavior of the derived bounds.

II. The Homogeneous Random Field Model

The presented random field model is derived based on the results of the Wold-type decomposition of 2-D regular and homogeneous random fields [1]. In this section we briefly summarize the results of [1]. Let $\{y(n,m),(n,m) \in \mathcal{Z}^2\}$ be a complex-valued homogeneous random field. Let $\hat{y}(n,m)$ be the projection of $y(n,m)$ on the Hilbert space spanned by those samples of the field that are in the "past" of the (n,m)th sample, where the "past" is defined with respect to the *totally ordered, nonsymmetrical-half-plane support*, i.e.

$$(i,j) \prec (s,t) \text{ iff } (i,j) \in \{(k,\ell)|k=s,\ell<t\} \bigcup \{(k,\ell)|k<s,-\infty<\ell<\infty\}. \quad (1)$$

The *innovation* with respect to the defined support and total order is given by $u(n,m)=y(n,m)-\hat{y}(n,m)$ and its variance is denoted by σ^2. If $E|y(n,m)-\hat{y}(n,m)|^2=\sigma^2>0$, the field $\{y(n,m)\}$ is called *regular*. The field is called *deterministic* if $E|y(n,m)-\hat{y}(n,m)|^2=0$. A regular field $\{y(n,m)\}$ is called *purely indeterministic* if $y(n,m) \in \overline{Sp}\{u(s,t)|(s,t) \preceq (n,m)\}$, where $\overline{Sp}\{\cdot\}$ denotes the closure of the span. These definitions result in the following decomposition theorem:

Theorem 1 [8]: Let $\{y(n,m),(n,m) \in \mathcal{Z}^2\}$ be a 2-D regular and homogeneous random field. Then there exist a deterministic random field $\{v(n,m)\}$ and an innovations field $\{u(n,m)\}$ such that $\{y(n,m)\}$ can be uniquely represented by the orthogonal decomposition

$$y(n,m)=w(n,m)+v(n,m) \quad (2)$$

where

$$w(n,m)=\sum_{(0,0)\preceq(k,\ell)} a(k,\ell)u(n-k,m-\ell) \quad (3)$$

and

$$\sum_{(0,0)\preceq(k,\ell)} |a(k,\ell)|^2<\infty, \quad a(0,0)=1.$$

The field $\{w(n,m)\}$ is purely indeterministic and regular. The fields $\{w(n,m)\}$ and $\{v(s,t)\}$ are mutually orthogonal for all (n,m) and (s,t).

Definition 1: A 2-D deterministic random field $\{e_o(n,m)\}$ is called *evanescent w.r.t. the NSHP total-order o* if it spans a Hilbert space identical to the one spanned by its *column-to-column innovations* at each coordinate (n,m) (w.r.t. the total order o).

The concept of column-to-column innovations of deterministic fields is best illustrated using the following example. Let $\{\xi(i)| -\infty < i < \infty\}$ be an infinite two-sided sequence of i.i.d. Gaussian random variables with zero mean and unit variance. Let also the 2-D random field $\{y(k,l)\}$ be defined by $y(k,l) = \xi(k)$. It is clear that $\hat{y}(k,l) = y(k,l-1) = \xi(k) = y(k,l)$. Therefore, the field $\{y(k,l)\}$ is deterministic. On the other hand, it is obvious that $y(k,l)$ is not a vector in the Hilbert space spanned by the field samples $y(s,t)$ for $s < k$ since this Hilbert space is spanned by $\{\xi(i)| -\infty < i < k\}$ and contains no information about $\xi(k)$. The innovation of $y(k,l)$ with respect to the Hilbert space spanned by the field samples $y(s,t)$ for $s < k$ is what we call the column-to-column innovation of the deterministic field y at the coordinate (k,l). Hence, in this example the field $\{y(k,l), (k,l) \in \mathcal{Z}^2\}$ is an evanescent field.

It can be shown that it is possible to define a family of NSHP total-order definitions such that the boundary line of the NSHP has rational slope. Let α, β be two coprime integers, such that $\alpha \neq 0$. The angle θ of the slope is given by $\tan\theta = \beta/\alpha$. (See, for example, Fig. 1.) Each of these supports is called *rational nonsymmetrical half-plane* (RNSHP). We denote by O the set of all possible RNSHP definitions on the 2-D lattice (i.e., the set of all NSHP definitions in which the boundary line of the NSHP has rational slope). For the special case in which $\theta = \pi/2$ the NSHP total order is defined by interchanging the roles of columns and rows. The introduction of the family of RNSHP total-ordering definitions results in a corresponding decomposition of the deterministic component of the random field:

Theorem 2 [1]: Let $\{v(n,m)\}$ be the deterministic component of a 2-D regular and homogeneous random field. Then $\{v(n,m)\}$ can be uniquely represented by the following countably infinite orthogonal decomposition:

$$v(n,m) = p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \qquad (4)$$

The random field $\{p(n,m)\}$ is *half-plane deterministic*, i.e., it has no column-to-column innovations w.r.t. any RNSHP total-ordering definition. The field $\{e_{(\alpha,\beta)}(n,m)\}$ is the evanescent component which generates the column-to-column innovations of the deterministic field w.r.t. the RNSHP total-ordering definition $(\alpha,\beta) \in O$.

Hence, if $\{y(n,m)\}$ is a 2-D regular and homogeneous random field, then $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \qquad (5)$$

In the following, all spectral measures are defined on the square region $K = [-1/2, 1/2] \times [-1/2, 1/2]$. The spectral

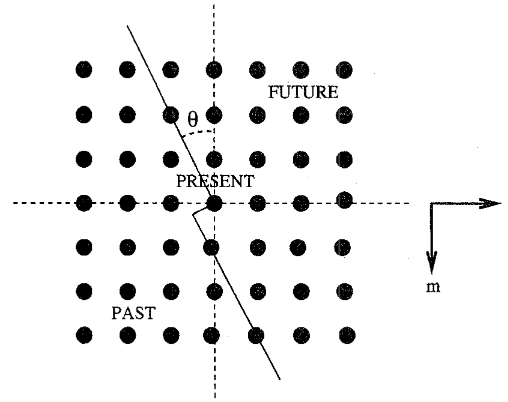

Fig. 1. RNSHP support.

representation of $y(n,m)$ is given by

$$y(n,m) = \int_K \exp\left[2\pi j(n\omega + m\nu)\right] dZ(\omega,\nu)$$

where $Z(\omega,\nu)$ is a doubly orthogonal increments process, such that $dF_y(\omega,\nu) = E[dZ(\omega,\nu)\ dZ^*(\omega,\nu)]$. $F_y(\omega,\nu)$ is the *spectral distribution function* of $\{y(n,m)\}$. Let $f(\omega,\nu)$ be the corresponding *spectral density function*, which is the 2-D derivative of $F_y(\omega,\nu)$. $F^s(\omega,\nu)$ denotes the singular part in the Lebesgue decomposition of $F_y(\omega,\nu)$. Let L be a set of Lebesgue measure zero in K, such that the measure defined by $F^s(\omega,\nu)$ is concentrated on L.

Theorem 3 [1]: The spectral measures of the decomposition components in (5) are mutually singular. The spectral distribution function of the purely indeterministic component is absolutely continuous, while the spectral measure of the deterministic component is concentrated on the set L of Lebesgue measure zero in K. Moreover, since both the half-plane deterministic field and all the evanescent fields in the decomposition (5) are components of the deterministic component of the regular field, their spectral measures are concentrated on subsets of the set L.

The definition of the evanescent field and Theorem 3 imply that the spectral measure of the evanescent component that generates the column-to-column innovations of the deterministic component for $(\alpha,\beta) = (1,0)$ is a linear combination of spectral measures of the form $dF_{e_{(1,0)}}(\omega,\nu) = k(\omega)\ d\omega\ dF^s(\nu)$, where $F^s(\nu)$ is a 1-D singular spectral distribution function and $k(\omega)$ is a 1-D spectral density function. In other words, the spectral distribution function of each evanescent component is separable: it is absolutely continuous in one dimension and singular in the orthogonal one (or a linear combination of such separable distribution functions).

From Theorem 3 we have that the spectral measure of each evanescent component of the regular field is concentrated on a set of Lebesgue measure zero. For practical applications we can exclude singular-continuous spectral distribution functions from the framework of our treatment. Hence, the "spectral density function" of the evanescent field $e_{(1,0)}$ has the countable sum form,

$$f_{e_{(1,0)}}(\omega, \nu) = \sum_i k_i^{(1,0)}(\omega)\delta(\nu - \nu_i^{(1,0)})$$

where $\delta(\cdot)$ is a Dirac delta function. A model for this evanescent field is given by [1]

$$e_{(1,0)}(n,m) = \sum_{i=1}^{I^{(1,0)}} s_i^{(1,0)}(n) e^{j2\pi m \nu_i^{(1,0)}} \quad (6)$$

where the 1-D purely indeterministic processes $\{s_i^{(1,0)}(n)\}$, $\{s_j^{(1,0)}(n)\}$ are mutually orthogonal for all $i \neq j$, and the spectral density function of the process $\{s_i^{(1,0)}(n)\}$ is $k_i^{(1,0)}(\omega)$. More generally, a model for the evanescent field which corresponds to the RNSHP defined by $(\alpha, \beta) \in O$ is given by

$$e_{(\alpha,\beta)}(n,m) = \sum_{i=1}^{I^{(\alpha,\beta)}} s_i^{(\alpha,\beta)}(n\alpha - m\beta) \cdot e^{j2\pi(\nu_i^{(\alpha,\beta)}/(\alpha^2+\beta^2))(n\beta+m\alpha)} \quad (7)$$

where the 1-D purely indeterministic processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}, \{s_j^{(\alpha,\beta)}(n\alpha - m\beta)\}$ are mutually orthogonal for all $i \neq j$, and $I^{(\alpha,\beta)}$ is infinite in general. Hence, the "spectral density function" of each evanescent field has the form of a countable sum of 1-D delta functions which are supported on lines of rational slope in the 2-D spectral domain.

In the following we assume that each of the 1-D purely indeterministic processes $s_i^{(\alpha,\beta)}$ obeys a finite-order autoregressive (AR) model. Thus for example, the purely indeterministic modulating process of $e_{(1,0)}$ is given by

$$s_i^{(1,0)}(n) = -\sum_{t=1}^{V_i^{(1,0)}} a_i^{(1,0)}(t) s_i^{(1,0)}(n-t) + \Gamma_i^{(1,0)}(n) \quad (8)$$

where $\Gamma_i^{(1,0)}(n)$ is a 1-D white innovations process.

One of the half-plane-deterministic field components, which is often found in physical problems, is the harmonic random field

$$h(n,m) = \sum_{p=1}^{P} C_p e^{j2\pi(n\omega_p + m\nu_p)} \quad (9)$$

where the C_p's are mutually orthogonal random variables, $E|C_p|^2 = \sigma_p^2$, and (ω_p, ν_p) are the spatial frequencies of the pth harmonic. In general, P is infinite. The parametric modeling of deterministic random fields whose spectral measures are concentrated on curves other than lines of rational slope, or discrete points in the frequency plane, is still an open question to the best of our knowledge.

Theorem 1 implies that the most general model for the purely indeterministic component of a regular homogeneous random field is the MA model (3). However, if its spectral density function is strictly positive on the unit bicircle and analytic in some neighborhood of it, a 2-D AR representation for the purely indeterministic field exists as well [10]. In the following, we assume that the above requirements are satisfied. Hence the purely indeterministic component *autoregressive model* is given by

$$w(n,m) = -\sum_{(0,0)\prec(k,\ell)} b(k,\ell) w(n-k, m-\ell) + u(n,m) \quad (10)$$

where $\{u(n,m)\}$ is the 2-D white innovations field whose variance is σ^2.

III. The Conditional Maximum-Likelihood Estimator

A. Problem Definition and Assumptions

The orthogonal decompositions of the previous section imply that if we exclude from the framework of our model those 2-D random fields whose spectral measures are singular continuous, or are concentrated on curves other than lines of rational slope, $y(n,m)$ is uniquely represented by

$$y(n,m) = w(n,m) + h(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m).$$

The problem of estimating the (α, β) pairs of the different evanescent components is beyond the scope of the present paper. In order to keep the notations as simple as possible, we restrict our attention to the case in which it is *a priori* known that $(\alpha, \beta) = (1, 0)$ for all the evanescent components. The more general problem of estimating the field parameters in the presence of evanescent fields which are characterized by unknown (α, β) parameters, will be discussed in a forthcoming paper. Hence, the problem faced here is the parameter estimation of the harmonic and evanescent components (those of $(\alpha, \beta) = (1, 0)$) of the field in the presence of an unknown colored noise generated by the purely indeterministic component, jointly with estimating the purely indeterministic component parameters.

When expressed in the general form (9), the coefficients $\{C_p\}$ of the harmonic component are complex-valued, mutually orthogonal random variables. However, since in general, only a single realization of the random field is observed we cannot infer anything about the variation of these coefficients over different realizations. The best we can do is to estimate the particular values which the C_p's take for the given realization; in other words, we might just as well treat the C_p's as unknown constants.

Finally, we note that a maximum-likelihood solution to our parameter estimation problem involves maximization of the *exact* likelihood function. However, this is a formidable task due to the complexity in representing the field covariance matrix in terms of the model parameters of the different components. For large enough data records the exact likelihood function can be approximated by the conditional likelihood function. Since this approach results in a more tractable solution, we have chosen it for the above parameter estimation problem.

We next state our assumptions and introduce some necessary notations. Let $\{y(n,m)\}, (n,m) \in D$ where

$$D = \{(i,j) | 0 \leq i \leq S-1, 0 \leq j \leq T-1\}$$

be the observed random field. Note, however, that the observed field just as well could have any *arbitrary* shape.

Assumption 1: The purely indeterministic component is a complex-valued Gaussian AR field, whose model is given by (10) with $(k,\ell) \in S_{N,M}\backslash\{(0,0)\}$, where

$$S_{N,M} = \{(i,j) | i = 0, 0 \le j \le M\} \cup \{(i,j) | 1 \le i \le N, -M \le j \le M\}$$

and N, M are *a priori* known. The driving noise of the AR model is a complex-valued Gaussian field such that its real and imaginary components are independent real Gaussian white noise fields each with zero mean and variance $\sigma^2/2$.

Assumption 2: The number P of harmonic components in (9), as well as the number $I^{(1,0)}$ of evanescent components in (6), are *a priori* known.

Assumption 3: The 1-D purely indeterministic processes $\{s_i^{(1,0)}\}$, of the type (8) are all assumed to be complex-valued Gaussian AR processes of known orders $V_i^{(1,0)}$. The driving noise of each of the AR models is an independent, zero-mean, complex-valued Gaussian process, such that its real and imaginary components are independent real Gaussian white noise processes each with zero mean and variance $(\sigma_i^{(1,0)})^2/2$.

In the proposed algorithm we take the approach of first estimating a *nonparametric* representation of the 1-D purely indeterministic processes $\{s_i^{(1,0)}\}$, and only in the second stage the AR models of these processes are estimated. Hence, in the first stage we estimate the particular values which the processes $\{s_i^{(1,0)}(n)\}_{n=0}^{S-1}$, $i = 1 \cdots I^{(1,0)}$ take for the given realization, i.e., we treat these as unknown constants.

To simplify the presentation of this section, we shall describe the solution for $I^{(1,0)} = 1$, i.e., in (6), $i \equiv 1$. Hence in the following we omit all the subindices i. Further, since we shall only deal with the case where $(\alpha, \beta) = (1,0)$, we shall omit the notation $(1,0)$ as superscripts and subscripts from our derivation, up to Section VII. Thus the parameters to be estimated are $\{C_p, \omega_p, \nu_p\}_{p=1}^{P}$, ν, $\{s(n)\}_{n=0}^{S-1}, \{b(k,\ell)\}_{(k,\ell)\in S_{N,M}}, \sigma^2$. We denote this vector of unknown parameters by $\boldsymbol{\theta}$.

Define

$$\boldsymbol{u} \triangleq [u(N,M), \cdots, u(N,T-1-M), u(N+1,M), \cdots, u(N+1,T-1-M), \cdots, u(S-1,T-1-M)]^T. \tag{11}$$

The vector $\boldsymbol{y}$ is similarly defined. Define (12) (at the bottom of this page). Also, we set (see (13) at the top of the following page)

$$\boldsymbol{W} \triangleq \begin{bmatrix} e^{j2\pi M\nu} \\ e^{j2\pi(M+1)\nu} \\ \vdots \\ e^{j2\pi(T-1-M)\nu} \end{bmatrix} \tag{14}$$

$$\boldsymbol{E_e} \triangleq \begin{bmatrix} \boldsymbol{W} & & & & \\ & \boldsymbol{W} & & \boldsymbol{0} & \\ & & \boldsymbol{W} & & \\ & \boldsymbol{0} & & \ddots & \\ & & & & \boldsymbol{W} \end{bmatrix} \tag{15}$$

and

$$\boldsymbol{b} \triangleq -[b(0,1), \cdots, b(0,M), b(1,-M), \cdots, b(1,M), \cdots, b(N,-M), \cdots, b(N,M)]^T. \tag{16}$$

B. Conditional ML Estimation in the Presence of a Single Evanescent Component

Since $u(n,m)$ is assumed to be Gaussian,

$$p(Y;\boldsymbol{\theta}, D \setminus D_1) = \frac{1}{(\pi\sigma^2)^{|D_1|}} \cdot \exp\left\{-\frac{1}{\sigma^2}\sum_{n=N}^{S-1}\sum_{m=M}^{T-1-M}|u(n,m)|^2\right\}. \tag{17}$$

The conditional MLE of $\boldsymbol{\theta}$ is found by maximizing (17), or equivalently by minimizing

$$J(\boldsymbol{\theta}) = \sum_{(n,m)\epsilon D_1} |u(n,m)|^2$$

$$Y \triangleq \begin{bmatrix} y(N,M-1) & \cdots & y(N,0) & y(N-1,2M) & \cdots & y(N-1,0) & \cdots & y(0,2M) & \cdots & y(0,0) \\ y(N,M) & \cdots & y(N,1) & y(N-1,2M+1) & \cdots & & \cdots & & \cdots & y(0,1) \\ \vdots & & & & \vdots & & & & \vdots & \\ y(N,T-M-2) & \cdots & y(N,T-1-2M) & y(N-1,T-1) & \cdots & y(N-1,T-1-2M) & \cdots & y(0,T-1) & \cdots & y(0,T-1-2M) \\ y(N+1,M-1) & \cdots & y(N+1,0) & y(N,2M) & \cdots & y(N,0) & \cdots & y(1,2M) & \cdots & y(1,0) \\ \vdots & & & & \vdots & & & & \vdots & \\ y(S-1,T-M-2) & \cdots & y(S-1,T-1-2M) & & & & \cdots & y(S-1-N,T-1) & \cdots & y(S-1-N,T-1-2M) \end{bmatrix}. \tag{12}$$

$$\boldsymbol{E_h} \triangleq \begin{bmatrix} e^{j2\pi[N\omega_1+M\nu_1]} & e^{j2\pi[N\omega_2+M\nu_2]} & \cdots & e^{j2\pi[N\omega_P+M\nu_P]} \\ \vdots & \vdots & \vdots & \vdots \\ e^{j2\pi[N\omega_1+(T-1-M)\nu_1]} & e^{j2\pi[N\omega_2+(T-1-M)\nu_2]} & \cdots & \\ e^{j2\pi[(N+1)\omega_1+M\nu_1]} & e^{j2\pi[(N+1)\omega_2+M\nu_2]} & \cdots & \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ e^{j2\pi[(S-1)\omega_1+(T-1-M)\nu_1]} & \cdots & \cdots & e^{j2\pi[(S-1)\omega_P+(T-1-M)\nu_P]} \end{bmatrix} \quad (13)$$

where $D_1 = \{(i,j)|N \le i \le S-1, M \le j \le T-1-M\}$, and $D \setminus D_1$ is the set of required initial conditions. Thus only actually occurring values of the observed field are used in the estimation procedure. Using this method we sum the squares of only $|D_1|$ values of $u(n,m)$, but this slight lost of information will be unimportant if the size of the observed field $|D|$ is large enough. Using (10), $u(n,m)$ is given by

$$u(n,m) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)w(n-k,m-\ell)$$

with $b(0,0) = 1$. Since $w = y - h - e$, we have

$$\begin{aligned} u(n,m) &= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\{y(n-k,m-\ell) \\ &\quad - h(n-k,m-\ell) - e(n-k,m-\ell)\} \\ &= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\Bigg\{y(n-k,m-\ell) \\ &\quad - \sum_{p=1}^{P} C_p e^{j2\pi[(n-k)w_p+(m-\ell)\nu_p]} \\ &\quad - s(n-k)e^{j2\pi\nu(m-\ell)}\Bigg\} \\ &= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)y(n-k,m-\ell) \\ &\quad - \sum_{p=1}^{P} C_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)e^{j2\pi[(n-k)w_p+(m-\ell)\nu_p]} \\ &\quad - \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)s(n-k)e^{j2\pi\nu(m-\ell)} \\ &= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)y(n-k,m-\ell) \\ &\quad - \sum_{p=1}^{P} C_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)e^{-j2\pi(k\omega_p+\ell\nu_p)}\right) \\ &\quad \cdot e^{j2\pi(n\omega_p+m\nu_p)} \\ &\quad - \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)s(n-k)e^{-j2\pi\nu\ell}\right)e^{j2\pi\nu m}. \end{aligned} \quad (18)$$

Define now the following transformations:

$$\mu_p \triangleq C_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)e^{-j2\pi(k\omega_p+\ell\nu_p)} \quad (19)$$

$$\eta(n) \triangleq \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)s(n-k)e^{-j2\pi\ell\nu}. \quad (20)$$

Let

$$B(e^{j2\pi\omega}, e^{j2\pi\nu}) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)e^{-j2\pi(\omega k+\nu\ell)}.$$

The assumptions made in Section II as to the properties of the spectral density function of the purely indeterministic field imply that the field AR model is such that $B(z_1, z_2)$ is minimum-phase. (We implicitly assume here that the finite support $B(z_1, z_2)$ defined above retains this property of the infinite support filter which corresponds to the AR model (10)). Since $B(e^{j2\pi\omega}, e^{j2\pi\nu})$ is nonzero on the unit bicircle, and in particular at the frequencies of the harmonic components, the transformation (19), of the C_p's to μ_p's is one-to-one. The transformation (20) is also one-to-one since given N initial values of the process $\{s(n)\}$, each newly introduced $s(n)$ results in a unique $\eta(n)$. The idea of using a transformation of the type (19) was developed in one dimension for estimating the parameters of harmonic signals in colored noise by Chatterjee *et al.* [14], as well as by Kay and Nagesha [15].

Let $S'_{N,M} = S_{N,M}\setminus\{(0,0)\}$. We can therefore rewrite (18) in the following form:

$$u(n,m) = y(n,m) + \sum_{(k,\ell)\in S'_{N,M}} b(k,\ell)y(n-k,m-\ell) - \sum_{p=1}^{P} \mu_p e^{j2\pi(n\omega_p+m\nu_p)} - \eta(n)e^{j2\pi\nu m} \quad (n,m)\in D_1. \quad (21)$$

Let

$$\boldsymbol{\mu} \triangleq [\mu_1, \quad \mu_2, \quad \cdots, \quad \mu_p]^T \quad (22)$$

and

$$\boldsymbol{\eta} \triangleq [\eta(N), \quad \eta(N+1), \quad \cdots \quad \eta(S-1)]^T. \quad (23)$$

Since $J(\boldsymbol{\theta}) = \boldsymbol{u^H u}$, we obtain by writing (21) for all $(n,m)\in D_1$, the following matrix representation for $J(\boldsymbol{\theta})$:

$$J(\boldsymbol{\theta}) = \|\boldsymbol{y} - \boldsymbol{Yb} - \boldsymbol{E_h\mu} - \boldsymbol{E_e\eta}\|^2. \quad (24)$$

Thus the transformations (19) and (20) allow us to minimize the objective function $J(\boldsymbol{\theta})$ with respect to $\boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\eta}$, and the deterministic component spectral support parameters, $\{\omega_p, \nu_p\}_{p=1}^P, \nu$, instead of minimizing it with respect to the original problem parameters. The properties of the above transformations guarantee that both minimizations will result in the same minima for J. Define $\boldsymbol{D} \triangleq [\boldsymbol{Y}\boldsymbol{E}_h\boldsymbol{E}_e]$ and $\boldsymbol{\theta}_1 \triangleq [\boldsymbol{b}^T\boldsymbol{\mu}^T\boldsymbol{\eta}^T]^T$. Then we can rewrite (24) as

$$J(\boldsymbol{\theta}) = \|\boldsymbol{y} - \boldsymbol{D}\boldsymbol{\theta}_1\|^2. \tag{25}$$

Because of the fact that the objective function is a quadratic function of $\boldsymbol{\theta}_1$, the minimization over $\boldsymbol{\theta}_1$ can be carried out analytically for any given value of $\boldsymbol{D}$. Using the well-known solution to the least squares problem we have that

$$\hat{\boldsymbol{\theta}}_1 = (\boldsymbol{D}^H\boldsymbol{D})^{-1}\boldsymbol{D}^H\boldsymbol{y} \tag{26}$$

will minimize $J(\boldsymbol{\theta})$ over $\boldsymbol{\theta}_1$. By inserting (26) into (25) we find that the minimum value of $J(\boldsymbol{\theta})$ is given by

$$J_{\min}(\{\omega_p, \nu_p\}_{p=1}^P, \nu) = \boldsymbol{y}^H(\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{D}^H\boldsymbol{D})^{-1}\boldsymbol{D}^H)\boldsymbol{y}. \tag{27}$$

Here $\boldsymbol{D}$ is assumed to be full rank so that $(\boldsymbol{D}^H\boldsymbol{D})^{-1}$ exists.

Thus maximization of the likelihood function is achieved by minimizing the new objective function $J_{\min}(\{\omega_p, \nu_p\}_{p=1}^P, \nu)$, which is a function only of the deterministic component spectral support parameters. We have thus shown that the minimization problem (24) which is obtained after taking the transformations (19), (20) is separable since its solution can be reduced to a minimization problem in the nonlinear deterministic component's spectral support parameters, $\{\omega_p, \nu_p\}_{p=1}^P, \nu$, only, while $\boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\eta}$ can then be determined by solving a *linear* least squares problem. This new minimization problem is of a considerably lower complexity. A broad discussion on the subject of separable, nonlinear, least squares minimization problems can be found in [11] and [12]. Since $J_{\min}(\{\omega_p, \nu_p\}_{p=1}^P, \nu)$ is a nonlinear function of $\{\omega_p, \nu_p\}_{p=1}^P, \nu$, this optimization problem cannot be solved analytically and we must resort to numerical methods. In order to avoid the enormous computational burden of an exhaustive search, we use the two-step procedure which is described in Section V.

In the discussion above we assumed that the noise variance σ^2 is known. If it is not known, it can be estimated. The maximum-likelihood estimate of σ^2 is derived by maximizing (17) with respect to σ^2. Using the estimated frequencies and (27) we have that

$$\hat{\sigma}^2 = \frac{\hat{J}_{\min}}{(S-N)(T-2M)}. \tag{28}$$

Thus (26) and (28) establish the estimate for the autoregressive model of the purely indeterministic component of the field. Using the estimated frequencies of the harmonic component and the transformation (19), a complete estimate for the parameters of the harmonic component is obtained. The solution for the parameters of the evanescent field is more involved and it is given in the next section.

IV. Estimating the Parameters of the Evanescent Component

Define

$$B(k) \triangleq \begin{cases} \sum_{\ell=0}^{M} b(0,\ell)e^{-j2\pi\ell\nu}, & k=0 \\ \sum_{\ell=-M}^{M} b(k,\ell)e^{-j2\pi\ell\nu}, & 1 \le k \le N. \end{cases} \tag{29}$$

Let us rewrite (20)

$$\begin{aligned} \eta(n) &= \left[\sum_{\ell=0}^{M} b(0,\ell)s(n)e^{-j2\pi\ell\nu} + \sum_{k=1}^{N}\sum_{\ell=-M}^{M} b(k,\ell)s(n-k)e^{-j2\pi\ell\nu}\right] \\ &= \left[s(n)B(0) + \sum_{k=1}^{N} B(k)s(n-k)\right] \\ &= \sum_{k=0}^{N} B(k)s(n-k), \qquad n = N,\cdots,S-1. \end{aligned} \tag{30}$$

Note, that as a result of the preceding stages of the algorithm, the *estimated* values of the $\eta(n)$'s and $B(k)$'s are available, rather than the *true* values. We next present two different approaches for estimating the parameters of the 1-D purely indeterministic process $\{s(n)\}$.

Following (8) and Assumption 3, we have that

$$s(n) = -\sum_{t=1}^{V} a(t)s(n-t) + \Gamma(n). \tag{31}$$

Substituting (31) into (30), we obtain

$$\begin{aligned} \eta(n) &= \sum_{k=0}^{N} B(k)\left[-\sum_{t=1}^{V} a(t)s(n-k-t) + \Gamma(n-k)\right] \\ &= -\sum_{t=1}^{V} a(t)\sum_{k=0}^{N} B(k)s(n-k-t) + \sum_{k=0}^{N} B(k)\Gamma(n-k) \\ &= -\sum_{t=1}^{V} a(t)\eta(n-t) + \sum_{k=0}^{N} B(k)\Gamma(n-k) \\ &\qquad n = N,\cdots,S-1. \end{aligned} \tag{32}$$

Hence, (32) implies that solving the problem of estimating the unknown parameters of the 1-D purely indeterministic process associated with the evanescent component, is equivalent to solving the above 1-D ARMA equation. In the equivalent problem, the MA parameters $\{B(k)\}_{k=0}^N$ have previously been estimated, the driving Gaussian noise source is of an unknown variance $(\sigma^{(1,0)})^2$, and the "observations" are the $\{\eta(n)\}$, for $n = N,\cdots,S-1$. In [17, pp. 205–208] it is shown how the exact likelihood function of an ARMA process can be computed, from the ARMA model parameters, by using the Kalman filter. Hence, maximization of the likelihood function with respect to the unknown parameters $\{a(t)\}_{t=1}^V, \sigma^{(1,0)}$ will result in an ML estimate of the ARMA parameters.

The alternative approach is simple to implement, but suboptimal. Moreover, this solution can be used to initialize the above ML algorithm. Define the 1-D function

$$\mathcal{B}(z_1) = \sum_{k=0}^{N} B(k) z_1^{-k}.$$

Note that the parameters of this function were previously estimated. Hence, if this estimated function is minimum-phase then, instead of solving (30), we can solve the equivalent problem obtained by deconvolving the sequence $\{B(k)\}_{k=0}^{N}$ out of the sequence $\{\eta(n)\}_{n=N}^{S-1}$. The result of this deconvolution is the sequence $\{s(n)\}_{n=0}^{S-1}$. However, the system (30) is clearly an underdetermined system having S unknowns with only $S-N$ equations. Hence, we obtain a solution for the required unknown sequence $\{s(n)\}_{n=0}^{S-1}$ by filtering the sequence $\{\eta(n)\}_{n=N}^{S-1}$ through the IIR filter $\mathcal{B}^{-1}(z_1)$ with zero initial conditions. Since $\{s(n)\}$ is an AR process, a standard AR model parameter estimation algorithm can now be applied to estimate its parameters. (In the present paper we use the conditional maximum-likelihood estimator [18]).

V. The Solution for the Spectral Support Parameters of the Deterministic Component

In Section III we concluded that the minimization problem (24) which is obtained after taking the transformations (19), (20) is *separable* since its solution can be reduced to a minimization problem in the nonlinear deterministic component's spectral support parameters $\{\omega_p, \nu_p\}_{p=1}^{P}, \nu$ only, while $\boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{\eta}$ can then be determined by solving a linear least squares problem. Hence, the first step in solving the presented estimation problem is the minimization of $J_{\min}$ with respect to the unknown spectral support parameters of the deterministic component. Since $J_{\min}$ is a nonlinear function of the deterministic component's spectral support parameters this optimization problem cannot be solved analytically and we must resort to numerical methods.

In general, $J_{\min}$ has a complicated multimodal shape. Hence, in order to avoid the enormous computational burden of an exhaustive search, we used the following two-step procedure. In the first stage, we obtain a suboptimal initial estimate for the parameters of the spectral support of the deterministic component. This stage is implemented by solving the system of overdetermined 2-D normal equations for the parameters of a high-order linear predictor of the observed data. In the second stage, we refine these initial estimates by an iterative numerical minimization of the objective function $J_{\min}$. In our experiments we used the conjugate gradient method of Fletcher and Reeves [19, p. 253]. Note that only for the case of a quadratic objective function, the conjugate gradient procedure is guaranteed to converge in at most N steps. For our problem, we simply restart the algorithm using new gradients, until the objective function becomes appreciably small. As is well known, this type of iterative optimization procedure converges to a local minimum, and does not guarantee global optimality, unless the initial estimate is sufficiently close to the global optimum. As we show in Section VII, the initial estimates provided by the solution of the overdetermined high-order normal equations appear to provide a good initial starting point (i.e., one which leads to convergence to the global minimum) as long as the local signal-to-noise ratio is sufficiently high, and the frequencies of the different deterministic components are not too close. We next describe the initialization algorithm.

Formalizing the 2-D linear prediction problem for some NSHP predictor with support

$$S = \{(k,\ell) | k = 0, 0 < \ell \le L\} \cup \{(k,\ell) | 1 \le k \le K, -L \le \ell \le L\}$$

results in the following linear system of equations:

$$\sum_{(k,\ell)\in S} a(k,\ell) r(i-k, j-\ell) = -r(i,j) \qquad (i,j) \succ (0,0) \tag{33}$$

where $\{a(k,\ell)\}_{(k,\ell)\in S}$ are the linear predictor coefficients. Rewriting the system (33) in a matrix form for all $(i,j) \in S$ results in the well-known 2-D Yule–Walker equations. Including in this system additional equations for $(i,j) \succ (0,0)$ such that $(i,j) \notin S$ results in an overdetermined system. The overdetermined Yule–Walker method is a modification of the basic Yule–Walker method, which was reported [4] to lead to a considerable increase in the estimation accuracy of the frequencies of harmonic signals in white noise for 1-D signals. It is further concluded in [4], that the asymptotic accuracy of the estimates will increase with the number of Yule–Walker equations used and with the model support. Intuitively, it can be expected that increasing the predictor support (i.e., increasing K and L), will improve the accuracy of the estimates of the deterministic component's spectral support, since the covariances for large lags contain "useful information" about the deterministic component. In order to solve (33) for the linear predictor parameters, the covariances of the observed field must be available. Since the covariance functions of the observed field are unknown they must be estimated from the data itself. Hence, in (33) we replace the true covariances by their estimates. However, as noted earlier, the ergodic property of the sample covariances does not generally hold in the present case. This is due to the result [6], that the sample covariances of a Gaussian process are consistent estimates of the covariance function if and only if the spectral distribution function of the process has no discontinuities. Clearly, this requirement does not hold in the present case. Nevertheless, since in practice the above method produces accurate estimates of the spectral support of the deterministic component, and since these estimates are used only to initialize the iterative minimization of the cost function $J_{\min}$, the above theoretical problem is avoided.

Since (33) is an overdetermined system, it is solved in the least squares sense for the linear predictor coefficients. We then look for the peaks of $1/|\hat{A}(e^{j2\pi\omega}, e^{j2\pi\nu})|^2$ to obtain the required initial estimates of the deterministic component spectral support parameters.

VI. The Cramer–Rao Lower Bound

A conditional ML algorithm for estimating the parameters of the harmonic and evanescent components in the presence of a 2-D circular Gaussian AR noise, jointly with estimating the AR model parameters, was suggested in the previous sections. In this section we derive the performance bound for this algorithm. As is well known, the Cramer–Rao Bound (CRB) provides a lower bound for the covariance matrix of any unbiased estimator of the model parameters. The bound is given by the inverse of the so-called Fisher Information Matrix (FIM) [16]. In the following derivation, the number of components of the evanescent field is $I^{(1,0)}$, the number of harmonic components is P, and the 2-D AR model support is $S_{N,M}$.

The conditional CRB is derived from the conditional probability density function of the observed process. Rewriting (17) we obtain

$$p(Y;\boldsymbol{\theta},D\setminus D_1)=\frac{1}{(\pi\sigma^2)^{|D_1|}}\exp\left\{-\frac{1}{\sigma^2}\sum_{(n,m)\in D_1}\left|\sum_{(k,\ell)\in S_{N,M}}b(k,\ell)[y(n-k,m-\ell)-d(n-k,m-\ell)]\right|^2\right\} \quad (34)$$

were we define

$$d(n,m)\triangleq h(n,m)+e(n,m). \quad (35)$$

Note that in the present framework $\{d(n,m)\}$ is the mean component of $\{y(n,m)\}$.

Collecting the parameters of the harmonic component into vector representations, we have

$$\boldsymbol{c}=[C_1,\cdots,C_P]^T \quad (36)$$
$$\boldsymbol{f}_i=[\omega_i,\nu_i]^T \quad (37)$$
$$\boldsymbol{\omega}_h=[\boldsymbol{f}_1^T,\cdots,\boldsymbol{f}_P^T]^T. \quad (38)$$

The parameter vectors of the evanescent components are defined by

$$\boldsymbol{\nu}_e=[\nu_1,\nu_2,\cdots,\nu_{I^{(1,0)}}]^T \quad (39)$$
$$\boldsymbol{s}_i^0=[s_i(0),s_i(1),\cdots,s_i(N-1)]^T \quad (40)$$
$$\boldsymbol{s}_i=[s_i(N),s_i(N+1),\cdots,s_i(S-1)]^T \quad (41)$$
$$\boldsymbol{x}_i=[\boldsymbol{s}_i^{0^T},\boldsymbol{s}_i^T]^T \quad (42)$$
$$\boldsymbol{s}=[\boldsymbol{s}_1^T,\boldsymbol{s}_2^T,\cdots,\boldsymbol{s}_{I^{(1,0)}}^T]^T. \quad (43)$$

Also let

$$\boldsymbol{d}=[d(0,0),\cdots,d(0,T-1),d(1,0),\cdots,d(1,T-1),\cdots,d(S-1,0),\cdots,d(S-1,T-1)]^T \quad (44)$$
$$\boldsymbol{h}=[h(0,0),\cdots,h(0,T-1),h(1,0),\cdots,h(1,T-1),\cdots,h(S-1,0),\cdots,h(S-1,T-1)]^T \quad (45)$$
$$\boldsymbol{e}_i=[e_i(0,0),\cdots,e_i(0,T-1),e_i(1,0),\cdots,e_i(1,T-1),\cdots,e_i(S-1,0),\cdots,e_i(S-1,T-1)]^T. \quad (46)$$

Rewriting (35) in vector form we have

$$\boldsymbol{d}=\boldsymbol{h}+\sum_{i=1}^{I^{(1,0)}}\boldsymbol{e}_i. \quad (47)$$

Let $\boldsymbol{d}^R=\mathrm{Re}\{\boldsymbol{d}\},\boldsymbol{d}^I=\mathrm{Im}\{\boldsymbol{d}\},\tilde{\boldsymbol{d}}=[\boldsymbol{d}^{R^T}\boldsymbol{d}^{I^T}]^T$. In a similar way we define the vectors $\boldsymbol{h}^R,\boldsymbol{h}^I,\tilde{\boldsymbol{h}},\boldsymbol{e}_i^R,\boldsymbol{e}_i^I,\tilde{\boldsymbol{e}}_i,\boldsymbol{x}_i^R,\boldsymbol{x}_i^I,\tilde{\boldsymbol{x}}_i,\boldsymbol{c}^R,\boldsymbol{c}^I,\tilde{\boldsymbol{c}},\boldsymbol{s}^R,\boldsymbol{s}^I,\tilde{\boldsymbol{s}}$. Also let $b^R(k,\ell)=\mathrm{Re}\{b(k,\ell)\}$, $b^I(k,\ell)=\mathrm{Im}\{b(k,\ell)\}$. Let us denote by $\overline{\boldsymbol{\theta}}$ the mean component parameter vector, i.e., $\overline{\boldsymbol{\theta}}=[\tilde{\boldsymbol{c}}^T\boldsymbol{\omega}_h^T\tilde{\boldsymbol{s}}^T\boldsymbol{\nu}_e^T]^T$. Note that the initial conditions vectors $\boldsymbol{s}_i^0, i=1,\cdots,I^{(1,0)}$, are not parameters to be estimated, as they are assumed known. Taking now the partial derivatives of the conditional log-likelihood function w.r.t. the elements of $\overline{\boldsymbol{\theta}}$, we have

$$\frac{\partial\ln P}{\partial\overline{\boldsymbol{\theta}}(i)}=\frac{1}{\sigma^2}\left\{\sum_{(n,m)\in D_1}\left(\sum_{(k,\ell)\in S_{N,M}}b(k,\ell)\frac{\partial d(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\cdot\left(\sum_{(k,\ell)\in S_{N,M}}b(k,\ell)[y(n-k,m-\ell)-d(n-k,m-\ell)]\right)^*\right.$$
$$\left.+\sum_{(n,m)\in D_1}\left(\sum_{(k,\ell)\in S_{N,M}}b(k,\ell)[y(n-k,m-\ell)-d(n-k,m-\ell)]\right)\cdot\left(\sum_{(k,\ell)\in S_{N,M}}b^*(k,\ell)\frac{\partial d^*(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\right\}. \quad (48)$$

Taking the partial derivative w.r.t. the AR process driving noise variance parameter yields

$$-E\left\{\frac{\partial^2\ln P}{\partial\sigma^2\partial\overline{\boldsymbol{\theta}}(i)}\right\}=\frac{1}{\sigma^4}E\left\{\sum_{(n,m)\in D_1}\left(\sum_{(k,\ell)\in S_{N,M}}b(k,\ell)\frac{\partial d(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\cdot\left(\sum_{(k,\ell)\in S_{N,M}}b(k,\ell)[y(n-k,m-\ell)-d(n-k,m-\ell)]\right)^*\right.$$
$$\left.+\sum_{(n,m)\in D_1}\left(\sum_{(k,\ell)\in S_{N,M}}b(k,\ell)[y(n-k,m-\ell)-d(n-k,m-\ell)]\right)\cdot\left(\sum_{(k,\ell)\in S_{N,M}}b^*(k,\ell)\frac{\partial d^*(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\right\}$$
$$=0. \quad (49)$$

Similarly, taking the partial derivatives w.r.t. the real and imaginary parts of the AR process parameters we find that

$$
\begin{aligned}
-E&\left\{\frac{\partial^2 \ln P}{\partial\overline{\boldsymbol{\theta}}(i)\partial b^R(k,\ell)}\right\}\\
&= -\frac{1}{\sigma^2}E\left\{\sum_{(n,m)\in D_1}\frac{\partial d(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right.\\
&\quad\cdot\left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)[y(n-k,m-\ell)\right.\\
&\quad\left.\left.-\,d(n-k,m-\ell)]\right)^*\right\}\\
&\quad-\frac{1}{\sigma^2}E\left\{\sum_{(n,m)\in D_1}\left(\sum_{(k,\ell)\in S_{N,M}}\right.\right.\\
&\quad\left.\cdot\, b(k,\ell)\frac{\partial d(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\\
&\quad\left.\cdot\,[y(n-k,m-\ell)-d(n-k,m-\ell)]^*\right\}\\
&\quad-\frac{1}{\sigma^2}E\left\{\sum_{(n,m)\in D_1}[y(n-k,m-\ell)\right.\\
&\quad-\,d(n-k,m-\ell)]\\
&\quad\left.\cdot\left(\sum_{(k,\ell)\in S_{N,M}} b^*(k,\ell)\frac{\partial d^*(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\right\}\\
&\quad-\frac{1}{\sigma^2}E\left\{\sum_{(n,m)\in D_1}\left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\right.\right.\\
&\quad\left.\cdot\,[y(n-k,m-\ell)-d(n-k,m-\ell)]\right)\\
&\quad\left.\cdot\,\frac{\partial d^*(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right\}=0.
\end{aligned}
\tag{50}
$$

Similar derivation w.r.t. $b^I(k,\ell)$, yields

$$
-E\left\{\frac{\partial^2 \ln P}{\partial\overline{\boldsymbol{\theta}}(i)\partial b^I(k,\ell)}\right\}=0. \tag{51}
$$

Thus the conditional FIM is block-diagonal. Hence, the conditional CRB on the harmonic and evanescent components parameters is decoupled from the bound on AR process parameters. Therefore, the conditional CRB's on the deterministic component, and on the AR component parameters are obtained by inverting the FIM blocks which correspond to the deterministic and the AR parameters, respectively. Asymptotic CRB on the parameters of an AR field with an NSHP support is given in [13]. In the following we concentrate on deriving the conditional CRB on the parameters of the harmonic and evanescent components. Taking the partial derivatives w.r.t. the mean component parameters we find that

$$
\begin{aligned}
-E&\left\{\frac{\partial^2 \ln P}{\partial\overline{\boldsymbol{\theta}}(i)\partial\overline{\boldsymbol{\theta}}(j)}\right\} = \frac{1}{\sigma^2}\sum_{(n,m)\in D_1}\\
&\cdot\left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\frac{\partial d(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\\
&\cdot\left(\sum_{(k,\ell)\in S_{N,M}} b^*(k,\ell)\frac{\partial d^*(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(j)}\right)\\
&+\frac{1}{\sigma^2}\sum_{(n,m)\in D_1}\\
&\cdot\left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\frac{\partial d(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(j)}\right)\\
&\cdot\left(\sum_{(k,\ell)\in S_{N,M}} b^*(k,\ell)\frac{\partial d^*(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\\
&=\frac{2}{\sigma^2}\mathrm{Re}\left\{\sum_{(n,m)\in D_1}\left(\sum_{(k,\ell)\in S_{N,M}}\right.\right.\\
&\quad\left.\cdot\, b^*(k,\ell)\frac{\partial d^*(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(i)}\right)\\
&\quad\left.\cdot\left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\frac{\partial d(n-k,m-\ell)}{\partial\overline{\boldsymbol{\theta}}(j)}\right)\right\}.
\end{aligned}
\tag{52}
$$

Thus the above derivation of the conditional FIM reveals that the bounds on both the amplitude and the frequency parameters of the harmonic components, as well as the bounds on the parameters of the evanescent components, are functions of the frequency response of the colored noise model at the frequencies of the spectral support of the deterministic components, and of the *derivative* of the frequency response at these frequencies.

Let $\overline{\boldsymbol{d}}$ be the "flipped-around" version of $\boldsymbol{d}$, i.e.,

$$
\begin{aligned}
\overline{\boldsymbol{d}} = [&d(S-1,T-1),\cdots,d(S-1,0),\cdots,\\
&d(1,T-1),\cdots,d(1,0),d(0,T-1),\cdots,d(0,0)]^T.
\end{aligned}
\tag{53}
$$

Note that $\overline{\boldsymbol{d}} = \boldsymbol{K}\boldsymbol{d}$, where

$$
\boldsymbol{K} = \begin{bmatrix} 0 & \cdots & 0 & 1\\ 0 & \cdots & 1 & 0\\ \vdots & & & \vdots\\ 1 & \cdots & 0 & 0\end{bmatrix} \tag{54}
$$

$$\overline{\boldsymbol{B}} = \left[\begin{array}{cccc|cccc|c|cc} \overline{\boldsymbol{b}}^* & 0 & \cdots & 0 & \boldsymbol{0}_T & \boldsymbol{0}_T & \cdots & \boldsymbol{0}_T & & \cdots & \boldsymbol{0}_{T(S-N-1)} \\ 0 & \overline{\boldsymbol{b}}^* & & \vdots & \overline{\boldsymbol{b}}^* & 0 & & 0 & & \cdots & 0 \\ 0 & 0 & & 0 & 0 & \overline{\boldsymbol{b}}^* & & \vdots & & & \\ 0 & 0 & & \overline{\boldsymbol{b}}^* & \vdots & 0 & & 0 & & & \\ \vdots & \vdots & & 0 & & & & \overline{\boldsymbol{b}}^* & \cdots & & \vdots \\ & & & \vdots & & & & 0 & & & \\ & & & \vdots & & \vdots & & \vdots & & & 0 \\ 0 & 0 & & 0 & 0 & 0 & & 0 & & & \overline{\boldsymbol{b}}^* \end{array}\right] \tag{57}$$

is the exchange matrix. Let $\boldsymbol{0}_k$ denote a k-dimensional column vector of zeros. Let also

$$\begin{aligned} \boldsymbol{b}_0 &= [\boldsymbol{0}_M^T, 1, b(0,1), \cdots, b(0,M), \boldsymbol{0}_{T-(2M+1)}^T]^T \\ \boldsymbol{b}_1 &= [b(1,-M), \cdots, b(1,0), \cdots, b(1,M), \boldsymbol{0}_{T-(2M+1)}^T]^T \\ &\vdots \qquad\qquad \vdots \\ \boldsymbol{b}_{N-1} &= [b(N-1,-M), \cdots, b(N-1,0), \cdots, \\ &\qquad b(N-1,M), \boldsymbol{0}_{T-(2M+1)}^T]^T, \\ \boldsymbol{b}_N &= [b(N,-M), \cdots, b(N,0), \cdots, b(N,M)]^T \end{aligned} \tag{55}$$

and

$$\overline{\boldsymbol{b}} = [\boldsymbol{b}_0^T, \boldsymbol{b}_1^T, \cdots, \boldsymbol{b}_N^T]^T. \tag{56}$$

Let $\overline{\boldsymbol{b}}^*$ denote the conjugate of $\overline{\boldsymbol{b}}$. Define the following $S \cdot T \times (S-N) \cdot (T-2M)$ matrix (see (57) at the top of this page) where each of the $S-N$ blocks is an $S \cdot T \times (T-2M)$ matrix.

Using these notations (52) can be written in the following matrix form:

$$-E\left\{\frac{\partial^2 \ln P}{\partial \overline{\boldsymbol{\theta}}(i) \partial \overline{\boldsymbol{\theta}}(j)}\right\} = \frac{2}{\sigma^2} \operatorname{Re}\left\{\frac{\partial \overline{\boldsymbol{d}}^H}{\partial \overline{\boldsymbol{\theta}}(i)} \overline{\boldsymbol{B}}\, \overline{\boldsymbol{B}}^H \frac{\partial \overline{\boldsymbol{d}}}{\partial \overline{\boldsymbol{\theta}}(j)}\right\}. \tag{58}$$

We next evaluate $\partial \boldsymbol{d}/\partial \overline{\boldsymbol{\theta}}(i)$ for each of the parameters of the deterministic component. We begin with the harmonic components. Let (see (59) at the bottom of this page), i.e., the ith column of $\overline{\boldsymbol{E}}_{\boldsymbol{h}}$ consists of the values of the ith harmonic component evaluated for all $(s,t) \in D$. We therefore have

$$\tilde{\boldsymbol{h}} = \boldsymbol{\rho} \tilde{\boldsymbol{c}} \tag{60}$$

where

$$\boldsymbol{\rho} = \begin{bmatrix} \overline{\boldsymbol{E}}_{\boldsymbol{h}}^R & -\overline{\boldsymbol{E}}_{\boldsymbol{h}}^I \\ \overline{\boldsymbol{E}}_{\boldsymbol{h}}^I & \overline{\boldsymbol{E}}_{\boldsymbol{h}}^R \end{bmatrix} \tag{61}$$

and $\overline{\boldsymbol{E}}_{\boldsymbol{h}}^R = \operatorname{Re}\{\overline{\boldsymbol{E}}_{\boldsymbol{h}}\}, \overline{\boldsymbol{E}}_{\boldsymbol{h}}^I = \operatorname{Im}\{\overline{\boldsymbol{E}}_{\boldsymbol{h}}\}$. Taking the partial derivatives of $\tilde{\boldsymbol{d}}$ with respect to the harmonic component amplitude parameters we find that

$$\frac{\partial \tilde{\boldsymbol{d}}}{\partial \tilde{\boldsymbol{c}}(\ell)} = \boldsymbol{\rho}_\ell \tag{62}$$

where $\tilde{\boldsymbol{c}}(\ell)$ is the ℓth element of $\tilde{\boldsymbol{c}}$, and $\boldsymbol{\rho}_\ell$ is the ℓth column of $\boldsymbol{\rho}$. Hence

$$\begin{aligned} \frac{\partial \boldsymbol{d}}{\partial \tilde{\boldsymbol{c}}(\ell)} &= \frac{\partial \boldsymbol{d}^R}{\partial \tilde{\boldsymbol{c}}(\ell)} + j \frac{\partial \boldsymbol{d}^I}{\partial \tilde{\boldsymbol{c}}(\ell)} \\ &= \boldsymbol{\rho}_\ell^R + j \boldsymbol{\rho}_\ell^I \end{aligned} \tag{63}$$

where

$$\begin{aligned} \boldsymbol{\rho}_\ell^R &= [\boldsymbol{\rho}_\ell(0), \cdots, \boldsymbol{\rho}_\ell(ST-1)]^T \\ \boldsymbol{\rho}_\ell^I &= [\boldsymbol{\rho}_\ell(ST), \cdots, \boldsymbol{\rho}_\ell(2ST-1)]^T. \end{aligned} \tag{64}$$

Let

$$\begin{aligned} \boldsymbol{\tau}_1 &= [0, 1, \cdots, (S-1)^T \otimes \boldsymbol{1_T} \\ \boldsymbol{\tau}_2 &= \boldsymbol{1}_S \otimes [0, 1, \cdots, (T-1)]^T \end{aligned} \tag{65}$$

where $\boldsymbol{1_T}$ and $\boldsymbol{1_S}$ are T-dimensional and S-dimensional column vectors of ones, respectively, and $\otimes$ is the Kronecker product. In other words, $\boldsymbol{\tau}_1$ is the vector of the first indices of

$$\overline{\boldsymbol{E}}_{\boldsymbol{h}} = \begin{bmatrix} e^{j2\pi[0\omega_1+0\nu_1]} & e^{j2\pi[0\omega_2+0\nu_2]} & \cdots & e^{j2\pi[0\omega_P+0\nu_P]} \\ e^{j2\pi[0\omega_1+1\nu_1]} & e^{j2\pi[0\omega_2+1\nu_2]} & \cdots & e^{j2\pi[0\omega_P+1\nu_P]} \\ \vdots & \vdots & \vdots & \vdots \\ e^{j2\pi[0\omega_1+(T-1)\nu_1]} & e^{j2\pi[0\omega_2+(T-1)\nu_2]} & \cdots & e^{j2\pi[0\omega_P+(T-1)\nu_P]} \\ e^{j2\pi[1\omega_1+0\nu_1]} & e^{j2\pi[1\omega_2+0\nu_2]} & \cdots & e^{j2\pi[1\omega_P+0\nu_P]} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ e^{j2\pi[(S-1)\omega_1+(T-1)\nu_1]} & \cdots & \cdots & e^{j2\pi[(S-1)\omega_P+(T-1)\nu_P]} \end{bmatrix} \tag{59}$$

the elements of $\boldsymbol{d}$ in (44), and $\boldsymbol{\tau}_2$ is the vector of the second indices of the elements of $\boldsymbol{d}$. Taking now the partial derivatives w.r.t. the harmonic frequencies yields

$$\begin{aligned}\frac{\partial \boldsymbol{h}^R}{\partial \omega_p} &= -2\pi \text{ diag}(\boldsymbol{\tau}_1)(\boldsymbol{c}^R(p)\overline{\boldsymbol{E}}^I_{\boldsymbol{h}_p} + \boldsymbol{c}^I(p)\overline{\boldsymbol{E}}^R_{\boldsymbol{h}_p})\\ \frac{\partial \boldsymbol{h}^R}{\partial \nu_p} &= -2\pi \text{ diag}(\boldsymbol{\tau}_2)(\boldsymbol{c}^R(p)\overline{\boldsymbol{E}}^I_{\boldsymbol{h}_p} + \boldsymbol{c}^I(p)\overline{\boldsymbol{E}}^R_{\boldsymbol{h}_p})\\ \frac{\partial \boldsymbol{h}^I}{\partial \omega_p} &= 2\pi \text{ diag}(\boldsymbol{\tau}_1)(-\boldsymbol{c}^I(p)\overline{\boldsymbol{E}}^I_{\boldsymbol{h}_p} + \boldsymbol{c}^R(p)\overline{\boldsymbol{E}}^R_{\boldsymbol{h}_p})\\ \frac{\partial \boldsymbol{h}^I}{\partial \nu_p} &= 2\pi \text{ diag}(\boldsymbol{\tau}_2)(-\boldsymbol{c}^I(p)\overline{\boldsymbol{E}}^I_{\boldsymbol{h}_p} + \boldsymbol{c}^R(p)\overline{\boldsymbol{E}}^R_{\boldsymbol{h}_p}) \end{aligned} \tag{66}$$

where diag$(\boldsymbol{\tau}_1)$, (diag$(\boldsymbol{\tau}_2)$) is an $ST \times ST$ matrix whose diagonal is the vector $\boldsymbol{\tau}_1, (\boldsymbol{\tau}_2)$. $\boldsymbol{c}^R(p), (\boldsymbol{c}^I(p))$, is the pth element of $\boldsymbol{c}_R, (\boldsymbol{c}^I)$, and $\overline{\boldsymbol{E}}^R_{\boldsymbol{h}_p}, (\overline{\boldsymbol{E}}^I_{\boldsymbol{h}_p})$ is the pth column of $\overline{\boldsymbol{E}}^R_{\boldsymbol{h}}, (\overline{\boldsymbol{E}}^I_{\boldsymbol{h}})$. Hence

$$\begin{aligned}\frac{\partial \boldsymbol{d}}{\partial \omega_p} &= \frac{\partial \boldsymbol{h}^R}{\partial \omega_p} + j\frac{\partial \boldsymbol{h}^I}{\partial \omega_p}\\ \frac{\partial \boldsymbol{d}}{\partial \nu_p} &= \frac{\partial \boldsymbol{h}^R}{\partial \nu_p} + j\frac{\partial \boldsymbol{h}^I}{\partial \nu_p}.\end{aligned} \tag{67}$$

Similarly, for the parameters of the ith evanescent component, we first define

$$\overline{\boldsymbol{W}}_i = \begin{bmatrix} e^{j2\pi 0\nu_i} \\ e^{j2\pi 1\nu_i} \\ e^{j2\pi 2\nu_i} \\ \vdots \\ e^{j2\pi(T-1)\nu_i}\end{bmatrix} \tag{68}$$

$$\overline{\boldsymbol{E}}_{\boldsymbol{e}_i} = \boldsymbol{I}_{S\times S} \otimes \overline{\boldsymbol{W}}_i \tag{69}$$

where $\boldsymbol{I}_{S\times S}$ is an $S \times S$ identity matrix.

We therefore have

$$\tilde{\boldsymbol{e}}_i = \boldsymbol{\psi}_i \tilde{\boldsymbol{x}}_i, \tag{70}$$

where

$$\boldsymbol{\psi}_i = \begin{bmatrix} \overline{\boldsymbol{E}}^R_{\boldsymbol{e}_i} & -\overline{\boldsymbol{E}}^I_{\boldsymbol{e}_i} \\ \overline{\boldsymbol{E}}^I_{\boldsymbol{e}_i} & \overline{\boldsymbol{E}}^R_{\boldsymbol{e}_i}\end{bmatrix} \tag{71}$$

and $\overline{\boldsymbol{E}}^R_{\boldsymbol{e}_i} = \text{Re}\{\overline{\boldsymbol{E}}_{\boldsymbol{e}_i}\}, \overline{\boldsymbol{E}}^I_{\boldsymbol{e}_i} = \text{Im}\{\overline{\boldsymbol{E}}_{\boldsymbol{e}_i}\}$. Taking now the partial derivatives w.r.t. the ith evanescent component frequency yields

$$\begin{aligned}\frac{\partial \boldsymbol{e}^R_i}{\partial \nu_{e_i}} &= -2\pi \text{ diag}(\boldsymbol{\tau}_2)(\overline{\boldsymbol{E}}^I_{\boldsymbol{e}_i}\boldsymbol{x}^R_i + \overline{\boldsymbol{E}}^R_{\boldsymbol{e}_i}\boldsymbol{x}^I_i)\\ &= -2\pi \text{ diag}(\boldsymbol{\tau}_2)\boldsymbol{e}^I_i\\ \frac{\partial \boldsymbol{e}^I_i}{\partial \nu_{ei}} &= 2\pi \text{ diag}(\boldsymbol{\tau}_2)(\overline{\boldsymbol{E}}^R_{\boldsymbol{e}_i}\boldsymbol{x}^R_i - \overline{\boldsymbol{E}}^I_{\boldsymbol{e}_i}\boldsymbol{x}^I_i)\\ &= 2\pi \text{ diag}(\boldsymbol{\tau}_2)\boldsymbol{e}^R_i.\end{aligned} \tag{72}$$

Hence

$$\begin{aligned}\frac{\partial \boldsymbol{d}}{\partial \nu_{ei}} &= \frac{\partial \boldsymbol{e}^R_i}{\partial \nu_{ei}} + j\frac{\partial \boldsymbol{e}^I_i}{\partial \nu_{ei}}\\ &= j2\pi \text{ diag}(\boldsymbol{\tau}_2)\boldsymbol{e}_i.\end{aligned} \tag{73}$$

Because the initial conditions vectors $\boldsymbol{s}^0_i, i = 1, \cdots, I^{(1,0)}$ are not parameters to be estimated (as they are assumed known), we find that taking the partial derivatives of $\tilde{\boldsymbol{d}}$ with respect to elements of $\tilde{\boldsymbol{s}}_i$ yields

$$\frac{\partial \tilde{\boldsymbol{d}}}{\partial \boldsymbol{s}^R_i(\ell)} = \frac{\partial \tilde{\boldsymbol{d}}}{\partial \tilde{\boldsymbol{x}}_i(\ell + N)} = \boldsymbol{\psi}_{i,\ell+N} \tag{74}$$

$$\frac{\partial \tilde{\boldsymbol{d}}}{\partial \boldsymbol{s}^I_i(\ell)} = \frac{\partial \tilde{\boldsymbol{d}}}{\partial \tilde{\boldsymbol{x}}_i(S + \ell + N)} = \boldsymbol{\psi}_{i,S+\ell+N} \tag{75}$$

where $\boldsymbol{\psi}_{i,\ell}$ is the ℓth column of $\boldsymbol{\psi}_i, \boldsymbol{s}^R_i(\ell)$ denotes the ℓth element of Re$\{\boldsymbol{s}_i\}$, and $\boldsymbol{s}^I_i(\ell)$ denotes the ℓth element of Im$\{\boldsymbol{s}_i\}$. Hence

$$\begin{aligned}\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{s}^R_i(\ell)} &= \frac{\partial \boldsymbol{d}^R}{\partial \boldsymbol{s}^R_i(\ell)} + j\frac{\partial \boldsymbol{d}^I}{\partial \boldsymbol{s}^R_i(\ell)}\\ &= \boldsymbol{\psi}^R_{i,\ell+N} + j\boldsymbol{\psi}^I_{i,\ell+N}\end{aligned} \tag{76}$$

and

$$\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{s}^I_i(\ell)} = \boldsymbol{\psi}^R_{i,S+\ell+N} + j\boldsymbol{\psi}^I_{i,S+\ell+N} \tag{77}$$

where

$$\begin{aligned}\boldsymbol{\psi}^R_{i,\ell} &= [\boldsymbol{\psi}_{i,\ell}(0), \cdots, \boldsymbol{\psi}_{i,\ell}(ST-1)]^T\\ \boldsymbol{\psi}^I_{i,\ell} &= [\boldsymbol{\psi}_{i,\ell}(ST), \cdots, \boldsymbol{\psi}_{i,\ell}(2ST-1)]^T.\end{aligned} \tag{78}$$

Note that the bound on the variance of estimating the parameters of the 1-D modulating purely indeterministic processes is given in terms of their nonparametric representation.

Substituting (63), (67), (73), (76), and (77) into (58), we obtain the FIM block which corresponds to the parameters of the deterministic component. Since the conditional FIM is block-diagonal, the lower bound on the accuracy of estimating the deterministic component parameters is obtained by inverting (58), after the above substitutions have been made.

VII. Numerical Examples

In this section, we investigate the behavior of the conditional CRB and the performance of the suggested ML algorithm using some specific examples. First, we investigate the CRB as a function of the spectral support parameters of the harmonic and evanescent components, and the shape of the purely indeterministic component spectral density. In the second part of this section we illustrate the performance of the ML algorithm by Monte Carlo simulations, and compare the variance of the ML algorithm estimation errors with the lower bound given by computing the CRB for these examples.

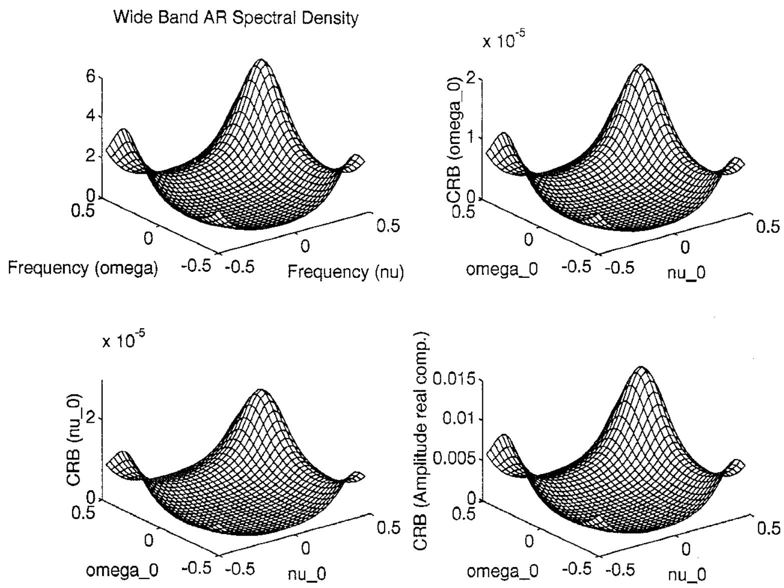

Fig. 2. The CRB on the parameters of a single exponential in a wideband 2-D AR field.

TABLE I
PURELY INDETERMINISTIC COMPONENTS PARAMETERS

	σ^2	b(0,1)	b(1,-1)	b(1,0)	b(1,1)
Narrowband AR	1	$0.9 \cdot \exp(j2\pi 0.4)$	0	$0.9 \cdot \exp(j2\pi 0.3)$	$0.81 \cdot \exp(-j2\pi 0.3)$
Wideband AR	1	$-0.4 \cdot \exp(j2\pi 0.4)$	0	$-0.3 \cdot \exp(j2\pi 0.3)$	$0.12 \cdot \exp(-j2\pi 0.3)$

A. The Bounds as Functions of the Harmonic and Evanescent Frequencies

In this section we first investigate the bound on the harmonic component parameters, as a function of frequency, for a fixed data size of 16×16 samples. The harmonic component comprises a single exponential of unit amplitude, and no evanescent components are present. For each of the two different AR models of the purely indeterministic component listed in Table I, the frequency of the exponential is varied in the square region $K = [-1/2, 1/2] \times [-1/2, 1/2]$, and the bound on the estimation error variance of each of the harmonic component parameters is computed for each spatial frequency the exponential assumes. The results are illustrated in Figs. 2 and 3. Note that both for the narrowband AR field, and the wideband AR field, the shape of the bound as a function of frequency matches the shape of the spectral density of the AR field. In other words, the lower bound on the estimation error variance of any of the exponential parameters becomes higher, and hence the estimation more difficult, as the local SNR given by

$$\text{SNR}\,(\omega_k, \nu_k) = \frac{|C_k|^2}{S(e^{j2\pi\omega_k}, e^{j2\pi\nu_k})} \tag{79}$$

decreases. Here

$$S(e^{j2\pi\omega}, e^{j2\pi\nu}) = \frac{\sigma^2}{|B(e^{j2\pi\omega}, e^{j2\pi\nu})|^2}$$

denotes the spectral density function of the 2-D AR field.

In the next example, we investigate the bound on the frequency parameter of a single evanescent component embedded in the above wideband, and narrowband AR modeled purely indeterministic components, as a function of the evanescent component frequency parameter, for a fixed data size of 16×16 samples. The evanescent component has spectral support parameters $(\alpha, \beta) = (1, 0)$, and its modulating 1-D process is a zero-mean, unit-variance Gaussian white noise process. No harmonic component is present. For each of the two different AR models of the purely indeterministic component listed in Table I, the frequency parameter of the evanescent component $\nu^{(1,0)}$ is varied in the interval $[-1/2, 1/2]$, and the bound on its estimation error variance is computed for each value $\nu^{(1,0)}$ assumes. The results are depicted in Fig. 4. Note that both for the narrowband AR field, and the wideband AR field, the shape of the bound as a function of frequency matches the spectral density of the AR field. In other words, the lower bound on the estimation error variance of $\nu^{(1,0)}$ becomes higher as the power of the purely indeterministic component increases relative to the power of the evanescent field embedded in it.

B. Performance Examples of the ML Algorithm

In this section, we illustrate the performance of the ML algorithm by Monte Carlo simulations, and by comparing the variance of its estimation errors with the CRB. The experimental results are based on 100 independent realizations

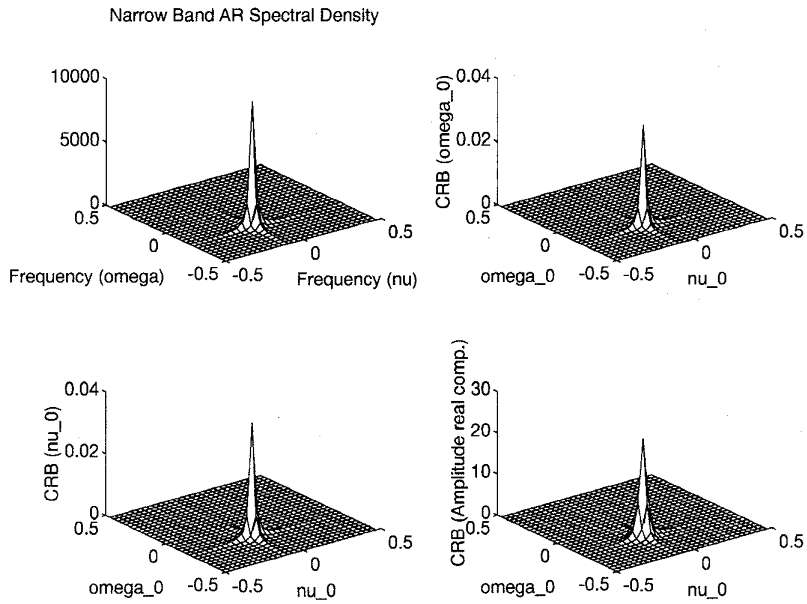

Fig. 3. The CRB on the parameters of a single exponential in a narrowband 2-D AR field.

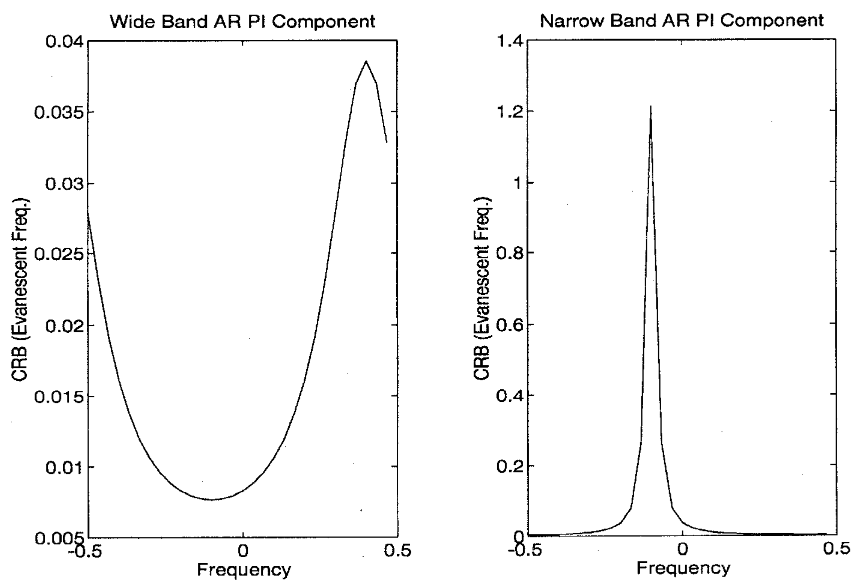

Fig. 4. The CRB on the frequency parameter of a single evanescent field in wideband and narrowband AR fields.

of the purely indeterministic component, and of the modulating 1-D purely indeterministic process of each evanescent field. Similar results were obtained when a single realization of the modulating 1-D purely indeterministic process of each evanescent field was used in all 100 experiments. We consider two sets of test data, represented as 32×32 realizations of the fields.

Example 1: Consider a field which consists of the sum of a purely indeterministic component modeled by the narrow bandwidth 2-D AR model with support $S_{1,1}$ whose parameters are listed in Table I, two exponentials of equal amplitudes, and two evanescent components. The frequencies of the two harmonic components are $(\omega_1, \nu_1) = (0.15, 0.25)$ and $(\omega_2, \nu_2) = (0.16, 0.26)$, which are far away from the peak of the spectral density of the purely indeterministic component. The frequencies of the two evanescent components are $\nu^{(0,1)} = 0.1$ and $\nu^{(1,0)} = -0.4$. The modulating 1-D purely indeterministic process $s^{(1,0)}(n)$ is a first-order AR process with parameter $0.9 \cdot \exp(j2\pi 0.4)$, whose input is a unit variance Gaussian white noise process. The process $s^{(0,1)}(m)$ is a unit variance Gaussian white noise.

Example 2: Consider the following case. The purely indeterministic component is the wideband AR field with support

TABLE II
DETERMINISTIC COMPONENTS ESTIMATION RESULTS

Parameters		*Example 1*				*Example 2*			
		Orig.	*CRB*	*Bias*	*Var*	*Orig.*	*CRB*	*Bias*	*Var.*
First harmonic component	ω_1	0.15	6.9865e-08	5.408e-05	7.6399e-08	0.25	3.3129e-05	1.210e-03	3.9733e-05
	ν_1	0.25	7.4476e-08	5.477e-05	7.8123e-08	0.35	3.3636e-05	1.234e-03	3.9882e-05
	Real	1	6.2165e-04	5.412e-03	1.0022e-03	0.3	3.3353e-02	1.960e-03	4.3187e-02
	Imag	0	3.6350e-03	9.799e-03	5.4130e-03	0	1.6959e-01	1.001e-01	3.5453e-01
Second harmonic component	ω_2	0.16	6.9287e-08	5.478e-05	7.8378e-08	0.26	1.2598e-05	7.828e-04	1.6616e-05
	ν_2	0.26	6.7335e-08	5.463e-05	7.7931e-08	0.36	1.2853e-05	7.998e-04	1.7350e-05
	Real	1	6.4339e-04	5.632e-03	1.0662e-03	0.5	3.2483e-02	1.915e-04	4.5418e-02
	Imag	0	3.4100e-03	1.052e-02	5.7620e-03	0	1.7993e-01	1.002e-01	3.0118e-01
First Evanescent component	α	0	-	-	-	1	-	-	-
	β	1	-	-	-	0	-	-	-
	$\nu^{(\alpha,\beta)}$	0.1	4.5160e-08	4.292e-05	4.8076e-08	-0.4	2.3338e-07	6.217e-05	3.0158e-07
Second Evanescent component	α	1	-	-	-	-	-	-	-
	β	0	-	-	-	-	-	-	-
	$\nu^{(\alpha,\beta)}$	-0.4	5.9676e-08	4.418e-05	6.1000e-08	-	-	-	-

$S_{1,1}$ whose parameters are listed in Table I, there is a single evanescent component and two exponentials of unequal amplitudes. Here, the harmonic component frequencies are $(\omega_1, \nu_1) = (0.25, 0.25)$ and $(\omega_2, \nu_2) = (0.26, 0.26)$, which are close to the peak of the spectral density of the purely indeterministic component. The evanescent component has spectral support parameters $(\alpha, \beta) = (1, 0)$, with $\nu_1^{(1,0)} = -0.4$. The modulating 1-D purely indeterministic process $s^{(1,0)}(n)$ is a first-order AR process with parameter $0.9 \cdot \exp(j2\pi 0.4)$, whose input is a unit variance Gaussian white noise process.

The experimental results, summarized in Table II, show that the estimates obtained by the proposed conditional ML algorithm are essentially unbiased as the experimental bias is much smaller than the standard deviation of the experimental results. Furthermore, the results show that the CRB on the error variance in estimating the spectral support parameters is within the 0.95 confidence interval of the experimental variance. Note, however, that the experimental error variances of the harmonic components' amplitudes are slightly higher than the CRB, probably due to the small data sizes used in these experiments.

VIII. CONCLUSIONS

In this paper we have presented a conditional maximum-likelihood solution to the general problem of fitting a parametric model to observations from a single realization of a 2-D homogeneous random field with mixed spectral distribution. The Cramer–Rao lower bound on the accuracy of jointly estimating the parameters of the different components was derived, and it was shown that the estimation of the purely indeterministic and deterministic components are decoupled. The results presented in this paper provide a useful tool for estimating the parameters of 2-D random fields with mixed spectral distributions, and a method for assessing the achievable accuracy of this estimation procedure.

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