Cramér–Rao Bound on the Estimation Accuracy of Complex-Valued Homogeneous Gaussian Random Fields

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***Abstract*—This paper considers the problem of the achievable accuracy in jointly estimating the parameters of a complex-valued two-dimensional (2-D) Gaussian and homogeneous random field from a single observed realization of it. Based on the 2-D Wold decomposition, the field is modeled as a sum of purely indeterministic, evanescent, and harmonic components. Using this parametric model, we first solve a key problem common to many open problems in parametric estimation of homogeneous random fields: that of expressing the field mean and covariance functions in terms of the model parameters. Employing the parametric representation of the observed field mean and covariance, we derive a closed-form expression for the Fisher information matrix (FIM) of complex-valued homogeneous Gaussian random fields with mixed spectral distribution. Consequently, the Cramér–Rao lower bound on the error variance in jointly estimating the model parameters is evaluated.**

***Index Terms*—Cramér–Rao bounds, Fisher information, homogeneous random fields, 2-D Wold decomposition.**

I. Introduction

IN THIS paper, we consider two fundamental problems in parametric modeling and estimation of two-dimensional (2-D) complex-valued homogeneous random fields with mixed spectral distribution. Employing the parametric model that follows from the 2-D Wold-like decomposition of homogeneous random fields, [1], we first obtain closed-form expressions for the field mean and covariance functions in terms of the model parameters. Assuming the observed random field is Gaussian, we then investigate the problem of the achievable accuracy in jointly estimating the parameters of the field model. These fundamental problems are of great theoretical and practical importance. They arise in various wave propagation estimation problems such as in space–time adaptive processing of radar signals [6] and the special case of real-valued 2-D random fields arises quite naturally in terms of texture modeling and estimation in images [16].

From the 2-D Wold-like decomposition, we have that any 2-D regular and homogeneous discrete random field can be represented as a sum of two mutually orthogonal components: a *purely indeterministic* field and a *deterministic* one. The purely indeterministic component has a unique white innovations driven nonsymmetrical half-plane (NSHP) moving average representation. In general, the support of the NSHP moving average model has infinite dimensions. The deterministic component is further orthogonally decomposed into a *harmonic* field and a countable number of mutually orthogonal *evanescent* fields. This decomposition results in a corresponding decomposition of the spectral measure of the regular random field into a countable sum of mutually singular spectral measures. The spectral distribution function of the purely indeterministic component is absolutely continuous. Furthermore, since the random field is regular, the spectral density of the purely indeterministic component is zero at most on a set of Lebesgue measure zero [2]. The spectral measure of the deterministic component, which is singular with respect to the spectral measure of the purely indeterministic component, is therefore concentrated on a set of Lebesgue measure zero in the frequency plane. It is shown in [1] that under some mild assumptions (that always hold in practice), each evanescent component can be modeled by a separable model, given by the product of a one-dimensional (1-D) purely indeterministic process in one dimension and an exponential in the orthogonal dimension (or a linear combination of such separable random fields). Hence, the spectral supports of the different evanescent components have the form of lines, where the slope of each line is a rational number. In [16], the 2-D Wold-like decomposition, and the resulting random field model, are employed for modeling, analysis, and synthesis of natural textures. Illustrative synthetic examples can be found in [17].

It is shown in [9] that the same parametric model that results from the above orthogonal decomposition naturally arises as the physical model in the problem of space–time processing of airborne radar data. In this problem, the target model is that of an harmonic component. The purely indeterministic component of the space–time field is the sum of a white noise field due to the internally generated receiver amplifier noise and a colored noise field due to the sky noise contribution. The presence of a jammer is modeled by an evanescent component whose 1-D modulating process is a white noise. In the angle-Doppler domain, the ground clutter produces a "clutter ridge," supported on a diagonal line (that generally wraps around). This ground clutter is modeled by an additional evanescent component of the observed 2-D space–time field. (See [6] and [7] for a detailed description of this problem.)

Assuming that the NSHP moving average model of the purely indeterministic component has finite support and that

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the 1-D purely indeterministic process of each evanescent component is a finite-order moving average process, we derive closed-form expressions for the mean and covariance functions of the field in terms of the model parameters. Due to the generality of this modeling, the derivation provides a solution to the problem of expressing the mean and covariance functions of essentially any complex-valued homogeneous random field in terms of the model parameters. This result opens the way for *parametric* solutions that can simplify and improve existing methods of space–time adaptive processing. In [9], we exploit the correspondence between the parametric model that follows from the 2-D Wold-like decomposition and the STAP physical model to derive a computationally efficient algorithm for parametric estimation and mitigation of the jamming and clutter fields.

Assuming the observed field is Gaussian, the parametric representation of the field mean and covariance is employed in this paper to derive a closed-form *exact* expression for the Fisher information matrix (FIM) of complex-valued homogeneous Gaussian random fields with mixed spectral distribution. Consequently, we obtain an expression for the Cramér–Rao lower bound on the error variance in *jointly* estimating the parameters of the harmonic, evanescent, and purely indeterministic components of the field from a *finite dimension single observed realization* of it. It is further shown that regardless of the parametric models of the purely indeterministic and evanescent components, the lower bound on the error variance in estimating the parameters of the harmonic component is decoupled from the bound on the parameters of the purely indeterministic and evanescent components. Moreover, the bound on the parameters of the purely indeterministic and evanescent components is independent of the harmonic component.

The asymptotic Cramér–Rao bound (CRB) on the parameters of a Gaussian purely indeterministic field was derived by Whittle [3]. In [5], a matrix enhancement and matrix pencil method for estimating the parameters of 2-D superimposed, complex-valued exponential signals was suggested. Assuming the noise field is *white*, the Cramér–Rao lower bound for this problem was derived as well. The problem of ML estimation of 2-D superimposed, complex-valued exponential signals has been recently considered in [8]. However, most of the literature on parametric modeling and estimation of 2-D random fields is concerned with the parameter estimation of real-valued 2-D AR fields, (see, e.g., [3], [4], and [10]–[12]), and the statistical inference of Markov random fields (MRFs) (see, e.g., [13], [14], and the references therein). The underlying assumption in these papers is that the random field is purely indeterministic, and hence, it can be fit with a white- or correlated-noise driven linear model. In the Gaussian case, all of the foregoing problems are only special cases of the general problem, which is addressed here.

In [17], we have developed a conditional ML algorithm for jointly estimating the parameters of the harmonic, evanescent, and purely indeterministic components of a complex-valued homogeneous random field from a single observed realization of it. The *conditional* Cramér–Rao lower bound on the covariance matrix of the estimates was derived as well, assuming the purely indeterministic component is a circular Gaussian field, and that the evanescent component is of a special type. In this paper, we derive an *exact* Cramér–Rao lower bound on the error variance in jointly estimating the parameters of essentially any complex-valued homogeneous Gaussian random field that can be modeled by a finite-order model. In this derivation, the field may contain all of the 2-D Wold decomposition components.

The paper is organized as follows. In Section II, we briefly review the results of the 2-D Wold-like decomposition. In Section III, we employ the parametric model that follows from the 2-D Wold-like decomposition to obtain a general expression for the covariance matrix of the field in terms of its model parameters. In Section IV, it is assumed that the 1-D purely indeterministic processes of the evanescent fields are MA processes. Using this assumption, we derive closed-form expressions for the covariance matrix of the evanescent field in terms of its parametric representation. Section V presents a derivation of a closed-form expression of the covariance matrix for a finite-support NSHP moving average purely indeterministic component in terms of the MA model parameters. Assuming a nil harmonic component and that the purely indeterministic and evanescent components of the field are Gaussian, we derive in Section VI a closed-form expression for the FIM of the observed field. Section VII generalizes the derivation of Section VI to include the case where an harmonic component exists in the observed field. It is shown that the lower bound on the error variance in estimating the parameters of the harmonic component is decoupled from the bound on the parameters of the purely indeterministic and evanescent components. In Section VIII, we present some numerical examples in order to get further insight into the properties of the bound. Additional examples can be found in [18], where the bound derived in this paper is employed to evaluate the performance of a computationally efficient algorithm for estimating the parameters of the evanescent and purely indeterministic components of the field.

II. Homogeneous Random Field Model

The considered random field model is based on the Wold-type decomposition of 2-D regular and homogeneous random fields, presented in [1], and briefly summarized in this section. Let $\{y(n, m), (n, m) \in \mathcal{Z}^2\}$ be a complex-valued, regular, homogeneous random field. Then, $y(n, m)$ can be uniquely represented by the orthogonal decomposition

$$y(n, m) = w(n, m) + v(n, m). \tag{1}$$

The field $\{v(n, m)\}$ is a deterministic random field. The field $\{w(n, m)\}$ is purely indeterministic and has a unique white innovations driven moving-average representation given by

$$w(n, m) = \sum_{(0,0)\preceq(k,\ell)} b(k, \ell)u(n-k, m-\ell) \tag{2}$$

where the relation $\preceq$ is defined with respect to some NSHP total-order definition, $\{u(n, m)\}$ is the innovations field of $\{y(n, m)\}$, $b(0, 0) = 1$, and $\sum_{(0,0)\preceq(k,\ell)} |b(k, \ell)|^2 < \infty$.

It is possible to define [1] a family of NSHP total-order definitions such that the boundary line of the NSHP has a rational slope. Let α and β be two coprime integers, such that $\alpha \neq 0$. The angle θ of the slope is given by $\tan\theta = \beta/\alpha$. Each of these supports is called *rational nonsymmetrical half-plane* (RNSHP). We denote by O the set of all possible RNSHP definitions on the 2-D lattice (i.e., the set of all NSHP definitions in which the boundary line of the NSHP has a rational slope). The introduction of the family of RNSHP total-ordering definitions results in a countably infinite orthogonal decomposition of the deterministic component of the random field: $v(n, m) = p(n, m) + \sum_{(\alpha, \beta) \in O} e_{(\alpha, \beta)}(n, m)$. The random field $\{p(n, m)\}$ is called *half-plane deterministic*. The fields $\{e_{(\alpha, \beta)}(n, m)\}$ are the evanescent components of the field $\{y(n, m)\}$.

It is shown in [1] that the model for the evanescent field which corresponds to the RNSHP defined by $(\alpha, \beta) \in O$ is given by

$$\begin{aligned} e_{(\alpha, \beta)}(n, m) &= \sum_{i=1}^{I^{(\alpha, \beta)}} e_i^{(\alpha, \beta)}(n, m) \\ &= \sum_{i=1}^{I^{(\alpha, \beta)}} s_i^{(\alpha, \beta)}(n\alpha - m\beta) \\ &\quad \cdot \exp\left(j2\pi \frac{\nu_i^{(\alpha, \beta)}}{\alpha^2 + \beta^2} (n\beta + m\alpha)\right) \end{aligned} \tag{3}$$

where the 1-D purely indeterministic, complex-valued processes $\{s_i^{(\alpha, \beta)}(n\alpha - m\beta)\}$, $\{s_j^{(\alpha, \beta)}(n\alpha - m\beta)\}$ are zero-mean and mutually orthogonal for all $i \neq j$. Hence, the "spectral density function" of each evanescent field has the form of a sum of 1-D delta functions supported on lines of rational slope in the 2-D spectral domain. Since interchanging the roles of past and future in any total-order definition results in identical evanescent components, it is sufficient to consider only $0 \leq \theta < \pi$. We therefore assume without limiting the generality of the derivation that $\alpha > 0$, whereas β can assume any integer value.

One of the half-plane-deterministic field components, which is often found in physical problems, is the harmonic random field

$$h(n, m) = \sum_{p=1}^{P} C_p e^{j2\pi(n\omega_p + m\nu_p)} \tag{4}$$

where the C_ps are mutually orthogonal random variables, and (ω_p, ν_p) are the spatial frequencies of the pth harmonic. In general, P is infinite. The parametric modeling of deterministic random fields whose spectral measures are concentrated on curves other than lines of rational slope, or discrete points in the frequency plane, is still an open question, to the best of our knowledge.

Thus, if we exclude from the framework of our model those 2-D random fields whose spectral measures are concentrated on curves other than lines of rational slope, $y(n, m)$ is uniquely represented by

$$y(n, m) = w(n, m) + h(n, m) + \sum_{(\alpha, \beta) \in O} e_{(\alpha, \beta)}(n, m) \tag{5}$$

where $\{w(n, m)\}$, $\{e_{(\alpha, \beta)}(n, m)\}$, and $\{h(n, m)\}$ are given by (2)–(4), respectively.

III. Parametric Representation of the Covariance Matrix of a Regular and Homogeneous Random Field

To simplify the presentation, we consider first the problem of estimating the parameters of an observed field where no harmonic component is present. In Section VII, this derivation is generalized to include the case where the deterministic component of the field comprises both harmonic and evanescent components. In this section, we employ the 2-D Wold decomposition-based parametric random field model to obtain closed-form expression of the field covariance matrix in terms of the parametric models of the decomposition components.

We next state our assumptions and introduce some necessary notations. Let $\{y(n, m)\}$, $(n, m) \in D$, where $D = \{(i, j) | 0 \leq i \leq S-1, 0 \leq j \leq T-1\}$ be the observed random field. Note, however, that the observed field could just as well have any *arbitrary* shape.

Assumption 1: The values of the (α, β) pairs, as well as the number $I^{(\alpha, \beta)}$ of evanescent components in (3), are *a-priori* known for all the evanescent components.

Assumption 2: The real and imaginary components of the purely indeterministic component are zero mean, jointly wide sense homogeneous fields. Let $\mathbf{\Gamma}_{\text{PI}}$ denote the covariance matrix of the purely indeterministic component. We assume that the covariance matrix has some known parametric form, where $\mathbf{b}$ is the parameter vector. At the moment, we will not specify the functional dependence of $\mathbf{\Gamma}_{\text{PI}}$ on $\mathbf{b}$ but, rather, leave it implicit.

Assumption 3: For each evanescent field $\{e_i^{(\alpha, \beta)}\}$, the modulating complex-valued 1-D purely indeterministic process $\{s_i^{(\alpha, \beta)}\}$ is a zero-mean process such that its real and imaginary components are jointly wide sense stationary. Let $\mathbf{R}_i^{(\alpha, \beta)}$ denote the covariance matrix of $\{s_i^{(\alpha, \beta)}\}$. We assume that the covariance matrix has some known parametric form, where $\mathbf{a}_i^{(\alpha, \beta)}$ is the parameter vector. At the moment, we will not specify the functional dependence of $\mathbf{R}_i^{(\alpha, \beta)}$ on $\mathbf{a}_i^{(\alpha, \beta)}$ but, rather, leave it implicit as well.

Thus, the parameter vector of each of the evanescent components $\{e_i^{(\alpha, \beta)}\}$ is given by $\boldsymbol{\phi}_i^{(\alpha, \beta)} = [\nu_i^{(\alpha, \beta)}, (\mathbf{a}_i^{(\alpha, \beta)})^T]^T$. Therefore, the parameter vector of the evanescent field $\{e^{(\alpha, \beta)}\}$ is obtained by collecting the vectors $\boldsymbol{\phi}_i^{(\alpha, \beta)}$ into a single column vector, i.e., $\boldsymbol{\phi}^{(\alpha, \beta)} = [(\boldsymbol{\phi}_1^{(\alpha, \beta)})^T, \ldots, (\boldsymbol{\phi}_{I^{(\alpha, \beta)}}^{(\alpha, \beta)})^T]^T$. The parameter vector of the observed field $\{y(n, m)\}$ is given by

$$\boldsymbol{\theta} = \left[\mathbf{b}^T, \left\{\left(\boldsymbol{\phi}^{(\alpha, \beta)}\right)^T\right\}_{(\alpha, \beta) \in O}\right]^T. \tag{6}$$

Let

$$\begin{aligned} \mathbf{y} = [&y(0, 0), \ldots, y(0, T-1), y(1, 0), \ldots \\ &y(1, T-1), \ldots, y(S-1, 0), \ldots \\ &y(S-1, T-1)]^T \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbf{e}_i^{(\alpha, \beta)} = \Big[&e_i^{(\alpha, \beta)}(0, 0), \ldots, e_i^{(\alpha, \beta)}(0, T-1), \\ &e_i^{(\alpha, \beta)}(1, 0), \ldots, e_i^{(\alpha, \beta)}(1, T-1), \ldots \\ &e_i^{(\alpha, \beta)}(S-1, 0), \ldots, e_i^{(\alpha, \beta)}(S-1, T-1)\Big]^T. \end{aligned} \tag{8}$$

Let

$$\boldsymbol{\xi}_i^{(\alpha,\beta)} = \Big[s_i^{(\alpha,\beta)}(0),\, s_i^{(\alpha,\beta)}(-\beta),\, \ldots,\, s_i^{(\alpha,\beta)}(-(T-1)\beta) \\ s_i^{(\alpha,\beta)}(\alpha),\, s_i^{(\alpha,\beta)}(\alpha-\beta),\, \ldots \\ s_i^{(\alpha,\beta)}(\alpha-(T-1)\beta),\, \ldots \\ s_i^{(\alpha,\beta)}((S-1)\alpha),\, s_i^{(\alpha,\beta)}((S-1)\alpha-\beta),\, \ldots \\ s_i^{(\alpha,\beta)}((S-1)\alpha-(T-1)\beta)\Big]^T \tag{9}$$

be the vector whose elements are the observed samples from the 1-D modulating process $\{s_i^{(\alpha,\beta)}\}$. In addition, let $\mathbf{y}^R = \mathrm{Re}\{\mathbf{y}\}$, $\mathbf{y}^I = \mathrm{Im}\{\mathbf{y}\}$, $\overline{\mathbf{y}} = [(\mathbf{y}^R)^T(\mathbf{y}^I)^T]^T$. In a similar way, we define the vectors $(\mathbf{e}_i^{(\alpha,\beta)})^R$, $(\mathbf{e}_i^{(\alpha,\beta)})^I$, $\overline{\mathbf{e}}_i^{(\alpha,\beta)}$, and $(\boldsymbol{\xi}_i^{(\alpha,\beta)})^R$, $(\boldsymbol{\xi}_i^{(\alpha,\beta)})^I$, $\overline{\boldsymbol{\xi}}_i^{(\alpha,\beta)}$. Define

$$\mathbf{v}^{(\alpha,\beta)} = [0,\, \alpha,\, \ldots,\, (T-1)\alpha,\, \beta,\, \beta+\alpha,\, \ldots \\ \beta+(T-1)\alpha,\, \ldots,\, \ldots,\, (S-1)\beta, \\ (S-1)\beta+\alpha, \ldots, (S-1)\beta+(T-1)\alpha]^T. \tag{10}$$

Given a scalar function $f(v)$, we will denote the matrix, or column vector, consisting of the values of $f(v)$ evaluated for all the elements of $\mathbf{v}$, where $\mathbf{v}$ is a matrix, or a column vector, by $f(\mathbf{v})$. Using this notation, we define

$$\tilde{\mathbf{f}}_i^{(\alpha,\beta)} = \cos\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}\, \mathbf{v}^{(\alpha,\beta)}\right) \tag{11}$$

$$\tilde{\mathbf{g}}_i^{(\alpha,\beta)} = \sin\left(2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}\, \mathbf{v}^{(\alpha,\beta)}\right). \tag{12}$$

Thus, using (3), we have that

$$\mathbf{e}_i^{(\alpha,\beta)} = \boldsymbol{\xi}_i^{(\alpha,\beta)} \odot \tilde{\mathbf{d}}_i^{(\alpha,\beta)} \tag{13}$$

where $\odot$ denotes an element-by-element product of the vectors, and $\tilde{\mathbf{d}}_i^{(\alpha,\beta)} = \tilde{\mathbf{f}}_i^{(\alpha,\beta)} + j\tilde{\mathbf{g}}_i^{(\alpha,\beta)}$. Rewriting (13) using real quantities, we obtain

$$\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^R = \left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^R \odot \tilde{\mathbf{f}}_i^{(\alpha,\beta)} - \left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^I \odot \tilde{\mathbf{g}}_i^{(\alpha,\beta)} \tag{14}$$

$$\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^I = \left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^R \odot \tilde{\mathbf{g}}_i^{(\alpha,\beta)} + \left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^I \odot \tilde{\mathbf{f}}_i^{(\alpha,\beta)}. \tag{15}$$

Note that whenever $n\alpha - m\beta = k\alpha - \ell\beta$ for some integers n, m, k, ℓ such that $0 \le n, k \le S-1$ and $0 \le m, \ell \le T-1$, the same sample from the process $\{s_i^{(\alpha,\beta)}\}$ is duplicated in the vector $\boldsymbol{\xi}_i^{(\alpha,\beta)}$. It can be shown that for a rectangular observed field of dimensions $S \times T$, the number of *distinct* samples from the random process $\{s_i^{(\alpha,\beta)}\}$ that are found in the observed field is $N_c = (S-1)|\alpha| + (T-1)|\beta| + 1 - (|\alpha|-1)(|\beta|-1)$. We therefore define the *concentrated version*, $\mathbf{s}_i^{(\alpha,\beta)}$ of $\boldsymbol{\xi}_i^{(\alpha,\beta)}$ to be an N_c-dimensional column vector of nonrepeating samples of the process $\{s_i^{(\alpha,\beta)}\}$. More specifically, for the case in which $\alpha > 0$ and $\beta \ge 0$, $\mathbf{s}_i^{(\alpha,\beta)}$ is given by

$$\mathbf{s}_i^{(\alpha,\beta)} = \left[s_i^{(\alpha,\beta)}(-(T-1)\beta), \ldots, \ldots, s_i^{(\alpha,\beta)}((S-1)\alpha)\right]^T \tag{16}$$

whereas for the case in which $\alpha \ge 0$ and $\beta < 0$, $\mathbf{s}_i^{(\alpha,\beta)}$ is given by

$$\mathbf{s}_i^{(\alpha,\beta)} = \left[s_i^{(\alpha,\beta)}(0), \ldots, \ldots, s_i^{(\alpha,\beta)}((S-1)\alpha - \beta(T-1))\right]^T. \tag{17}$$

Note, however, that due to boundary effects, the vector $\mathbf{s}_i^{(\alpha,\beta)}$ is not composed of consecutive samples from the process $\{s_i^{(\alpha,\beta)}\}$ unless $|\alpha| \le 1$ or $|\beta| \le 1$. In other words, for some arbitrary α and β, there are missing samples in $\mathbf{s}_i^{(\alpha,\beta)}$. Thus, for any (α, β), we have that $\boldsymbol{\xi}_i^{(\alpha,\beta)} = \mathbf{A}_i^{(\alpha,\beta)}\mathbf{s}_i^{(\alpha,\beta)}$, where $\mathbf{A}_i^{(\alpha,\beta)}$ is rectangular matrix of zeros and ones that replicates rows of $\mathbf{s}_i^{(\alpha,\beta)}$. Similarly to the foregoing definitions, we define $(\mathbf{s}_i^{(\alpha,\beta)})^R = \mathrm{Re}\{(\mathbf{s}_i^{(\alpha,\beta)})\}$, $(\mathbf{s}_i^{(\alpha,\beta)})^I = \mathrm{Im}\{(\mathbf{s}_i^{(\alpha,\beta)})\}$, and

$$\overline{\mathbf{s}}_i^{(\alpha,\beta)} = \left[\left(\left(\mathbf{s}_i^{(\alpha,\beta)}\right)^R\right)^T \left(\left(\mathbf{s}_i^{(\alpha,\beta)}\right)^I\right)^T\right]^T. \tag{18}$$

We note that the covariance matrix $\mathbf{R}_i^{(\alpha,\beta)}$ that characterizes the process $\{s_i^{(\alpha,\beta)}\}$ is defined in terms of the concentrated version vector $\mathbf{s}_i^{(\alpha,\beta)}$, i.e.,

$$\mathbf{R}_i^{(\alpha,\beta)} = E\left[\overline{\mathbf{s}}_i^{(\alpha,\beta)}\left(\overline{\mathbf{s}}_i^{(\alpha,\beta)}\right)^T\right] \tag{19}$$

and not in terms of the covariance matrix

$$\tilde{\mathbf{R}}_i^{(\alpha,\beta)} = E\left[\overline{\boldsymbol{\xi}}_i^{(\alpha,\beta)}\left(\overline{\boldsymbol{\xi}}_i^{(\alpha,\beta)}\right)^T\right] \tag{20}$$

of the vector $\overline{\boldsymbol{\xi}}_i^{(\alpha,\beta)}$. The matrix $\tilde{\mathbf{R}}_i^{(\alpha,\beta)}$ is a singular matrix, which is also given by

$$\tilde{\mathbf{R}}_i^{(\alpha,\beta)} = \boldsymbol{\Sigma}_i^{(\alpha,\beta)}\mathbf{R}_i^{(\alpha,\beta)}\left(\boldsymbol{\Sigma}_i^{(\alpha,\beta)}\right)^T \tag{21}$$

where

$$\boldsymbol{\Sigma}_i^{(\alpha,\beta)} = \begin{bmatrix} \mathbf{A}_i^{(\alpha,\beta)} & 0 \\ 0 & \mathbf{A}_i^{(\alpha,\beta)} \end{bmatrix}. \tag{22}$$

In terms of the Fisher information, both $\tilde{\mathbf{R}}_i^{(\alpha,\beta)}$ and $\mathbf{R}_i^{(\alpha,\beta)}$ represent the same information on the process $\{s_i^{(\alpha,\beta)}\}$.

Since the evanescent components $\{e_i^{(\alpha,\beta)}\}$ are mutually orthogonal and since all the evanescent components are orthogonal to the purely indeterministic component, we conclude that $\boldsymbol{\Gamma}$, which is the covariance matrix of $\mathbf{y}$, has the form

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{\mathrm{PI}} + \sum_{(\alpha,\beta)\in O} \sum_{i=1}^{I^{(\alpha,\beta)}} \boldsymbol{\Gamma}_i^{(\alpha,\beta)} \tag{23}$$

where $\boldsymbol{\Gamma}_i^{(\alpha,\beta)}$ is the covariance matrix of $\overline{\mathbf{e}}_i^{(\alpha,\beta)}$.

In the following, we use a 2×2 partitioned matrix notation for the covariance matrix of any complex-valued random vector that is expressed using real quantities only. Hence, for example, we let

$$\mathbf{\Gamma}_i^{(\alpha,\beta)} = E\left[\overline{\mathbf{e}}_i^{(\alpha,\beta)}\left(\overline{\mathbf{e}}_i^{(\alpha,\beta)}\right)^T\right] = \begin{bmatrix} \left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,1} & \left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,2} \\ \left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{2,1} & \left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{2,2} \end{bmatrix} \tag{24}$$

where all four sub-blocks of the matrix are of identical dimensions.

Using (3) and (13), we find that the four blocks of the symmetric matrix $\mathbf{\Gamma}_i^{(\alpha,\beta)}$ are given by

$$\begin{aligned}\left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,1} &= E\left[\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^R\left(\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^R\right)^T\right] \\ &= E\left[\left(\left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^R\left(\left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^R\right)^T\right) \odot \left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right)\right] \\ &\quad - E\left[\left(\left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^R\left(\left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^I\right)^T\right) \odot \left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right)\right] \\ &\quad - E\left[\left(\left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^I\left((\boldsymbol{\xi}_i^{(\alpha,\beta)})^R\right)^T\right) \odot \left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right)\right] \\ &\quad + E\left[\left(\left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^I\left(\left(\boldsymbol{\xi}_i^{(\alpha,\beta)}\right)^I\right)^T\right) \odot \left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right)\right] \\ &= \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,1} \odot \left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right] - \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,2} \odot \left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right] \\ &\quad - \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,1} \odot \left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right] + \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,2} \odot \left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right]\end{aligned} \tag{25}$$

and similarly

$$\begin{aligned}\left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,2} &= E\left[\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^R\left(\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^I\right)^T\right] \\ &= \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,1} \odot \left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right] + \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,2} \odot \left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right] \\ &\quad - \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,1} \odot \left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right] - \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,2} \odot \left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right]\end{aligned} \tag{26}$$

$$\begin{aligned}\left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{2,2} &= E\left[\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^I\left(\left(\mathbf{e}_i^{(\alpha,\beta)}\right)^I\right)^T\right] \\ &= \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,1} \odot \left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right] + \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,2} \odot \left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right] \\ &\quad + \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,1} \odot \left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right] + \left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,2} \odot \left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right].\end{aligned} \tag{27}$$

A compact matrix representation of $\mathbf{\Gamma}_i^{(\alpha,\beta)}$ for any (α,β) cannot be derived due to the dependence of the matrix structure on (α,β). However, for the case in which $(\alpha,\beta) = (1,0)$ [and similarly for $(\alpha,\beta) = (0,1)$], a somewhat more compact representation is possible, using Kronecker products instead of the Hadamard products. This special case is beyond the scope of the paper.

IV. Covariance Matrix of Evanescent Fields With MA Modulating Processes

In the previous section, we have derived a general expression for the covariance matrix of a complex-valued evanescent random field. It was assumed that each of the 1-D purely indeterministic processes $\{s_i^{(\alpha,\beta)}\}$ is a zero mean process whose covariance matrix has some known, but unspecified, parametric form, where $\mathbf{a}_i^{(\alpha,\beta)}$ is the parameter vector. In this section, we specialize the results of the previous section. We consider the case in which the modulating 1-D processes are moving average processes. Using this derivation, we obtain a *closed-form* expression of the evanescent field covariance matrix in terms of its model parameters.

Let $n^{(\alpha,\beta)} = n\alpha - m\beta$. Assume that the modulating 1-D process $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ of each evanescent field can be modeled by a finite-order MA model, i.e.,

$$s_i^{(\alpha,\beta)}\left(n^{(\alpha,\beta)}\right) = \sum_{\tau=0}^{Q_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)\zeta_i^{(\alpha,\beta)}\left[n^{(\alpha,\beta)} - \tau\right] \tag{28}$$

where

$$n^{(\alpha,\beta)} = \begin{cases} -(T-1)\beta, \ldots, (S-1)\alpha, & \alpha > 0 \text{ and } \beta \ge 0 \\ 0, \ldots, (S-1)\alpha - (T-1)\beta, & \alpha \ge 0 \text{ and } \beta < 0 \end{cases} \tag{29}$$

and $a_i^{(\alpha,\beta)}(0) = 1$. The driving noise processes $\{\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ are mutually orthogonal, complex-valued processes such that the real and imaginary components of each process are orthogonal real-valued white noise processes with zero mean and variances $(\sigma_i^{(\alpha,\beta)})^2$, and $(\rho_i^{(\alpha,\beta)})^2$, respectively. We further assume that the MA processes are of known orders $Q_i^{(\alpha,\beta)}$, where $Q_i^{(\alpha,\beta)} \le N_c$.

For the case in which $\alpha > 0$ and $\beta \geq 0$, define the $(S-1)|\alpha| + (T-1)|\beta| + 1 + Q_i^{(\alpha,\beta)}$-dimensional vector of *consecutive* samples

$$\boldsymbol{\zeta}_i^{(\alpha,\beta)} = \Big[\zeta_i^{(\alpha,\beta)}\left(-(T-1)\beta - Q_i^{(\alpha,\beta)}\right) \; \zeta_i^{(\alpha,\beta)}\left(-(T-1)\beta - Q_i^{(\alpha,\beta)} + 1\right) \ldots, \ldots, \zeta_i^{(\alpha,\beta)}((S-1)\alpha)\Big]^T \tag{30}$$

whereas for the case in which $\alpha \geq 0$ and $\beta < 0$

$$\boldsymbol{\zeta}_i^{(\alpha,\beta)} = \Big[\zeta_i^{(\alpha,\beta)}\left(-Q_i^{(\alpha,\beta)}\right), \zeta_i^{(\alpha,\beta)}\left(-Q_i^{(\alpha,\beta)} + 1\right), \ldots \ldots, \zeta_i^{(\alpha,\beta)}((S-1)\alpha - \beta(T-1))\Big]^T. \tag{31}$$

Hence, for both cases, we have

$$\mathbf{s}_i^{(\alpha,\beta)} = \left(\mathbf{W}_i^{(\alpha,\beta)}\mathbf{D}_i^{(\alpha,\beta)}\right)\boldsymbol{\zeta}_i^{(\alpha,\beta)} \tag{32}$$

where $\mathbf{D}_i^{(\alpha,\beta)}$ is the

$$((S-1)|\alpha| + (T-1)|\beta| + 1) \times \left((S-1)|\alpha| + (T-1)|\beta| + 1 + Q_i^{(\alpha,\beta)}\right)$$

Toeplitz matrix, shown in (33) at the bottom of the page, and $\mathbf{W}_i^{(\alpha,\beta)}$ is a rectangular matrix of zeros and ones that eliminates rows that correspond to the $(|\alpha|-1)(|\beta|-1)$ samples that are missing from $\mathbf{s}_i^{(\alpha,\beta)}$ due to the edge effects. These missing samples result in $\mathbf{s}_i^{(\alpha,\beta)}$ being composed of nonconsecutive samples in its top and bottom.

Let $\mathbf{B}_i^{(\alpha,\beta)} = \text{Re}\{\mathbf{D}_i^{(\alpha,\beta)}\}$ and $\mathbf{C}_i^{(\alpha,\beta)} = \text{Im}\{\mathbf{D}_i^{(\alpha,\beta)}\}$. Thus, the four blocks of the covariance matrix $\mathbf{R}_i^{(\alpha,\beta)}$ of the $Q_i^{(\alpha,\beta)}$-order MA process are given by

$$\begin{aligned}\left(\mathbf{R}_i^{(\alpha,\beta)}\right)_{1,1} &= \left(\sigma_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)}\mathbf{B}_i^{(\alpha,\beta)}\left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \\ &\quad + \left(\rho_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)}\mathbf{C}_i^{(\alpha,\beta)}\left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T\end{aligned} \tag{34}$$

$$\begin{aligned}\left(\mathbf{R}_i^{(\alpha,\beta)}\right)_{1,2} &= \left(\sigma_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)}\mathbf{B}_i^{(\alpha,\beta)}\left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \\ &\quad - \left(\rho_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)}\mathbf{C}_i^{(\alpha,\beta)}\left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T\end{aligned} \tag{35}$$

$$\begin{aligned}\left(\mathbf{R}_i^{(\alpha,\beta)}\right)_{2,2} &= \left(\sigma_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)}\mathbf{C}_i^{(\alpha,\beta)}\left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \\ &\quad + \left(\rho_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)}\mathbf{B}_i^{(\alpha,\beta)}\left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T\end{aligned} \tag{36}$$

and $(\mathbf{R}_i^{(\alpha,\beta)})_{2,1} = ((\mathbf{R}_i^{(\alpha,\beta)})_{1,2})^T$.

V. Covariance Matrix of an MA Purely Indeterministic Field

From the Wold-type decomposition (1), it is known that the purely indeterministic component of the field has a unique white innovations-driven NSHP moving average representation. In practice, the observed field is of finite dimensions. Hence, we restrict our attention to NSHP MA models with finite-dimensional support. More specifically, we assume that the purely indeterministic field is a complex-valued MA field, whose model is given by (2) with $(k,\ell) \in S_{N,M}$, where

$$S_{N,M} = \{(i,j) | i = 0, 0 \leq j \leq M\} \bigcup \{(i,j) | 1 \leq i \leq N, -M \leq j \leq M\} \tag{37}$$

and N, M are *a priori* known. The driving noise of the MA model is a complex-valued white noise field such that its real and imaginary components are orthogonal real-valued white noise fields, each with zero mean and variance σ^2, and ρ^2, respectively. Thus, (2) is replaced by

$$w(n,m) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)u(n-k, m-\ell). \tag{38}$$

In this section, we consider the representation of the covariance matrix of a complex-valued 2-D MA random field in terms of the MA model parameters for finite-order MA models. Let

$$\tilde{\mathbf{b}} = [b(0,1), \ldots, b(0,M), b(1,-M), \ldots b(1,M), \ldots, b(N,-M), \ldots, b(N,M)]^T. \tag{39}$$

Thus, the parameter vector of the purely indeterministic component of the field is given by

$$\mathbf{b} = \left[\sigma^2, \rho^2, \text{Re}\{\tilde{\mathbf{b}}^T\}, \text{Im}\{\tilde{\mathbf{b}}^T\}\right]^T. \tag{40}$$

Let

$$\mathbf{w} = [w(0,0), \ldots, w(0,T-1), w(1,0), \ldots, w(1,T-1) \ldots, w(S-1,0), \ldots, w(S-1,T-1)]^T. \tag{41}$$

$$\mathbf{D}_i^{(\alpha,\beta)} = \begin{bmatrix} a_i^{(\alpha,\beta)}\left(Q_i^{(\alpha,\beta)}\right) & \cdots & a_i^{(\alpha,\beta)}(1) & 1 & & & \\ & a_i^{(\alpha,\beta)}\left(Q_i^{(\alpha,\beta)}\right) & & \cdots & a_i^{(\alpha,\beta)}(1) & 1 & \mathbf{0} \\ \mathbf{0} & & & \ddots & & \ddots & \\ & & a_i^{(\alpha,\beta)}\left(Q_i^{(\alpha,\beta)}\right) & & \cdots & a_i^{(\alpha,\beta)}(1) & 1 \end{bmatrix} \tag{33}$$

Similarly, let the driving noise vector be defined by

$$\begin{aligned}\mathbf{u} = [&u(-N, -M), \ldots, u(-N, T+M-1), \ldots \\ &u(0, -M), \ldots, u(0, T+M-1), \ldots \\ &u(S-1, -M), \ldots, u(S-1, T+M-1)]^T. \end{aligned} \quad (42)$$

Let $\mathbf{0}_k$ denote a k-dimensional row vector of zeros. In addition, let

$$\begin{aligned}\mathbf{b}_0 &= [b(0, M), \ldots, b(0, 1), 1, \mathbf{0}_{T-1+M}] \\ \mathbf{b}_1 &= [b(1, M), \ldots, b(1, 0), \ldots, b(1, -M), \mathbf{0}_{T-1}] \\ &\vdots \qquad\qquad \vdots \\ \mathbf{b}_N &= [b(N, M), \ldots, b(N, 0), \ldots, b(N, -M), \mathbf{0}_{T-1}] \end{aligned} \quad (43)$$

and

$$\overline{\mathbf{b}} = [\mathbf{b}_N, \mathbf{b}_{N-1}, \ldots, \mathbf{b}_0]. \quad (44)$$

Note that $\overline{\mathbf{b}}$ is a $(T+2M) \cdot (N+1)$-dimensional row vector.

Define the following $T \times (T+2M) \cdot (N+1)$ banded Toeplitz matrix

$$[\overline{\mathbf{B}}]_{i,j} = \begin{cases} \overline{\mathbf{b}}(j-i+1), & j \geq i \\ 0, & i < j \end{cases} \quad (45)$$

where $\overline{\mathbf{b}}(i) = 0$ for $i < 0$, and $i > (T+2M) \cdot (N+1)$. Finally, we define the following $ST \times (T+2M)(S+N)$ block matrix, shown in (46) at the bottom of the page.

Thus we can rewrite the observations equation (38) in matrix form as

$$\mathbf{w} = \mathbf{B}\mathbf{u}. \quad (47)$$

Rewriting (47) using real quantities, we have

$$\overline{\mathbf{w}} = \tilde{\mathbf{B}}\overline{\mathbf{u}} \quad (48)$$

where $\overline{\mathbf{w}} = [(\mathbf{w}^R)^T (\mathbf{w}^I)^T]^T$, $\mathbf{w}^R = \text{Re}\{\mathbf{w}\}$, $\mathbf{w}^I = \text{Im}\{\mathbf{w}\}$, $\overline{\mathbf{u}} = [(\mathbf{u}^R)^T (\mathbf{u}^I)^T]^T$, $\mathbf{u}^R = \text{Re}\{\mathbf{u}\}$, $\mathbf{u}^I = \text{Im}\{\mathbf{u}\}$, and

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^R & -\mathbf{B}^I \\ \mathbf{B}^I & \mathbf{B}^R \end{bmatrix}. \quad (49)$$

Here, $\mathbf{B}^R = \text{Re}\{\mathbf{B}\}$, and $\mathbf{B}^I = \text{Im}\{\mathbf{B}\}$. Thus, the covariance matrix of the purely indeterministic component is given in terms of the MA model parameters by

$$\mathbf{\Gamma}_{\text{PI}} = \tilde{\mathbf{B}} \begin{bmatrix} \sigma^2 \mathbf{I}_{(T+2M)(S+N)} & \mathbf{0} \\ \mathbf{0} & \rho^2 \mathbf{I}_{(T+2M)(S+N)} \end{bmatrix} \tilde{\mathbf{B}}^T \quad (50)$$

where $\mathbf{I}_{(T+2M)(S+N)}$ is a $(T+2M)(S+N) \times (T+2M)(S+N)$ identity matrix.

Substituting (21), (22), (25)–(27), (34)–(36), and (50) into (23), we obtain a closed-form expression for the covariance matrix of the observed homogeneous random field in terms of the parametric models of its components.

VI. General Form of the CRB

Assume that the real and imaginary components of the purely indeterministic component are jointly Gaussian and that for each evanescent field $\{e_i^{(\alpha,\beta)}\}$, the modulating complex-valued 1-D purely indeterministic process $\{s_i^{(\alpha,\beta)}\}$ is an independent zero-mean process such that its real and imaginary components are jointly Gaussian. Hence, the observed field $\{y(n, m)\}$ is Gaussian as well.

The general expression for the Fisher information matrix of a real Gaussian process is given by (e.g., [20])

$$[\mathbf{J}(\boldsymbol{\theta})]_{k,\ell} = \frac{\partial \boldsymbol{\mu}^T}{\partial \theta_k} \mathbf{\Gamma}^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \theta_\ell} + \frac{1}{2} \text{tr} \left\{ \mathbf{\Gamma}^{-1} \frac{\partial \mathbf{\Gamma}}{\partial \theta_k} \mathbf{\Gamma}^{-1} \frac{\partial \mathbf{\Gamma}}{\partial \theta_\ell} \right\} \quad (51)$$

where

$\boldsymbol{\mu}$	mean of the observation vector;
$\mathbf{\Gamma}$	observation vector covariance matrix;
$[\mathbf{J}(\boldsymbol{\theta})]_{k,\ell}$	(k, ℓ) entry of the matrix $\mathbf{J}$.

Since the purely indeterministic and evanescent components of the 2-D Wold-like decomposition (5) have zero mean, we have $\boldsymbol{\mu} \equiv 0$. Hence, the first term of (51) vanishes. Thus, in this section, we study the problem of the achievable accuracy in *jointly* estimating the parameters of the evanescent and purely indeterministic components using a finite-size, single observed realization of the field. In this framework, the purely indeterministic component can be viewed as an unknown colored noise field.

Note from (21), (22), and (25)–(27) that dependence of the observed field covariance function on the (α, β) parameters exists both through the dependence of the exponential frequency on these parameters as well as through the dependence of the indices of the modulating process $s_i^{(\alpha,\beta)}(n\alpha - m\beta)$ on α and β. Therefore, we must assume that the (α, β) pair of each evanescent component is known and derive the CRB under this assumption. Indeed, since in the space–time adaptive radar problem the interference-to-background noise ratio is quite high [6] and since the dimensions of the observed field are limited, the integer pair (α, β) can be estimated with very low probability of error.

$$\mathbf{B} = \begin{bmatrix} \overline{\mathbf{B}} & \mathbf{0}_{T\times(T+2M)} & \cdots & & \mathbf{0}_{T\times(T+2M)} \\ \mathbf{0}_{T\times(T+2M)} & \overline{\mathbf{B}} & \mathbf{0}_{T\times(T+2M)} & \cdots & \mathbf{0}_{T\times(T+2M)} \\ & & \ddots & & \\ \mathbf{0}_{T\times(T+2M)} & \cdots & & \overline{\mathbf{B}} & \mathbf{0}_{T\times(T+2M)} \\ \mathbf{0}_{T\times(T+2M)} & \cdots & & \mathbf{0}_{T\times(T+2M)} & \overline{\mathbf{B}} \end{bmatrix} \quad (46)$$

Using the orthogonality of the evanescent components, their orthogonality to the purely indeterministic component, and (23), we find that

$$\frac{\partial \mathbf{\Gamma}}{\partial \mathbf{b}_k} = \frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial \mathbf{b}_k} \tag{52}$$

and for all $(\alpha, \beta) \in O$ and i

$$\frac{\partial \mathbf{\Gamma}}{\partial \left[\boldsymbol{\phi}_i^{(\alpha,\beta)}\right]_k} = \frac{\partial \mathbf{\Gamma}_i^{(\alpha,\beta)}}{\partial \left[\boldsymbol{\phi}_i^{(\alpha,\beta)}\right]_k}. \tag{53}$$

Substituting (52) and (53) into (51), we find that the FIM entries that correspond to parameters of the purely indeterministic and evanescent components are given by

$$\left[\mathbf{J}^{\mathbf{b},\mathbf{b}}\right]_{k,\ell} = \frac{1}{2}\,\mathrm{tr}\left\{\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial \mathbf{b}_k}\,\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial \mathbf{b}_\ell}\right\} \tag{54}$$

$$\left[\mathbf{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\mathbf{b}}\right]_{k,\ell} = \frac{1}{2}\,\mathrm{tr}\left\{\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}_i^{(\alpha,\beta)}}{\partial \left[\boldsymbol{\phi}_i^{(\alpha,\beta)}\right]_k}\,\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial \mathbf{b}_\ell}\right\} \tag{55}$$

and

$$\left[\mathbf{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{\phi}_j^{(\epsilon,\eta)}}\right]_{k,\ell} = \frac{1}{2}\,\mathrm{tr}\left\{\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}_i^{(\alpha,\beta)}}{\partial \left[\boldsymbol{\phi}_i^{(\alpha,\beta)}\right]_k}\,\mathbf{\Gamma}^{-1}\frac{\partial \mathbf{\Gamma}_j^{(\epsilon,\eta)}}{\partial \left[\boldsymbol{\phi}_j^{(\epsilon,\eta)}\right]_\ell}\right\} \tag{56}$$

where $(\epsilon, \eta) \in O$, and $1 \le j \le I^{(\epsilon,\eta)}$.

Using (25)–(27), and since $\tilde{\mathbf{R}}_i^{(\alpha,\beta)}$ is independent of $\nu_i^{(\alpha,\beta)}$, we find that

$$\begin{aligned}\frac{\partial \left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,1}}{\partial \nu_i^{(\alpha,\beta)}} &= \frac{2\pi}{\alpha^2+\beta^2}\left\{\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,1}\right.\\ &\odot\left[-\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right.\\ &\quad\left.-\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right]-\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,2}\\ &\odot\left[-\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right.\\ &\quad\left.+\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right]-\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,1}\\ &\odot\left[\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right.\\ &\quad\left.-\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right]+\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,2}\\ &\odot\left[\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right.\\ &\quad\left.\left.+\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\mathbf{v}^{(\alpha,\beta)}\odot\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right]\right\}.\end{aligned} \tag{57}$$

Similar derivations produce expressions for

$$\frac{\partial \left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,2}}{\partial \nu_i^{(\alpha,\beta)}} \quad \text{and} \quad \frac{\partial \left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{2,2}}{\partial \nu_i^{(\alpha,\beta)}}$$

as well. Finally

$$\begin{aligned}\frac{\left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,1}}{\partial\left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k} &= \frac{\partial\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,1}}{\partial\left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k}\odot\left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right]\\ &-\frac{\partial\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{1,2}}{\partial\left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k}\odot\left[\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right]\\ &-\frac{\partial\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,1}}{\partial\left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k}\odot\left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{f}}_i^{(\alpha,\beta)}\right)^T\right]\\ &+\frac{\partial\left(\tilde{\mathbf{R}}_i^{(\alpha,\beta)}\right)_{2,2}}{\partial\left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k}\odot\left[\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\left(\tilde{\mathbf{g}}_i^{(\alpha,\beta)}\right)^T\right].\end{aligned} \tag{58}$$

Similar derivations produce expressions for

$$\frac{\left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{1,2}}{\partial\left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k} \quad \text{and} \quad \frac{\left(\mathbf{\Gamma}_i^{(\alpha,\beta)}\right)_{2,2}}{\partial\left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k}.$$

A. FIM for Evanescent Components With Gaussian MA Modulating Processes

The parameter vector of the 1-D purely indeterministic modulating MA process $\mathbf{a}_i^{(\alpha,\beta)}$ is the $2(Q_i^{(\alpha,\beta)}+1)$-dimensional vector

$$\begin{aligned}\mathbf{a}_i^{(\alpha,\beta)} = \Big[&\left(\sigma_i^{(\alpha,\beta)}\right)^2, \left(\rho_i^{(\alpha,\beta)}\right)^2, \mathrm{Re}\left\{a_i^{(\alpha,\beta)}(1)\right\}\\ &\mathrm{Re}\left\{a_i^{(\alpha,\beta)}(2)\right\}, \ldots, \mathrm{Re}\left\{a_i^{(\alpha,\beta)}\left(Q_i^{(\alpha,\beta)}\right)\right\}\\ &\mathrm{Im}\left\{a_i^{(\alpha,\beta)}(1)\right\}, \mathrm{Im}\left\{a_i^{(\alpha,\beta)}(2)\right\}, \ldots\\ &\mathrm{Im}\left\{a_i^{(\alpha,\beta)}\left(Q_i^{(\alpha,\beta)}\right)\right\}\Big]^T.\end{aligned} \tag{59}$$

Assume that the real and imaginary components of the driving noise process $\{\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$, which are defined in Section IV, are also Gaussian. Thus, taking the partial derivatives of $\mathbf{R}_i^{(\alpha,\beta)}$, we have (60) and (61), shown at the bottom of the next page, and for all $n = 1, \ldots, Q_i^{(\alpha,\beta)}$

$$\frac{\partial \mathbf{R}_i^{(\alpha,\beta)}}{\partial\left[\mathrm{Re}\left\{a_i^{(\alpha,\beta)}(n)\right\}\right]} = \begin{bmatrix}\Delta_{1,1} & \Delta_{1,2}\\ \Delta_{2,1} & \Delta_{2,2}\end{bmatrix} \tag{62}$$

where

$$\begin{aligned}\Delta_{1,1} = \left(\sigma_i^{(\alpha,\beta)}\right)^2\Big[&\mathbf{W}_i^{(\alpha,\beta)}\mathbf{U}_n\left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T\\ &+\mathbf{W}_i^{(\alpha,\beta)}\mathbf{B}_i^{(\alpha,\beta)}\mathbf{U}_n^T\left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T\Big]\end{aligned} \tag{63}$$

$$\Delta_{1,2} = \left(\sigma_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)} \mathbf{U}_n \left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T - \left(\rho_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)} \mathbf{C}_i^{(\alpha,\beta)} \mathbf{U}_n^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \quad (64)$$

$$\Delta_{2,2} = \frac{\left(\rho_i^{(\alpha,\beta)}\right)^2}{\left(\sigma_i^{(\alpha,\beta)}\right)^2} \Delta_{1,1} \quad (65)$$

and $\Delta_{2,1} = \Delta_{1,2}^T$, where $\mathbf{U}_n$ is the up shift matrix

$$[\mathbf{U}_n]_{k,\ell} = \begin{cases} 1, & \ell - k = Q_i^{(\alpha,\beta)} - n \\ 0, & \text{otherwise.} \end{cases} \quad (66)$$

In a similar way, we have for all $n = 1, \ldots, Q_i^{(\alpha,\beta)}$

$$\frac{\partial \mathbf{R}_i^{(\alpha,\beta)}}{\partial \left[\mathrm{Im}\left\{a_i^{(\alpha,\beta)}(n)\right\}\right]} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{bmatrix} \quad (67)$$

where

$$\Phi_{1,1} = \left(\rho_i^{(\alpha,\beta)}\right)^2 \left[\mathbf{W}_i^{(\alpha,\beta)} \mathbf{U}_n \left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T + \mathbf{W}_i^{(\alpha,\beta)} \mathbf{C}_i^{(\alpha,\beta)} \mathbf{U}_n^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T\right] \quad (68)$$

$$\Phi_{1,2} = \left(\sigma_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)} \mathbf{B}_i^{(\alpha,\beta)} \mathbf{U}_n^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T - \left(\rho_i^{(\alpha,\beta)}\right)^2 \mathbf{W}_i^{(\alpha,\beta)} \mathbf{U}_n \left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \quad (69)$$

$$\Phi_{2,2} = \frac{\left(\sigma_i^{(\alpha,\beta)}\right)^2}{\left(\rho_i^{(\alpha,\beta)}\right)^2} \Phi_{1,1} \quad (70)$$

and $\Phi_{2,1} = \Phi_{1,2}^T$.

Substituting (21) and (22), (25)–(27) and (60)–(70) into (57) and (58), we obtain closed-form expressions for

$$\frac{\partial \mathbf{\Gamma}_i^{(\alpha,\beta)}}{\partial \left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k} \quad \text{and} \quad \frac{\partial \mathbf{\Gamma}_i^{(\alpha,\beta)}}{\partial \nu_i^{(\alpha,\beta)}}$$

for the case in which the modulating 1-D processes of the evanescent fields are MA processes.

B. FIM for an MA Purely Indeterministic Component

Assume that the real and imaginary components of the MA model driving noise field $\{u(n,m)\}$ defined in Section V are also Gaussian. Thus, taking the partial derivatives with respect to the MA model parameters, we get

$$\frac{\partial \overline{\mathbf{b}}}{\partial [\mathrm{Re}\{b(k,\ell)\}]} = \mathbf{e}_{(N-k)(T+2M)+M+1-\ell} \quad (71)$$

where $\mathbf{e}_{(N-k)(T+2M)+M+1-\ell}$ is a $(T+2M)\cdot(N+1)$-dimensional row vector whose $(N-k)(T+2M)+M+1-\ell$ element equals one, whereas all its other elements are zero. Hence

$$\frac{\partial \overline{\mathbf{B}}^R}{\partial [\mathrm{Re}\{b(k,\ell)\}]} = \overline{\mathbf{U}}_{(k,\ell)} \quad (72)$$

where $\overline{\mathbf{U}}_{(k,\ell)}$ is the upshift matrix

$$\left[\overline{\mathbf{U}}_{(k,\ell)}\right]_{i,j} = \begin{cases} 1, & j - i = (N-k)(T+2M) + M - \ell \\ 0, & \text{otherwise.} \end{cases} \quad (73)$$

Similarly

$$\frac{\partial \overline{\mathbf{B}}^I}{\partial [\mathrm{Im}\{b(k,\ell)\}]} = \overline{\mathbf{U}}_{(k,\ell)}. \quad (74)$$

Taking the partial derivatives of $\mathbf{\Gamma}_{\mathrm{PI}}$ with respect to the MA model parameters, we have for $(k,\ell) \in S_{N,M} \setminus \{(0,0)\}$

$$\frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial [\mathrm{Re}\{b(k,\ell)\}]} = \begin{bmatrix} \sigma^2 \mathbf{U}_{(k,\ell)} & \mathbf{0} \\ \mathbf{0} & \rho^2 \mathbf{U}_{(k,\ell)} \end{bmatrix} \tilde{\mathbf{B}}^T + \tilde{\mathbf{B}} \begin{bmatrix} \sigma^2 \mathbf{U}_{(k,\ell)} & \mathbf{0} \\ \mathbf{0} & \rho^2 \mathbf{U}_{(k,\ell)} \end{bmatrix}^T \quad (75)$$

where $\mathbf{U}_{(k,\ell)}$, in (76), shown at the bottom of the next page, is a $ST \times (T+2M)(S+N)$ matrix. Similarly

$$\frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial [\mathrm{Im}\{b(k,\ell)\}]} = \begin{bmatrix} \mathbf{0} & -\rho^2 \mathbf{U}_{(k,\ell)} \\ \sigma^2 \mathbf{U}_{(k,\ell)} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{B}}^T + \tilde{\mathbf{B}} \begin{bmatrix} \mathbf{0} & -\rho^2 \mathbf{U}_{(k,\ell)} \\ \sigma^2 \mathbf{U}_{(k,\ell)} & \mathbf{0} \end{bmatrix}^T. \quad (77)$$

In addition, let

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{B}^R \\ \mathbf{B}^I \end{bmatrix} \quad (78)$$

$$\mathbf{B}_2 = \begin{bmatrix} -\mathbf{B}^I \\ \mathbf{B}^R \end{bmatrix}. \quad (79)$$

$$\frac{\partial \mathbf{R}_i^{(\alpha,\beta)}}{\partial \left(\sigma_i^{(\alpha,\beta)}\right)^2} = \begin{bmatrix} \mathbf{W}_i^{(\alpha,\beta)} \mathbf{B}_i^{(\alpha,\beta)} \left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T & \mathbf{W}_i^{(\alpha,\beta)} \mathbf{B}_i^{(\alpha,\beta)} \left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \\ \mathbf{W}_i^{(\alpha,\beta)} \mathbf{C}_i^{(\alpha,\beta)} \left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T & \mathbf{W}_i^{(\alpha,\beta)} \mathbf{C}_i^{(\alpha,\beta)} \left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \end{bmatrix} \quad (60)$$

$$\frac{\partial \mathbf{R}_i^{(\alpha,\beta)}}{\partial \left(\rho_i^{(\alpha,\beta)}\right)^2} = \begin{bmatrix} \mathbf{W}_i^{(\alpha,\beta)} \mathbf{C}_i^{(\alpha,\beta)} \left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T & -\mathbf{W}_i^{(\alpha,\beta)} \mathbf{C}_i^{(\alpha,\beta)} \left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \\ -\mathbf{W}_i^{(\alpha,\beta)} \mathbf{B}_i^{(\alpha,\beta)} \left(\mathbf{C}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T & \mathbf{W}_i^{(\alpha,\beta)} \mathbf{B}_i^{(\alpha,\beta)} \left(\mathbf{B}_i^{(\alpha,\beta)}\right)^T \left(\mathbf{W}_i^{(\alpha,\beta)}\right)^T \end{bmatrix} \quad (61)$$

Then

$$\frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial \sigma^2} = \mathbf{B}_1 \mathbf{B}_1^T \quad (80)$$

$$\frac{\partial \mathbf{\Gamma}_{\mathrm{PI}}}{\partial \rho^2} = \mathbf{B}_2 \mathbf{B}_2^T. \quad (81)$$

Substituting (23), (24), (50), (75), (77), (80), and (81) into (54), we obtain a closed-form expression for the FIM block that corresponds to the 2-D Gaussian MA random field. In the case where the observed field comprises only a moving-average purely indeterministic component, inversion of (54) after the above substitutions have been made provides the exact CRB on the error variance in jointly estimating the parameters of the purely indeterministic field. When evanescent components are present, substituting (21), (22), (25)–(27) and (60)–(70) into (57) and (58), we obtain closed-form expressions for

$$\frac{\partial \mathbf{\Gamma}_i^{(\alpha,\beta)}}{\partial \left[\mathbf{a}_i^{(\alpha,\beta)}\right]_k} \quad \text{and} \quad \frac{\partial \mathbf{\Gamma}_i^{(\alpha,\beta)}}{\partial \nu_i^{(\alpha,\beta)}}$$

for the case in which the modulating 1-D processes of the evanescent fields are MA processes. A final substitution into (54)–(56) provides the CRB on the error variance in estimating the parameters of essentially any homogeneous Gaussian random field with nil harmonic component. In the next section, we extend this derivation and consider the problem of the achievable estimation accuracy of both the harmonic, evanescent, and purely indeterministic components of a homogeneous random field.

VII. Fisher Information Matrix in the Presence of the Harmonic Component

In this section, we extend the derivation of the previous sections and address the problem of the achievable estimation accuracy of both the harmonic, evanescent, and purely indeterministic components of a homogeneous Gaussian random field. Note that when expressed in the general form (4), the coefficients $\{C_p\}$ of the harmonic component are complex-valued, mutually orthogonal random variables. However, in general, only a single realization of the random field is available. Hence, we cannot infer anything about the variation of these coefficients over different realizations. The best we can do is to estimate the particular values that the C_p's take for the given realization; in other words, we might just as well treat the C_p's as unknown constants and the harmonic component as the unknown mean of the observed realization. In the following, we assume that the number P of harmonic components in the observed field is *a priori* known.

Let

$$\mathbf{h} = [h(0,0), \ldots, h(0,T-1), h(1,0), \ldots, h(1,T-1) \\ \ldots, h(S-1,0), \ldots, h(S-1,T-1)]^T. \quad (82)$$

In addition, let $\mathbf{c} = [C_1, \ldots, C_P]^T$, $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_P]^T$, $\boldsymbol{\nu} = [\nu_1, \ldots, \nu_P]^T$. Similarly to the definitions of Section III, we define the vectors $\mathbf{c}^R$, $\mathbf{c}^I$, $\overline{\mathbf{c}}$, $\mathbf{h}^R$, $\mathbf{h}^I$, and $\overline{\mathbf{h}}$. Hence, in this case, $\boldsymbol{\mu} \equiv \overline{\mathbf{h}}$ in (51). The parameter vector of the observed field $\{y(n,m)\}$ is now given by

$$\boldsymbol{\theta} = \left[\overline{\mathbf{c}}^T \boldsymbol{\omega}^T \boldsymbol{\nu}^T \mathbf{b}^T \left\{\left(\boldsymbol{\phi}^{(\alpha,\beta)}\right)^T\right\}_{(\alpha,\beta)\in O}\right]^T. \quad (83)$$

Define (84), shown at the bottom of the next page, where the ith column of $\mathbf{H}$ consists of the values of the ith harmonic component evaluated for all $(s,t) \in D$. We therefore have

$$\overline{\mathbf{h}} = \mathbf{\Lambda} \overline{\mathbf{c}} \quad (85)$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{H}^R & -\mathbf{H}^I \\ \mathbf{H}^I & \mathbf{H}^R \end{bmatrix} \quad (86)$$

and $\mathbf{H}^R = \mathrm{Re}\{\mathbf{H}\}$, $\mathbf{H}^I = \mathrm{Im}\{\mathbf{H}\}$. Taking the partial derivatives of $\overline{\mathbf{h}}$, we get

$$\frac{\partial \overline{\mathbf{h}}}{\partial \overline{\mathbf{c}}_\ell} = \mathbf{\Lambda}_\ell \quad (87)$$

where $\mathbf{\Lambda}_\ell$ is the ℓth column of $\mathbf{\Lambda}$. Since the evanescent components, as well as the purely indeterministic component, are zero mean fields, the mean vector is independent of their parameters. Hence

$$\frac{\partial \overline{\mathbf{h}}}{\partial \mathbf{b}_k} = 0 \quad (88)$$

$$\frac{\partial \overline{\mathbf{h}}}{\partial \left[\boldsymbol{\phi}_i^{(\alpha,\beta)}\right]_k} = 0. \quad (89)$$

$$\mathbf{U}_{(k,\ell)} = \begin{bmatrix} \overline{\mathbf{U}}_{(k,\ell)} & \mathbf{0}_{T\times(T+2M)} & \cdots & & \mathbf{0}_{T\times(T+2M)} \\ \mathbf{0}_{T\times(T+2M)} & \overline{\mathbf{U}}_{(k,\ell)} & \mathbf{0}_{T\times(T+2M)} & \cdots & \mathbf{0}_{T\times(T+2M)} \\ & & \ddots & & \\ \mathbf{0}_{T\times(T+2M)} & \cdots & & \overline{\mathbf{U}}_{(k,\ell)} & \mathbf{0}_{T\times(T+2M)} \\ \mathbf{0}_{T\times(T+2M)} & \cdots & & \mathbf{0}_{T\times(T+2M)} & \overline{\mathbf{U}}_{(k,\ell)} \end{bmatrix} \quad (76)$$

In addition, note that since the field covariance function $\mathbf{\Gamma}$ is independent of the mean

$$\frac{\partial \mathbf{\Gamma}}{\partial \overline{\mathbf{c}}_k} = 0 \tag{90}$$

$$\frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{\omega}_k} = 0 \tag{91}$$

and

$$\frac{\partial \mathbf{\Gamma}}{\partial \boldsymbol{\nu}_k} = 0. \tag{92}$$

Hence, the $(1/2)\,\mathrm{tr}\{\cdot\}$ term in (51) vanishes for all the FIM entries that correspond to parameters of the harmonic mean. Therefore, $\mathbf{J}^{\mathbf{b},\overline{\mathbf{c}}} = 0$, $\mathbf{J}^{\mathbf{b},\boldsymbol{\omega}} = 0$, $\mathbf{J}^{\mathbf{b},\boldsymbol{\nu}} = 0$, and for the evanescent components, we have that for all (α, β) and i, $\mathbf{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\overline{\mathbf{c}}} = 0$, $\mathbf{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{\omega}} = 0$, $\mathbf{J}^{\boldsymbol{\phi}_i^{(\alpha,\beta)},\boldsymbol{\nu}} = 0$. Hence, we conclude that the lower bound on the error variance in estimating the parameters of the harmonic component is decoupled from the bound on the parameters of the purely indeterministic and evanescent components.

Using (87) and (90), we conclude that the FIM elements that correspond to the amplitude parameters of the harmonic component are given by

$$\left[\mathbf{J}^{\overline{\mathbf{c}},\overline{\mathbf{c}}}\right]_{k,\ell} = \mathbf{\Lambda}_k^T \mathbf{\Gamma}^{-1} \mathbf{\Lambda}_\ell. \tag{93}$$

Let

$$\boldsymbol{\tau}_1 = [0, 1, \ldots, (S-1)]^T \otimes \mathbf{1_T} \tag{94}$$

$$\boldsymbol{\tau}_2 = \mathbf{1}_S \otimes [0, 1, \ldots, (T-1)]^T \tag{95}$$

where $\mathbf{1_T}$ and $\mathbf{1_S}$ are T-dimensional and S-dimensional column vectors of ones, respectively. In other words, $\boldsymbol{\tau}_1$ is the vector of the first indices of the elements of $\mathbf{h}$ in (82), and $\boldsymbol{\tau}_2$ is the vector of the second indices of the elements of $\mathbf{h}$. Taking now the partial derivatives w.r.t. the harmonic frequencies yields

$$\frac{\partial \overline{\mathbf{h}}}{\partial \omega_p} = 2\pi \begin{bmatrix} -\mathrm{diag}(\boldsymbol{\tau}_1) & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}(\boldsymbol{\tau}_1) \end{bmatrix} \begin{bmatrix} \mathbf{c}_p^R \mathbf{H}_p^I + \mathbf{c}_p^I \mathbf{H}_p^R \\ -\mathbf{c}_p^I \mathbf{H}_p^I + \mathbf{c}_p^R \mathbf{H}_p^R \end{bmatrix} \tag{96}$$

$$\frac{\partial \overline{\mathbf{h}}}{\partial \nu_p} = 2\pi \begin{bmatrix} -\mathrm{diag}(\boldsymbol{\tau}_2) & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}(\boldsymbol{\tau}_2) \end{bmatrix} \begin{bmatrix} \mathbf{c}_p^R \mathbf{H}_p^I + \mathbf{c}_p^I \mathbf{H}_p^R \\ -\mathbf{c}_p^I \mathbf{H}_p^I + \mathbf{c}_p^R \mathbf{H}_p^R \end{bmatrix} \tag{97}$$

where $\mathrm{diag}(\boldsymbol{\tau}_1)$, $(\mathrm{diag}(\boldsymbol{\tau}_2))$ is a $ST \times ST$ matrix whose diagonal is the vector $\boldsymbol{\tau}_1$, $(\boldsymbol{\tau}_2)$, and $\mathbf{H}_p^R$, $(\mathbf{H}_p^I)$ is the pth column of $\mathbf{H}^R$, $(\mathbf{H}^I)$.

Substituting (87)–(92), (96), and (97) into (51)

$$\left[\mathbf{J}^{\overline{\mathbf{c}},\boldsymbol{\omega}}\right]_{k,\ell} = \mathbf{\Lambda}_k^T \mathbf{\Gamma}^{-1} \frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\omega}_\ell} \tag{98}$$

$$\left[\mathbf{J}^{\overline{\mathbf{c}},\boldsymbol{\nu}}\right]_{k,\ell} = \mathbf{\Lambda}_k^T \mathbf{\Gamma}^{-1} \frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\nu}_\ell} \tag{99}$$

$$\left[\mathbf{J}^{\boldsymbol{\omega},\boldsymbol{\omega}}\right]_{k,\ell} = \left(\frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\omega}_k}\right)^T \mathbf{\Gamma}^{-1} \frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\omega}_\ell} \tag{100}$$

$$\left[\mathbf{J}^{\boldsymbol{\omega},\boldsymbol{\nu}}\right]_{k,\ell} = \left(\frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\omega}_k}\right)^T \mathbf{\Gamma}^{-1} \frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\nu}_\ell} \tag{101}$$

$$\left[\mathbf{J}^{\boldsymbol{\nu},\boldsymbol{\nu}}\right]_{k,\ell} = \left(\frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\nu}_k}\right)^T \mathbf{\Gamma}^{-1} \frac{\partial \overline{\mathbf{h}}}{\partial \boldsymbol{\nu}_\ell}. \tag{102}$$

We have previously concluded that the lower bound on the error variance in estimating the parameters of the harmonic component is decoupled from the bound on the parameters of the purely indeterministic and evanescent components. Using (51), (54)–(56), and (88)–(92), we further conclude that the bound on the purely indeterministic and evanescent components is found by inverting the FIM block that corresponds to the parameters of these components, and it is *independent* of the harmonic component parameters. Therefore, this bound is identical to the one obtained for the case in which no harmonic component exists. We have thus completed the derivation of closed-form expressions for the *exact* Cramér–Rao lower bound on the error variance in jointly estimating the parameters of essentially any complex-valued homogeneous Gaussian random field that can be modeled by a finite-order model. In this derivation, the field may contain all of the 2-D Wold decomposition components.

Finally, in [19], the large sample Cramér-Rao bound on the parameters of the harmonic component in the presence of the purely indeterministic field has been recently derived. This model is a special case of the general model considered in this paper. Assuming that the covariance sequence of the purely indeterministic component satisfies certain conditions on its rate of decay, it is shown that the large sample CRB on the parameters of each exponential is decoupled from the bound

$$\mathbf{H} = \begin{bmatrix} e^{j2\pi[0\omega_1+0\nu_1]} & e^{j2\pi[0\omega_2+0\nu_2]} & \cdots & e^{j2\pi[0\omega_P+0\nu_P]} \\ e^{j2\pi[0\omega_1+1\nu_1]} & e^{j2\pi[0\omega_2+1\nu_2]} & \cdots & e^{j2\pi[0\omega_P+1\nu_P]} \\ \vdots & \vdots & \vdots & \vdots \\ e^{j2\pi[0\omega_1+(T-1)\nu_1]} & e^{j2\pi[0\omega_2+(T-1)\nu_2]} & \cdots & e^{j2\pi[0\omega_P+(T-1)\nu_P]} \\ e^{j2\pi[1\omega_1+0\nu_1]} & e^{j2\pi[1\omega_2+0\nu_2]} & \cdots & e^{j2\pi[1\omega_P+0\nu_P]} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ e^{j2\pi[(S-1)\omega_1+(T-1)\nu_1]} & \cdots & \cdots & e^{j2\pi[(S-1)\omega_P+(T-1)\nu_P]} \end{bmatrix} \tag{84}$$

TABLE I
CRB ON THE PARAMETERS OF THE MA FIELD FOR EXAMPLE 1 AND EXAMPLE 2

	Parameters	Ex.1: CRB	Ex.2: CRB
σ^2	1	0.00885	0.00891
ρ^2	4	0.12204	0.12216
$\mathrm{Re}\{b(0,1)\}$	-0.6364	0.00143	0.00143
$\mathrm{Im}\{b(0,1)\}$	-0.6364	0.00124	0.00124
$\mathrm{Re}\{b(1,-1)\}$	0.0309	0.00156	0.00156
$\mathrm{Im}\{b(1,-1)\}$	0.0951	0.00112	0.00113
$\mathrm{Re}\{b(1,0)\}$	0.4045	0.00210	0.00211
$\mathrm{Im}\{b(1,0)\}$	-0.2939	0.00154	0.00155
$\mathrm{Re}\{b(1,1)\}$	0.3236	0.00129	0.00129
$\mathrm{Im}\{b(1,1)\}$	-0.2351	0.00129	0.00129

on the parameters of the other exponentials. It is further shown that asymptotically

$$\mathrm{CRB}(\omega_p) = \frac{1}{T^3 S} \frac{6\mathcal{S}\left(e^{j2\pi\omega_p},\, e^{j2\pi\nu_p}\right)}{|C_p|^2}$$

$$\mathrm{CRB}(\nu_p) = \frac{1}{T S^3} \frac{6\mathcal{S}\left(e^{j2\pi\omega_p},\, e^{j2\pi\nu_p}\right)}{|C_p|^2}$$

$$\mathrm{CRB}(|C_p|) = \frac{1}{TS} \frac{\mathcal{S}\left(e^{j2\pi\omega_p},\, e^{j2\pi\nu_p}\right)}{2}$$

$$\mathrm{CRB}(\phi_p) = \frac{1}{TS} \frac{7\mathcal{S}\left(e^{j2\pi\omega_p},\, e^{j2\pi\nu_p}\right)}{2|C_p|^2}$$

where $\mathcal{S}(e^{j2\pi\omega}, e^{j2\pi\nu})$ is the spectral density of the colored noise field, and ϕ_p denotes the phase of C_p.

VIII. NUMERICAL EXAMPLES

To gain more insight into the behavior of the bound on the different components, we resort to numerical evaluation of some specific examples. In this section, we present several such examples that illustrate the dependence of the bound on various parameters of the field. In all of the examples, the dimensions of the observed field are relatively small: $S = T = 20$.

Example 1: Consider a 2-D homogeneous, purely indeterministic random field modeled by a NSHP MA model with support $S_{1,1}$. The model parameters are listed in Table I. In this example, we evaluate the Cramér–Rao lower bound on the error variance in estimating the model parameters, as well as the bound on the error variance in estimating the spectral density of the field. The values of the CRB on the 2-D MA model parameters are also listed in Table I.

The spectral density function of this purely indeterministic field, and the CR lower bound on the error variance in estimating it, are depicted in Fig. 1. Note that the shape of the bound as a function of frequency matches the shape of the MA field spectral density function. It can be further shown by considering the normalized CRB, i.e., the ratio of the squared root of the CRB to the spectral density function of the MA field, that the lower bound on the error variance in estimating the MA field spectral density function is relatively higher in those frequency regions where the MA model transfer function is close to zero. In other words, the estimation of the MA field spectral density function is relatively less accurate in frequency regions where the spectral density function is close to zero than in regions of higher spectral density.

Example 2: Consider a 2-D homogeneous random field consisting of a sum of a purely indeterministic component and a single evanescent component. The purely indeterministic component is the same NSHP MA field with support $S_{1,1}$ whose parameters are listed in Table I. The evanescent component spectral support parameters are $(\alpha, \beta) = (1, 1)$, $\nu^{(1,1)} = 0.1$. The modulating 1-D purely indeterministic process of this evanescent component is a second-order Gaussian MA process such that $\sigma^{(1,1)} = 2$, $\rho^{(1,1)} = 1$, $a^{(1,1)}(1) = -0.55\exp(j\pi/4)$, and $a^{(1,1)}(2) = 0.1\exp(j\pi/4)$.

In this example, we evaluate the CR lower bound on the error variance in estimating the two components of the field. Note from Table I that the bounds on the parameters of the purely indeterministic field remain essentially identical to their values for the case in which no evanescent component was present in the field (Example 1). Next, we investigate the bound on the spectral density function of the modulating 1-D MA process of the evanescent field. Fig. 2 depicts the spectral density function of the modulating MA process. The mean value of the spectral density (dashed line) and the mean plus and minus the standard deviation computed from the CRB (dashed-dotted line) are shown. As a reference, the solid line denotes the mean value of the spectral density plus and minus the standard deviation computed from the CRB of the same 1-D MA process for the case in which this MA process is observed directly as a 1-D process, (i.e., a standard 1-D problem). It is concluded that the presence of the 2-D purely indeterministic field causes the bound on the spectral density of the evanescent field 1-D MA process to be higher than in the standard 1-D case. However, its shape (as a function of frequency) remains similar to its shape in the standard 1-D case.

Example 3: Consider a 2-D homogeneous random field consisting of a sum of a purely indeterministic component, a single evanescent component, and an harmonic component. The purely indeterministic component is the same NSHP MA field with support $S_{1,1}$ whose parameters are listed in Table I. The evanescent component is the same as in Example 2. The harmonic component comprises a single exponential whose frequency varies from experiment to experiment. In this example, we investigate the bound on the error variance in estimating the parameters of the exponential as a function of its frequency. Since the estimation problem of the purely indeterministic and evanescent components is independent of the estimation problem of the harmonic component, the bounds on the parameters of both these components are not affected by the presence of the harmonic component. These bounds are therefore identical to the bounds computed in Example 2.

Note using Figs. 1 and 2 that the bound on the frequency parameter of the harmonic component, which is depicted in Fig. 3, matches the shape of the spectral density of the purely indeterministic component as well as the shape of the spectral density of the modulating 1-D MA process of the evanescent field. In other words, the bound is higher at those frequencies where the spectral density of the 2-D MA field is higher and at those frequencies along the spectral support of the evanescent field

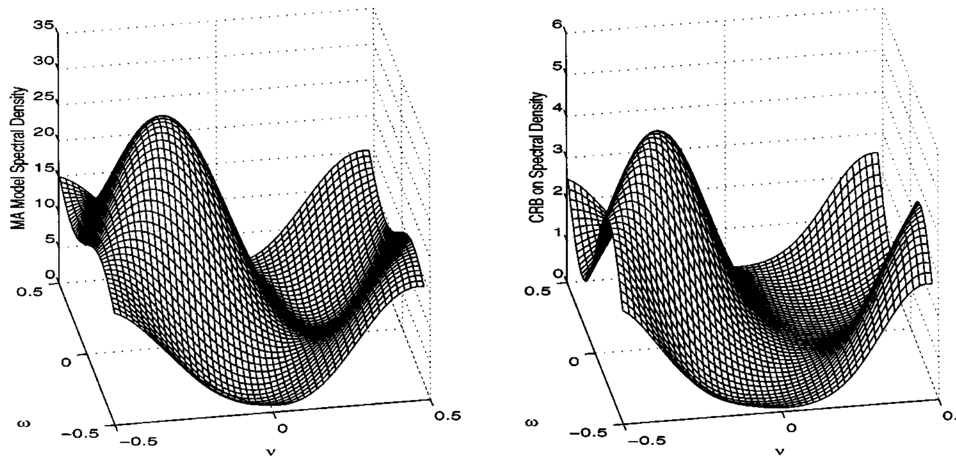

Fig. 1. Spectral density function of the MA field and the CR lower bound on the error variance in estimating it.

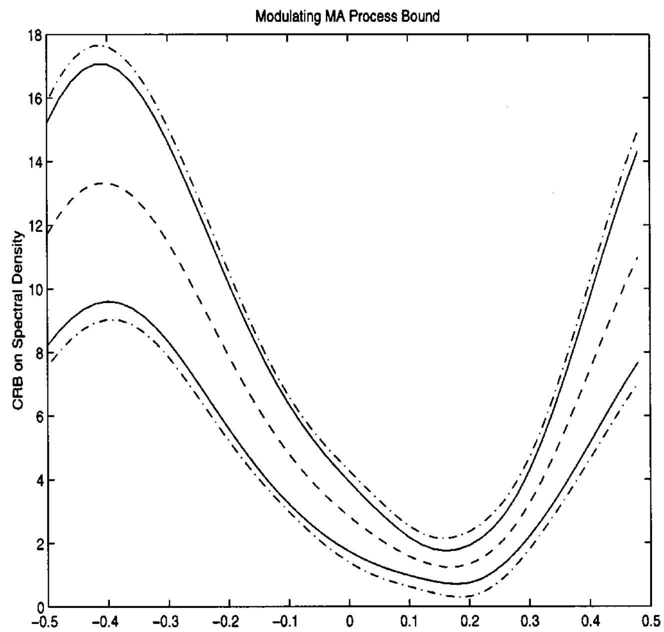

Fig. 2. CRBs on the spectral density function of the evanescent component 1-D MA process.

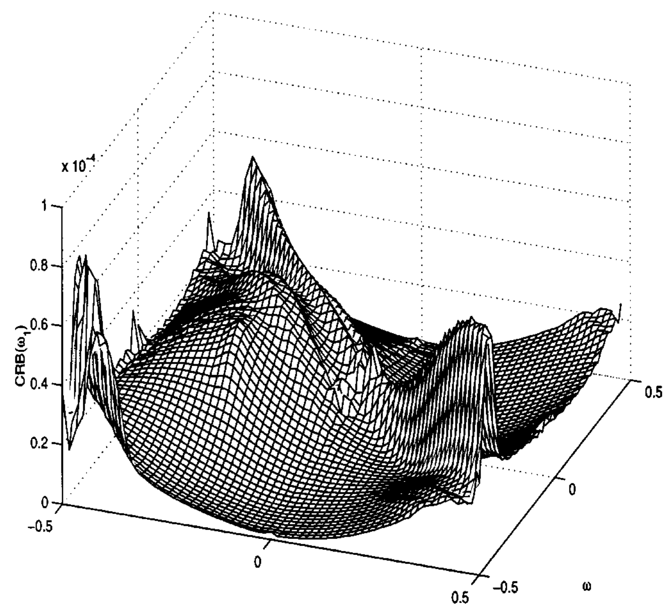

Fig. 3. CRB on the frequency parameter ω_1 of the exponential as a function of the exponential's frequency.

where the spectral density of the modulating 1-D purely indeterministic process is higher. The effect of the wraparound of the spectral support of the evanescent field is clearly seen in Fig. 3. Similar results are obtained for the other parameters of the exponential as well.

Example 4: Here, we consider a case in which the observed random field has two evanescent components whose spectral supports intersect. We compare the CRB on the accuracy of estimating the field parameters with the CRB for the same field, when only one of the two evanescent components is present. More specifically, we consider a 2-D homogeneous random field consisting of a sum of two evanescent components $e^{(1,1)}(n, m)$, $e^{(2,1)}(n, m)$ and a Gaussian purely indeterministic component whose real and imaginary components are independent zero mean white Gaussian fields with variance σ^2 and ρ^2, respectively. The parameters of the different components of the field are listed in Table II.

We consider three cases.

Case 1) Both evanescent components exist in the observed field.

Case 2) The only evanescent component of the observed field is $e^{(1,1)}(n, m)$.

Case 3) The only evanescent component of the observed field is $e^{(2,1)}(n, m)$.

In Table II, we list the bounds on the error variance in estimating the parameters of the observed field for the three cases. The results indicate that the lower bounds on the error variance in estimating the parameters of the different components are essentially the same in all three cases being considered here.

Using this example, we conclude that in general, the presence of evanescent random fields with intersecting spectral supports has only a negligible effect on the CRB of each component parameters, compared with the case in which this component is the only evanescent component of the field. Using the conclusions of Examples 2 and 3, we finally conclude that the presence of an

TABLE II
CRB FOR A FIELD WITH EVANESCENT COMPONENTS OF INTERSECTING SPECTRAL SUPPORTS

			CRB: Case I	CRB: Case II	CRB: Case III
First evanescent component $e^{(1,1)}(n,m)$	$\nu^{(1,1)}$	0.1	9.1314565e-07	9.1313922e-07	-
	$(\sigma^{(1,1)})^2$	4	1.0352698	1.0346625	-
	$(\rho^{(1,1)})^2$	1	9.6123217e-02	9.6031324e-02	-
	$\mathrm{Re}\{a^{(1,1)}(1)\}$	-0.3889	2.0063218e-02	2.0038275e-02	-
	$\mathrm{Im}\{a^{(1,1)}(1)\}$	-0.3889	9.5612309e-03	9.5578983e-03	-
	$\mathrm{Re}\{a^{(1,1)}(2)\}$	0.0707	1.6381512e-02	1.6375244e-02	-
	$\mathrm{Im}\{a^{(1,1)}(2)\}$	0.0707	1.1910025e-02	1.1895710e-02	-
Second evanescent component $e^{(2,1)}(n,m)$	$\nu^{(2,1)}$	0.25	4.4280236e-06	-	4.4276767e-06
	$(\sigma^{(2,1)})^2$	2.25	2.6050751e-01	-	2.6041051e-01
	$(\rho^{(2,1)})^2$	1	7.1620562e-02	-	7.1538973e-02
	$\mathrm{Re}\{a^{(2,1)}(1)\}$	0.3536	1.2907384e-02	-	1.2890388e-02
	$\mathrm{Im}\{a^{(2,1)}(1)\}$	0.3536	1.0290456e-02	-	1.0290081e-02
Purely indeterministic	σ^2	1	8.0427776e-03	7.9775468e-03	8.0199510e-03
	ρ^2	4	9.4522371e-02	9.4419934e-02	9.4491849e-02

evanescent component in the field has only a negligible effect on the lower bound on the accuracy of estimating the parameters of the other components of the field, unless the spectral support of the evanescent component is very close to the spectral support of a harmonic component.

IX. CONCLUSIONS

In this paper, we have elaborated on two fundamental problems in parametric modeling and estimation of 2-D complex-valued homogeneous random fields with mixed spectral distribution. Employing the parametric model that follows from the 2-D Wold-like decomposition of homogeneous random fields, we have obtained closed-form expressions for the field mean and covariance functions in terms of the model parameters. Assuming the observed random field is Gaussian, we have then investigated the problem of the achievable accuracy in jointly estimating the field parameters from a single observed realization of it. It is shown that the estimation of the harmonic component is decoupled from that of the purely indeterministic and evanescent components; furthermore, the bound on the purely indeterministic and evanescent components is independent of the harmonic component. Due to the generality of the Wold decomposition-based field model, the derivation in this paper provides a closed-form expression of the Fisher information matrix of essentially any complex-valued homogeneous Gaussian random field and, hence, of the corresponding Cramér–Rao lower bound.

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