noisy images with extremely low SNR have been considered. In all cases, the results are successful. Restored images with considerable high SNR, even in the worst conditions, are obtained by the DDLF. The performance and quality of the DDLF depend on the order of the 2-D lattice filter, the choice of delta parameters Δ_1 and Δ_2, and the initial values.

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Optimal Parameter Selection in the Phase Differencing Algorithm for 2-D Phase Estimation

Joseph M. Francos and Benjamin Friedlander

***Abstract*— A parametric model and a corresponding algorithm for estimating two-dimensional (2-D) phase functions are presented in a previous paper. The performance of the phase estimation algorithm and, hence, the performance of any algorithm that employs it, strongly depends on the choice of the two free parameters of the algorithm. In this correspondence, we systematically analyze the performance of the phase estimation algorithm and derive rules for selecting the algorithm parameters such that the mean squared error in estimating the signal phase is minimized. It is shown analytically and verified using Monte-Carlo simulations that this choice of parameters results in unbiased estimates of the phase and spatial frequency functions. The variances of both the estimated phase and frequency functions are very close to the corresponding Cramér–Rao lower bounds.**

I. INTRODUCTION

Phase information has fundamental importance in many one- and two-dimensional (1-D and 2-D) signal processing problems. In one dimension, the first derivative of the phase is the instantaneous frequency of the signal, whereas for multidimensional data, the partial derivatives of the phase along each of the spatial axes provide the local spatial frequency of the analyzed field. When dealing with 2-D signals, estimates of the phase are required in different applications such as 2-D homomorphic signal processing, magnetic resonance imaging (MRI) [3], optical imaging, [6], estimation of shape from texture [10], [11], and interferometric synthetic aperture radar (INSAR) [4], [5], where the signal phase is proportional to the elevation of the scattering point on the ground. Hence, ground elevations and terrain maps can be produced from the INSAR data. See also [2] and [9] and the references therein.

A critical problem in analyzing the phase information is the need to unwrap the phase of the observed 2-D signal to enable a meaningful interpretation of the data. Ideally, in the absence of noise and phase aliasing, we could unwrap the phase function by following an integration path and adding multiples of 2π to the phase whenever a sudden drop from π to $-\pi$ occurs. To ensure that no phase-aliasing occurs, the original scene must be properly sampled so that phase differences between two adjacent samples are smaller than π rad. This requirement cannot be generally satisfied, and hence, in the presence of noise and phase aliasing, this simple phase unwrapping method is inadequate. Many of the existing 2-D phase unwrapping techniques involve *local* analysis of the phase image by means of sequential processing of the differences between adjacent pixels (see, e.g., [7]) or by employing edge detection techniques (see, e.g., [5]). Since in those schemes local errors result in global errors, their usefulness in the presence of noise is limited. An alternative, global method for 2-D phase unwrapping is to obtain a least squares estimate of the true

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phase by minimizing the differences between the first-order discrete partial derivatives of the wrapped phase function and those of the unknown unwrapped solution function (see, [8] and the references therein). However, using this method, any error in estimating the phase gradient at the boundaries (for example, due to noise) would influence the results of the entire phase unwrapping procedure. In [9], we proposed a parametric approach for 2-D phase unwrapping. The algorithm employs *global* analysis of the observed signal, and hence, it is insensitive to local errors. In the proposed model, the phase and amplitude are unrelated. This 2-D phase unwrapping algorithm is based on the phase differencing (PD) algorithm [1].

The PD algorithm extends the high-order ambiguity function (HAF) based estimation algorithm (see [12] for a detailed exposition as well as [13] for a ML estimator) to the case of 2-D signals. It is therefore suggested that the reader be familiar with the derivations in [12]. Assuming the phase of the observed field is a continuous function of the field coordinates so that it can be approximated by a 2-D polynomial function of these coordinates, the PD algorithm provides estimates of all the phase parameters and, thus, of the phase function itself. It is therefore an efficient tool for instantaneous spatial frequency (IF) estimation, as well. In its initial step, the phase-unwrapping algorithm fits a 2-D polynomial model to the phase of the observed signal. Note that the algorithm attempts to fit a 2-D polynomial phase model to the data itself and is not at all concerned with the wrapped phase image as some of the existing phase unwrapping techniques. Since the model inherently assumes the phase to be a smooth function of the coordinates, it is not concerned with the 2π ambiguities of the phase function. In this method, the phase function model can be estimated, even for low SNR and phase aliasing scenarios in which the local edge detection-based algorithms are clearly not effective. In the unwrapping step, the estimated phase is used as a reference information, which directs the actual phase unwrapping process. The phase of each sample of the observed field is unwrapped by increasing (decreasing) it by the multiple of 2π that is the nearest to the difference between the principle value of the phase and the estimated phase value at this coordinate.

In [11], we derive an algorithm for estimating the orientation in space of a planar surface from its texture information. The algorithm employs the fact that the perspective projection transforms the phase function of any sinusoidal component of the homogeneous surface texture from a linear function of the surface coordinates to a nonlinear, yet continuous, function of the image coordinates. Thus, the first step of the algorithm is to obtain an estimate of the phase of the dominant sinusoidal component in the image plane using the PD algorithm. By substituting the estimated phase into the equation that relates it, through the physical model of the perspective projection, with the phase function on the homogeneous surface, we obtain highly accurate estimates of the unknown tilt and slant angles of the surface.

Next, we introduce some definitions and notations. Let

$$\begin{aligned} &y(n,m) = v(n,m) + w(n,m), \\ &n = 0,1,\ldots,N-1, \quad m = 0,1,\ldots,M-1 \end{aligned} \quad (1)$$

be the observed field, where $\{w(n,m)\}$ is an additive white Gaussian noise with variance σ^2, $v(n,m) = A\exp\{j\phi_{S+1}(n,m)\}$, and

$$\phi_{S+1}(n,m) = \sum_{\{0\le k,\ell:0\le k+\ell\le S+1\}} c(k,\ell)n^k m^\ell. \quad (2)$$

Applying the phase differencing operator $\text{PD}_{n^{(P)},m^{(S-P)}}[\cdot]$ (see [1] and [2]) to $v(n,m)$ results in a 2-D exponential whose spatial frequency (ω_S, ν_s) is given by

$$\omega_S = (-1)^S c(P+1, S-P)(P+1)!(S-P)!\tau_n^P \tau_m^{S-P} \quad (3)$$

$$\nu_S = (-1)^S c(P, S+1-P)P!(S+1-P)!\tau_n^P \tau_m^{S-P}. \quad (4)$$

Hence, estimating (ω_S, ν_S) using any standard frequency estimation technique results in an estimate of $c(P+1, S-P)$ and $c(P, S+1-P)$. Based on this result, all the parameters of the polynomial phase signal are estimated, as explained in [1].

Note from (3) and (4) that since the phase coefficients are estimated form the exponential's frequency, they can be estimated unambiguously (i.e., with no aliasing) as long as $|\omega_S| \le \pi$ and $|\nu_S| \le \pi$, which implies that

$$|c(P+1, S-P)| \le \frac{\pi}{(P+1)!(S-P)!\tau_n^P \tau_m^{S-P}} \quad (5)$$

and similarly for $c(P, S+1-P)$. However, since a parametric model is fitted to the observed signal, the phase function itself can be sampled *under* the Nyquist rate because the phase estimation is not performed through a waveform-based procedure. Therefore, phase differences between adjacent samples may be greater than π rad without affecting the ability of the algorithm to estimate the phase parameters, as long as the constraint (5) is satisfied. In other words, the proposed phase estimation algorithm can perform very well in the presence of phase aliasing due to a low sampling rate and noise. This point is further illustrated in Section III as well as in [9].

The performance of the phase estimation algorithm and, hence, the performance of any procedure that employs it, such as the phase unwrapping procedure [9] or the tilt and slant estimation [11], strongly depends on the choice of the two free parameters of the algorithm. In this correspondence, we systematically analyze the performance of the phase estimation algorithm and derive selection rules for the algorithm parameters such that the mean squared error (MSE) in estimating the signal phase is minimized. From (5), it is concluded that each selection of the algorithm parameters τ_n and τ_m has implications on the range of values the polynomial phase coefficients may assume.

II. The Optimal Selection of τ_n and τ_m

In Appendix A, we obtain approximate expressions for the mean squared error in estimating the parameters of the highest layer of the phase model when the observed field is given by (1). Since these expressions are difficult to evaluate, we restrict our attention in Appendix B to the case where the SNR is high. This assumption allows us to derive closed-form expressions for the mean squared errors. Our goal in this section is to analyze these expressions and to find the values of the algorithm parameters τ_n and τ_m that minimize the error variance of the phase estimates. Through numerical evaluation of the mean squared error expressions for polynomial phase signals of different total degrees, we determine the optimal values of τ_n and τ_m such that the mean squared error is minimized.

Let SNR $= \frac{A^2}{\sigma^2}$. In Appendix B, it is shown that under the high SNR assumption, $E\{\Delta c(P+1, S-P)\} \approx 0$, and $E\{\Delta c(P, S+1-P)\} \approx 0$, i.e., the estimates of the coefficients in the $S+1$ layer of a polynomial phase signal of total degree $S+1$ are unbiased for high SNR. We note that the expressions derived in Appendices A and B are valid only for the coefficients in the $S+1$ layer of a polynomial phase signal of total degree $S+1$. Complete performance analysis of the estimation algorithm is beyond the scope of the present work since the errors in estimating the parameters of lower layers are, in part, due to the propagation of errors in estimating the coefficients of higher layers. However, as we show next, this derivation is very useful for the purpose of choosing the optimal set of the algorithm parameters: τ_n and τ_m.

To illustrate the behavior of the MSE in estimating the phase parameters for high SNR, we present a typical example. Consider a constant amplitude polynomial phase signal of total degree

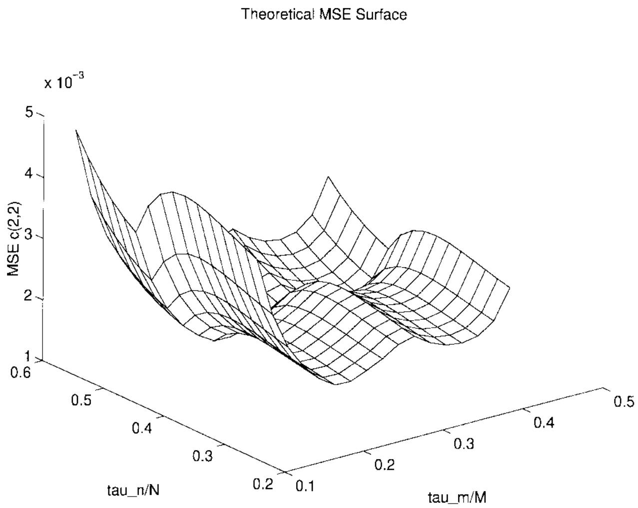

Fig. 1. MSE of $\hat{c}(2, 2)$ as a function of the ratios τ_n/N, and τ_m/N.

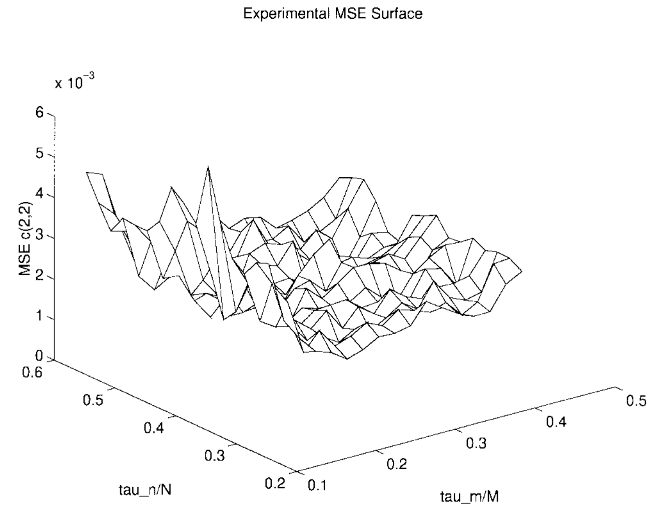

Fig. 2. Experimentally computed MSE of $\hat{c}(2, 2)$ as a function of the ratios τ_n/N, and τ_m/M.

4 in additive white noise such that the SNR is 25 dB. The field dimensions are $N = M = 50$. The phase parameters are $\mathbf{c} = [1; 0.5, 0.64; -0.008, 0.002, -0.0048; 2.4 \cdot 10^{-5}, 0, 1.6 \cdot 10^{-5}, 1.6 \cdot 10^{-5}; 0, 0, 8 \cdot 10^{-7}, 0, 0]^T$, where $\mathbf{c}$ is defined by $\mathbf{c} = [c(0,0); c(0,1), c(1,0); c(0,2), c(1,1), c(2,0); \ldots, \ldots; c(0, S+1), \ldots, c(S+1, 0)]^T$. [Thus, for example, $c(2,2) = 8 \cdot 10^{-7}$]. Using (17) and (18), we depict in Fig. 1 the MSE surface of $\hat{c}(2,2)$ (normalized with respect to $c^2(2,2)$) as a function of the ratios τ_n/N and τ_m/M. Note that the MSE surface is rather "flat" in the vicinity of its global minimum point. To verify the validly of the theoretical analysis, we computed the MSE in estimating the phase coefficients using Monte Carlo simulations for some specific examples for a wide range of τ_n and τ_m values. The results match well the theoretical results. For example, the experimentally evaluated normalized MSE surface of $\hat{c}(2,2)$ of the foregoing example is depicted in Fig. 2. The results are based on 100 independent realizations of the signal for each τ_n and τ_m.

Considering the simulation results for polynomial phase signals of total degree 2 to total degree 5, we find that the global minimum of the MSE in estimating the parameters $c(P+1, S-P)$ and $c(P, S+1-P)$, i.e., the parameters in the $S+1$ layer of the polynomial phase model, occur for the ratios of τ_n/N, which are given in Table I. In this table, we use the notation ϕ to denote that any value of τ_n, such that $0 \leq \tau_n \leq N-1$ can be chosen. Similar results are obtained for the ratio τ_m/M. These are summarized in Table II. We therefore conclude that the selection of an optimal τ_n is not a function of the optimal selection of τ_m, and vice versa.

III. Approximate Analysis of the MSE for an Arbitrary SNR

In the previous section, we concluded that for high SNR, the MSE surfaces are essentially flat in the vicinity of the minimum point. On the basis of this conclusion, we replace in this section the optimal τ_n and τ_m with the close-to-optimal choice $\tau_n = \frac{N}{P+1}$, and $\tau_m = \frac{M}{S-P+1}$. This choice of τ_n and τ_m enables us to analyze the performance of the phase estimation algorithm for an *arbitrary* SNR and to derive simple expressions of the MSE in estimating the model parameters and functions thereof. The details of the derivation are

TABLE I
Optimal τ_n/N Ratios That Minimize the MSE

	$P=0$	$P=1$	$P=2$	$P=3$	$P=4$
$c(P+1, S-P)$	ϕ	$\frac{1}{P+1}$	$\frac{1}{P+3}$	$\frac{1}{P+3}$	$\frac{1}{P+3}$
$c(P, S+1-P)$	ϕ	$\frac{1}{P+2}$	$\frac{1}{P+2}$	$\frac{1}{P+2}$	$\frac{1}{P+2}$

presented in Appendix C. Using these derivations, we conclude that for $\tau_n = \frac{N}{P+1}$ and $\tau_m = \frac{M}{S-P+1}$, the MSE in estimating a parameter in the highest layer of the phase model is given by (6), shown at the bottom of the page, where $\mathcal{C}(P, S, \text{SNR})$ is a function of P, S, and the SNR only, which is given by

$$\mathcal{C}(P, S, \text{SNR}) \triangleq \prod_{q=0}^{S-P} \prod_{p=0}^{P} \sum_{i=0}^{\binom{P}{p}\binom{S-P}{q}} \left(\begin{array}{c} \binom{P}{p}\binom{S-P}{q} \\ i \end{array} \right)^2 i! \left(\frac{1}{\text{SNR}} \right)^i - 1. \tag{7}$$

For the case in which $N \gg P$, we obtain

$$E\{[\Delta c(P+1, S-P)]^2\} \approx \frac{6\mathcal{C}(P, S, \text{SNR})}{[(P+1)!(S-P)!]^2 \left(\frac{N}{P+1}\right)^{2P+3} \left(\frac{M}{S-P+1}\right)^{2S-2P+1}}. \tag{8}$$

Similar MSE expressions can be derived for $E\{[\Delta c(P, S+1-P)]^2\}$.

To further illustrate the effectiveness of the proposed parameter selection rule, we consider the 30 × 30 polynomial phase signal of total-degree 2, whose noise-free phase function, as well as the observed phase, are shown in Fig. 3. In this example, the SNR of the observed signal is very low (SNR = 0 dB), and the sampling rate is low as well; therefore, the phase function is severely aliased. The phase parameter vector is $\mathbf{c} = [0; 0.2513, 0.2513; -0.1, -0.15, -0.1]^T$. Using Monte Carlo simulations, we investigate the behavior of the variance, bias, and MSE of the phase estimate for two distinct sets of τ_n and τ_m. Since, in many cases, the local spatial frequency of the signal is of interest, we investigate the performance of its estimate as well. The Monte Carlo simulations are based on 200 independent realizations of the observed field. In one case, which is denoted by solid lines

$$E\{[\Delta c(P+1, S-P)]^2\} \approx \frac{6\mathcal{C}(P, S, \text{SNR})(P+1)^2(N^2 - (P+1)^2)}{((P+1)!(S-P)!)^2 \left(\frac{N}{P+1}\right)^{2P+1} \left(\frac{M}{S-P+1}\right)^{2S-2P+1} (N-P+1)^2(N+P+1)^2} \tag{6}$$

TABLE II
OPTIMAL τ_m/M RATIOS THAT MINIMIZE THE MSE

	$S-P=0$	$S-P=1$	$S-P=2$	$S-P=3$	$S-P=4$
$c(P+1, S-P)$	ϕ	$\frac{1}{S-P+2}$	$\frac{1}{S-P+2}$	$\frac{1}{S-P+2}$	$\frac{1}{S-P+2}$
$c(P, S+1-P)$	ϕ	$\frac{1}{S-P+1}$	$\frac{1}{S-P+3}$	$\frac{1}{S-P+3}$	$\frac{1}{S-P+3}$

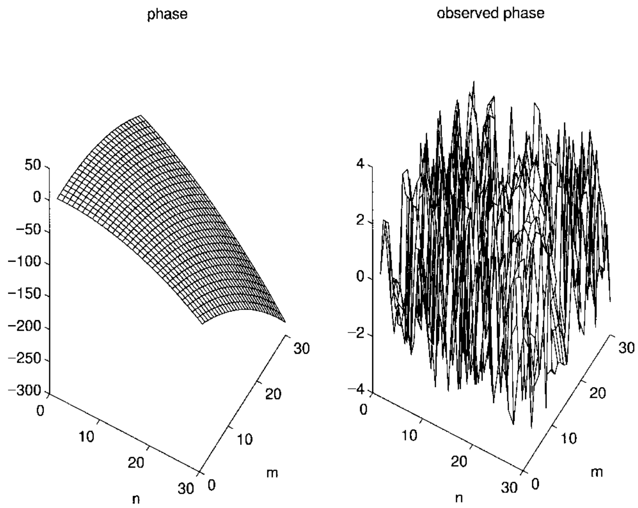

Fig. 3. Observed and noise-free phase functions of the observed signal.

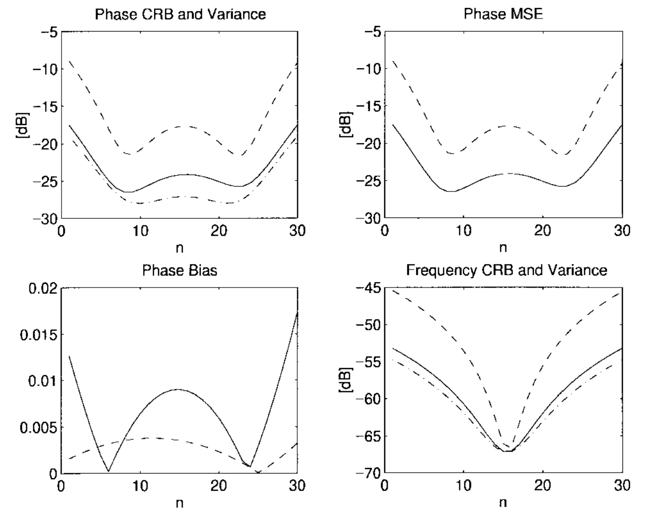

Fig. 4. Variance, bias, and MSE of the phase and frequency estimates for optimal and nonoptimal choice of the algorithm parameters at SNR $= 0$ dB and an undersampled phase.

in Fig. 4, we choose $\tau_n = \frac{N}{P+1}$ and $\tau_m = \frac{M}{S-P+1}$. Note that these values of τ_n and τ_m change when different parameters in different layers are estimated. By adopting this selection rule, we implicitly assume that there is no error propagation due to errors in estimating the coefficients of higher layers. In the second case, which is denoted by dashed lines in Fig. 4, we set $\tau_n = \tau_m = 2$ for all the algorithm iterations. In Fig. 4, we show the variance, the bias, and the MSE of the phase function estimate along a diagonal cross-section of the field from $(0, 0)$ to $(N-1, M-1)$. The error variance of the n-axis frequency estimate along the same diagonal cross section is depicted as well. The experimental results indicate that indeed, as suggested by the theoretical analysis, this choice of the algorithm parameters provides an essentially unbiased estimate of both the phase and the frequency functions. Both for the phase and the frequency estimates, the bias is considerably smaller than the standard deviation of the estimates, and hence, the MSE and the error variance curves are essentially identical.

In [2], we derived the Cramér–Rao lower bound (CRB) on the parameter estimates of the model defined in (1) and (2). The CRB provides the lower bound on the error variance in estimating the phase parameters for any unbiased estimator of the phase model. Since the phase and frequency estimates were found to be unbiased, the variance of these estimates can be compared with the corresponding CRB. The CRB on the phase and frequency estimates is depicted in Fig. 4 using dashed-dotted lines. The numerical results demonstrate that the nearly optimal selection rule of τ_n and τ_m yields estimation error variance that is very close to the bound, both for the phase and the frequency estimates, despite the severe phase aliasing and the small dimensions of the observed field. However, the arbitrary choice of the algorithm parameters $\tau_n = \tau_m = 2$ produces estimates with a much higher bias, variance, and MSE.

IV. CONCLUSION

In [9], a parametric modeling approach is proposed as the basic building block in an algorithm for unwrapping the phase of 2-D signals. In [11], we derive an algorithm for estimating the orientation in space of a planar surface from its texture information. The algorithms derived for both problems employ the PD algorithm [1], [2] to estimate the phase function of the observed signal. In this correspondence, we provide a performance analysis of the phase estimation algorithm. This analysis is essential since it provides guidelines for optimal selection of the algorithm parameters τ_n and τ_m. The parameter selection has a strong influence on the PD algorithm performance and, hence, on the performance of any procedure that employs it. Comparison of the theoretical performance of the algorithm with the CRB verifies the experimental observation that it provides accurate estimates at a relatively low computational cost.

We finally note that in order to satisfy the constraint (5), it may be necessary to use nonoptimal choices of the algorithm parameters when the observed data is small in dimensions to allow a wider range of phase parameters to be accommodated.

APPENDIX A
STATISTICAL ANALYSIS OF THE HIGHEST LAYER COEFFICIENTS

Define

$$y^{(*^{(p+q)})}(n + p\tau_n, m + q\tau_m) = \begin{cases} y(n + p\tau_n, m + q\tau_m), & p+q \text{ even} \\ y^*(n + p\tau_n, m + q\tau_m), & p+q \text{ odd} \end{cases}. \tag{9}$$

Let $g_{N,M}(\omega, \nu)$ be a 2-D complex valued function depending on two real variables ω and ν and on two positive integers N and M. Here, ω and ν are the 2-D frequency domain variables, whereas N and M are the observed field dimensions. Define $f_{N,M}(\omega, \nu) = |g_{N,M}(\omega, \nu)|^2$ and assume that $f_{N,M}(\omega, \nu)$ has its global maximum at $(\omega, \nu) = (\omega_S, \nu_S)$. More specifically

$$\begin{aligned} g_{N,M}(\omega, \nu) &= \mathrm{DFT}\{\mathrm{PD}_{n^{(P)}, m^{(S-P)}}[v(n, m)]\} \\ &= \exp\{j\gamma_S(\tau_n, \tau_m)\} \sum_{n=0}^{N-1-P\tau_n} \sum_{m=0}^{M-1-(S-P)\tau_m} \\ &\quad \times \exp\{j[(\omega_S - \omega)n + (\nu_S - \nu)m)\} \end{aligned} \tag{10}$$

where the last equality follows from [1, Th. 1]. Using [2, Lemma 2], we have that the perturbation of $g_{N,M}(\omega,\nu)$ from its true value due to the additive noise is given by

$$\begin{aligned}
&\Delta g_{N,M}(\omega,\nu)\\
&\quad= \mathrm{DFT}\{\mathrm{PD}_{n^{(P)},m^{(S-P)}}[y(n,m)]\}\\
&\quad\quad - \mathrm{DFT}\{\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]\}\\
&\quad= \sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\Bigg\{\prod_{q=0}^{S-P}\Bigg\{\prod_{p=0}^{P}\{y^{(*^{(p+q)})}\\
&\quad\quad\times (n+p\tau_n, m+q\tau_m)\}^{\binom{P}{p}}\Bigg\}^{\binom{S-P}{q}}\\
&\quad\quad - \prod_{q=0}^{S-P}\Bigg\{\prod_{p=0}^{P}\{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)\}^{\binom{P}{p}}\Bigg\}^{\binom{S-P}{q}}\Bigg\}\\
&\quad\quad \times \exp\{-j(\omega n+\nu m)\}\\
&= \sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\eta(n,m)\\
&\quad \times \exp\{j[(\omega_S-\omega)n+(\nu_S-\nu)m)\}\cdot\exp\{j\gamma_S(\tau_n,\tau_m)\} \qquad (11)
\end{aligned}$$

where we define

$$\begin{aligned}
\eta(n,m) = \prod_{q=0}^{S-P}\Bigg\{\prod_{p=0}^{P}\Bigg\{1\\
+\frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}\Bigg\}^{\binom{P}{p}}\Bigg\}^{\binom{S-P}{q}} - 1.
\end{aligned} \qquad (12)$$

Let $K=(N-P\tau_n)[M-(S-P)\tau_m]$. In addition, let $\mathbf{1}(k)$ denote the unit step function. Evaluating $g_{N,M}(\omega,\nu)$ and $\Delta g_{N,M}(\omega_S,\nu_S)$ and their partial derivatives with respect to ω and ν at (ω_S,ν_S), we have for the second derivatives of $f_{N,M}(\omega,\nu)$ that $\frac{\partial^2 f_{N,M}(\omega_S,\nu_S)}{\partial\omega\partial\nu}=0$ and that

$$\begin{aligned}
\frac{\partial^2 f_{N,M}(\omega,\nu)s}{\partial\omega^2} &= 2K^2\Bigg\{\frac{1}{4}(N-P\tau_n-1)^2\\
&\quad - \frac{1}{3}(N-P\tau_n-1)(N-P\tau_n-0.5)\Bigg\}\\
&\quad \cdot\mathbf{1}(N-1-P\tau_n)\mathbf{1}(M-1-(S-P)\tau_m)
\end{aligned} \qquad (13)$$

$$\begin{aligned}
\frac{\partial^2 f_{N,M}(\omega_S,\nu_S)}{\partial\nu^2} &= 2K^2\Bigg\{\frac{1}{4}[M-(S-P)\tau_m-1]^2\\
&\quad - \frac{1}{3}[M-(S-P)\tau_m-1]\\
&\quad \cdot[M-(S-P)\tau_m-0.5]\Bigg\}\mathbf{1}(N-1-P\tau_n)\\
&\quad \times\mathbf{1}(M-1-(S-P)\tau_m).
\end{aligned} \qquad (14)$$

Hence, the Hessian matrix is diagonal, and therefore, the estimation of $c(P+1,S-P)$ via ω_S is approximately decoupled from the estimation of $c(P,S+1-P)$ via ν_S. In other words, we have the following set of independent equations:

$$\Delta\omega \approx -\left[\frac{\partial^2 f_{N,M}(\omega_S,\nu_S)}{\partial\omega^2}\right]^{-1}\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\omega} \qquad (15)$$

$$\Delta\nu \approx -\left[\frac{\partial^2 f_{N,M}(\omega_S,\nu_S)}{\partial\nu^2}\right]^{-1}\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\nu}. \qquad (16)$$

The mean squared value of $\Delta c(P+1,S-P)$ due to the noise is given by

$$\begin{aligned}
&E\{[\Delta c(P+1,S-P)]^2\}\\
&\quad\approx \frac{1}{\left((P+1)!(S-P)!\tau_n^P\tau_m^{S-P}\right)^2}\left[\frac{\partial^2 f_{N,M}(\omega_S,\nu_S)}{\partial\omega^2}\right]^{-2}\\
&\quad\quad\times E\left\{\left[\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\omega}\right]^2\right\}
\end{aligned} \qquad (17)$$

and the mean squared value of $\Delta c(P,S+1-P)$ due to the noise is given by

$$\begin{aligned}
&E\{[\Delta c(P,S+1-P)]^2\}\\
&\quad\approx \frac{1}{\left(P!(S+1-P)!\tau_n^P\tau_m^{S-P}\right)^2}\left[\frac{\partial^2 f_{N,M}(\omega_S,\nu_S)}{\partial\nu^2}\right]^{-2}\\
&\quad\quad\times E\left\{\left[\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\nu}\right]^2\right\}
\end{aligned} \qquad (18)$$

where

$$\begin{aligned}
&E\left\{\left[\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\omega}\right]^2\right\}\\
&\quad= -K^2 E\Bigg\{\Bigg[\sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\left(\frac{N-P\tau_n-1}{2}-n\right)\\
&\quad\quad\times(\eta^*(n,m)c-\eta(n,m))\Bigg]\\
&\quad\quad\cdot\Bigg[\sum_{\ell=0}^{N-1-P\tau_n}\sum_{k=0}^{M-1-(S-P)\tau_m}\left(\frac{N-P\tau_n-1}{2}-\ell\right)\\
&\quad\quad\times(\eta^*(\ell,k)-\eta(\ell,k))\Bigg]\Bigg\}\\
&\quad= -K^2\sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\sum_{\ell=0}^{N-1-P\tau_n}\sum_{k=0}^{M-1-(S-P)\tau_m}\\
&\quad\quad\times\left(\frac{N-P\tau_n-1}{2}-n\right)\left(\frac{N-P\tau_n-1}{2}-\ell\right)\\
&\quad\quad\cdot\{E[\eta^*(n,m)\eta^*(\ell,k)]-2\,\mathrm{Re}\{E[\eta^*(n,m)\eta(\ell,k)]\}\\
&\quad\quad+E[\eta(n,m)\eta(\ell,k)]\}
\end{aligned} \qquad (19)$$

and

$$\begin{aligned}
&E\left\{\left[\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\nu}\right]^2\right\}\\
&\quad= -K^2\sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\sum_{\ell=0}^{N-1-P\tau_n}\sum_{k=0}^{M-1-(S-P)\tau_m}\\
&\quad\quad\times\left(\frac{M-(S-P)\tau_m-1}{2}-m\right)\\
&\quad\quad\times\left(\frac{M-(S-P)\tau_m-1}{2}-k\right)\\
&\quad\quad\cdot\{E[\eta^*(n,m)\eta^*(\ell,k)]-2\,\mathrm{Re}\{E[\eta^*(n,m)\eta(\ell,k)]\}\\
&\quad\quad+E[\eta(n,m)\eta(\ell,k)]\}.
\end{aligned} \qquad (20)$$

APPENDIX B
THE HIGH SNR CASE

Rewriting (12), we have

$$\begin{aligned}\eta(n,m) &= \prod_{q=0}^{S-P}\left\{\prod_{p=0}^{P}\left\{1 + \frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}\right\}^{\binom{P}{p}}\right\}^{\binom{S-P}{q}} - 1 \\ &= \prod_{q=0}^{S-P}\prod_{p=0}^{P}\sum_{i=0}^{\binom{P}{p}\binom{S-P}{q}}\binom{\binom{P}{p}\binom{S-P}{q}}{i} \\ &\quad\times\left[\frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}\right]^i - 1 \\ &\approx \sum_{q=0}^{S-P}\sum_{p=0}^{P}\binom{P}{p}\binom{S-P}{q} \\ &\quad\times\frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)} \end{aligned} \tag{21}$$

where the second equality results from expanding the first expression into a series form, whereas the above first-order approximation is valid as long as the noise variance is small relative to the energy of $v(n,m)$, and hence, all the high powers of $\frac{w^{(*^{(p+q)})}(n+p\tau_n,m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n,m+q\tau_m)}$, can be neglected.

Since $E\{\frac{w^{(*^{(p+q)})}(n+p\tau_n,m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n,m+q\tau_m)}\} = 0$, we have substituting (10)–(14) into (15) and (16) that under the high SNR assumption $E\{\Delta c(P+1, S-P)\} \approx 0$ and $E\{\Delta c(P, S+1-P)\} \approx 0$, i.e., the estimates of $c(P+1, S-P)$ and $c(P, S+1-P)$ are unbiased for high SNR. Next, we give a first-order approximation of the MSE (and variance) of the estimated parameters, i.e., we wish to evaluate $E\{[\Delta c(P+1, S-P)]^2\}$ and $E\{[\Delta c(P, S+1-P)]^2\}$. Using the first-order approximation of $\eta(n,m)$ (21), we have (22), shown at the bottom of the page. Since $w(n,m)$ is a complex valued, circular Gaussian, white noise

$$E\big[w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)w^{(*^{(s+t)})}(\ell+s\tau_n, k+t\tau_m)\big] = \begin{cases}\sigma^2, & n+p\tau_n = \ell+s\tau_n, \text{ and} \\ & m+q\tau_m = k+t\tau_m, \text{ and} \\ & |p+q-s-t| \text{ odd} \\ 0, & \text{otherwise}\end{cases}. \tag{23}$$

After some algebraic manipulations, we find that

$$\begin{aligned}&E\left\{\left[\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\omega}\right]^2\right\} \\ &\approx \frac{1}{6}\sigma^2K^2\cdot\sum_{r=-(S-P)}^{S-P}(-1)^r\binom{2S-2P}{S-P-|r|} \\ &\quad\times[M-(S-P+|r|)\tau_m] \\ &\quad\times\mathbf{1}(M-1-(S-P)\tau_m-|r|\tau_m) \\ &\quad\cdot\sum_{z=-P}^{P}(-1)^z\binom{2P}{P-|z|}(N-P\tau_n-|z|\tau_n) \\ &\quad\cdot[(N-P\tau_n)^2-2(N-P\tau_n)|z|\tau_n-2(z\tau_n)^2-1] \\ &\quad\times\mathbf{1}(N-1-P\tau_n-|z|\tau_n).\end{aligned} \tag{24}$$

Substitution of (24) and (13) into (17) gives the desired variance $E\{[\Delta c(P+1, S-P)]^2\}$. Following a similar procedure

$$\begin{aligned}&E\left\{\left[\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\nu}\right]^2\right\} \\ &\approx \frac{1}{6}\sigma^2K^2\cdot\sum_{r=-P}^{P}(-1)^r\binom{2P}{P-|r|} \\ &\quad\times[N-(P+|r|)\tau_n]\mathbf{1}(N-1-P\tau_n-|r|\tau_n) \\ &\quad\cdot\sum_{z=-(S-P)}^{S-P}(-1)^z\binom{2S-2P}{S-P-|z|} \\ &\quad\times(M-(S-P)\tau_m-|z|\tau_m) \\ &\quad\cdot[(M-(S-P)\tau_m)^2-2(M-(S-P)\tau_m)|z|\tau_m \\ &\quad-2(z\tau_m)^2-1]\cdot\mathbf{1}(M-1-(S-P)\tau_m-|z|\tau_m).\end{aligned} \tag{25}$$

Substitution of this expression and (14) into (18) gives the desired variance $E\{[\Delta c(P, S+1-P)]^2\}$. Note that for the high SNR case, both expressions in (17) and (18) are independent of the signal parameters.

APPENDIX C
DETAILS OF THE APPROXIMATE ANALYSIS OF THE MSE FOR AN ARBITRARY SNR

Using the choice of $\tau_n/N = \frac{1}{P+1}$ and $\tau_m/M = \frac{1}{S-P+1}$, we wish to analyze the performance of the phase estimation algorithm for an *arbitrary* SNR. This choice of τ_n and τ_m has the advantage of grouping the field samples into subgroups in such a way that the subgroups are all mutually exclusive, whereas each contains uniformly spaced samples from the entire observed field. Thus, expectations of the type

$$\begin{aligned}E\left\{\left[\frac{\partial\Delta f_{N,M}(\omega_S,\nu_S)}{\partial\omega}\right]^2\right\} &\approx -K^2\sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\sum_{\ell=0}^{N-1-P\tau_n}\sum_{k=0}^{M-1-(S-P)\tau_m} \\ &\quad\times\left(\frac{N-P\tau_n-1}{2}-n\right)\left(\frac{N-P\tau_n-1}{2}-\ell\right)\sum_{q=0}^{S-P}\sum_{p=0}^{P}\sum_{t=0}^{S-P}\sum_{s=0}^{P}\binom{P}{p}\binom{P}{s}\binom{S-P}{q}\binom{S-P}{t} \\ &\quad\times\left\{E\left[\frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}\frac{w^{(*^{(s+t)})}(\ell+s\tau_n, k+t\tau_m)}{v^{(*^{(s+t)})}(\ell+s\tau_n, k+t\tau_m)}\right]\right. \\ &\quad-2\,\mathrm{Re}\left\{E\left[\frac{w^{(*^{(p+q+1)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q+1)})}(n+p\tau_n, m+q\tau_m)}\frac{w^{(*^{(s+t)})}(\ell+s\tau_n, k+t\tau_m)}{v^{(*^{(s+t)})}(\ell+s\tau_n, k+t\tau_m)}\right]\right\} \\ &\quad\left.+E\left[\frac{w^{(*^{(p+q+1)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q+1)})}(n+p\tau_n, m+q\tau_m)}\frac{w^{(*^{(s+t+1)})}(\ell+s\tau_n, k+t\tau_m)}{v^{(*^{(s+t+1)})}(\ell+s\tau_n, k+t\tau_m)}\right]\right\}.\end{aligned} \tag{22}$$

$E\{\eta(n,m)\eta(\ell,k)\}$, where $\eta(n,m)$ is defined in (12), can be easily computed. This is due to the property that $\eta(n,m)$ is a function of only one subgroup of field samples, whereas $\eta(\ell,k)$ is a function of a different and mutually exclusive subgroup of field samples. Because the noise is white and circular Gaussian, the expected values are zero, unless $(\ell,k)=(n,m)$.

More specifically, since $\{w(n,m)\}$ is a circular white Gaussian noise, $E\{[w(n,m)]^r\}=0$ for any positive integer r, whereas the expected values of cross products that involve more than one sample are also zero. Using (12), we find that

$$\begin{aligned} E[\eta(n,m)] = E\Bigg\{\prod_{q=0}^{S-P}\prod_{p=0}^{P}\Bigg\{1 &+ \frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}\Bigg\}^{\binom{P}{p}\binom{S-P}{q}} - 1\Bigg\} \\ &= 0. \end{aligned} \tag{26}$$

Hence, we have, using (15) and (16), that $E\{\Delta c(P+1,S-P)\}\approx 0$, and $E\{\Delta c(P,S+1-P)\}\approx 0$, i.e., for the case of a circular white Gaussian observation noise the estimates of $c(P+1,S-P)$ and $c(P,S+1-P)$ are unbiased for *any* SNR.

Finally, we wish to evaluate (17) and (18) when we choose $\tau_n/N=\frac{1}{P+1}$, and $\tau_m/M=\frac{1}{S-P+1}$. We begin by evaluating (19) and (20).

$$\begin{aligned} &E\left\{\left[\frac{\partial \Delta f_{N,M}(\omega_S,\nu_S)}{\partial \omega}\right]^2\right\} \\ &\quad = -K^2 \sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\left(\frac{N-P\tau_n-1}{2}-n\right)^2 \\ &\qquad \cdot\{E[\eta^*(n,m)]^2 - 2E[\eta^*(n,m)\eta(n,m)] + E[\eta(n,m)]^2\} \end{aligned} \tag{27}$$

$$\begin{aligned} &E\left\{\left[\frac{\partial \Delta f_{N,M}(\omega_S,\nu_S)}{\partial \nu}\right]^2\right\} \\ &\quad = -K^2 \sum_{n=0}^{N-1-P\tau_n}\sum_{m=0}^{M-1-(S-P)\tau_m}\left(\frac{M-(S-P)\tau_m-1}{2}-m\right)^2 \\ &\qquad \cdot\{E[\eta^*(n,m)]^2 - 2E[\eta^*(n,m)\eta(n,m)] + E[\eta(n,m)]^2\}. \end{aligned} \tag{28}$$

Since $\{w(n,m)\}$ is a circular white Gaussian noise, we have using (12) and (21) that $E[\eta(n,m)]^2=0$, $E[\eta^*(n,m)]^2=0$ and that

$$\begin{aligned} E\Bigg\{\prod_{q=0}^{S-P}\prod_{p=0}^{P}\sum_{i=0}^{\binom{P}{p}\binom{S-P}{q}}\binom{\binom{P}{p}\binom{S-P}{q}}{i} \\ \times\left[\frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}\right]^i\Bigg\} = 1. \end{aligned} \tag{29}$$

In addition, $E\{[w(n,m)]^r[w^*(n,m)]^s\}=0$ for any non-negative distinct integers r and s, whereas $E\{[w(n,m)]^r[w^*(n,m)]^r\}=r!(\sigma^2)^r$ for any nonnegative integer r, (e.g., [12]). Therefore, using the first equality in (21)

$$\begin{aligned} &E[\eta(n,m)\eta^*(n,m)] \\ &= \prod_{q=0}^{S-P}\prod_{p=0}^{P}\prod_{t=0}^{S-P}\prod_{s=0}^{P}\sum_{i=0}^{\binom{P}{p}\binom{S-P}{q}}\sum_{j=0}^{\binom{P}{s}\binom{S-P}{t}} \\ &\quad\times\binom{\binom{P}{p}\binom{S-P}{q}}{i}\binom{\binom{P}{s}\binom{S-P}{t}}{j} \\ &\quad\cdot E\Bigg\{\left[\frac{w^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}{v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)}\right]^i \\ &\quad\times\left[\frac{w^{(*^{(s+t+1)})}(n+s\tau_n, m+t\tau_m)}{v^{(*^{(s+t+1)})}(n+s\tau_n, m+t\tau_m)}\right]^j\Bigg\} - 1 - 1 + 1 \\ &= \prod_{q=0}^{S-P}\prod_{p=0}^{P}\sum_{i=0}^{\binom{P}{p}\binom{S-P}{q}}\binom{\binom{P}{p}\binom{S-P}{q}}{i}^2 i! \\ &\quad\times\left(\frac{\sigma^2}{|v(n+p\tau_n, m+q\tau_m)|^2}\right)^i - 1 \\ &= \prod_{q=0}^{S-P}\prod_{p=0}^{P}\sum_{i=0}^{\binom{P}{p}\binom{S-P}{q}}\binom{\binom{P}{p}\binom{S-P}{q}}{i}^2 i!\left(\frac{1}{\mathrm{SNR}}\right)^i - 1 \end{aligned} \tag{30}$$

where the second equality is due to (30) and due to $\{w(n,m)\}$ being a white noise field.

Since for $\tau_m=\frac{M}{S-P+1}$, $M-(S-P)\tau_m=\frac{M}{S-P+1}$ and for $\tau_n=\frac{N}{P+1}$, $N-P\tau_n=\frac{N}{P+1}$, we have that $\sum_{n=0}^{N-1-P\tau_n}(\frac{N-P\tau_n-1}{2}-n)^2=\frac{1}{12}\frac{N}{P+1}[(\frac{N}{P+1})^2-1]$. Thus, using the equalities $E[\eta(n,m)]^2=0$ and $E[\eta^*(n,m)]^2=0$, we obtain (6) by substituting (30) into (27), followed by substitution of (27) and (13) into (17).

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