Asymptotic Normality of the Sample Mean and Covariances of Evanescent Fields in Noise

Mark Kliger and Joseph M. Francos *

June 8, 2007

Abstract

We consider the asymptotic properties of the sample mean and the sample covariance sequence of a field composed of the sum of a purely-indeterministic and evanescent components. The asymptotic normality of the sample mean and sample covariances is established. A Bartlett-type formula for the asymptotic covariance matrix of the sample covariances of this field, is derived.

Keywords: Homogeneous random fields, evanescent random fields, asymptotic normality.

*M. Kliger is with the Department of Electrical Engineering and Computer Science , University of Michigan, Ann Arbor, MI, 48109-2122, USA. Tel: (734) 647-8389 FAX: (734) 763-8041, email: mkliger@umich.edu.
J. M. Francos is with the Department of Electrical and Computer Engineering, Ben-Gurion University, Beer-Sheva 84105, Israel. Tel: +972 8 6461842, FAX: +972 8 6472949, email: francos@ee.bgu.ac.il

1 Introduction

The problem of linear prediction of homogenous random fields in two or more variables was first introduced rigorously in Helson and Lowdenslager [10]. The problem of defining past and future on the two-dimensional lattice (i.e., $\mathbb{Z}^2$) was defined in [10] in terms of "half plane" total-ordering. Further analysis of the prediction problem led to a generalization of the Wold decomposition [11]. The well known Wold decomposition of stationary complex valued processes indexed by $\mathbb{Z}$ (see Doob [4, p. 576]) contains two *stationary* parts: the purely-indeterministic process (which is producing the innovations) and the deterministic process. This decomposition can be equivalently reformulated using spectral notations: the spectral measure of the purely-indeterministic process is absolutely continuous with respect to the Lebesgue measure, and the spectral measure of the deterministic process is singular (i.e., the spectral measures of these orthogonal components yield the Lebesgue decomposition of the spectral measure of the process). When we consider homogenous random fields indexed by other groups (like those indexed by $\mathbb{Z}^2$) we obtain a Wold decomposition with respect to any given total order on the group. When the group is not $\mathbb{Z}$ (like $\mathbb{R}$ or $\mathbb{Z}^2$) the deterministic process can have as a direct summand a deterministic process of a special type, the *evanescent process*.

Evanescent processes were first introduced in [11] (on $\mathbb{R}$). In Korezlioglu and Loubaton [15], "horizontal" and "vertical" total-orders and the corresponding horizontally and vertically evanescent components of a homogeneous random field on $\mathbb{Z}^2$ are defined. In Kallianpur [13], as well as in Chiang [2], similar techniques are employed to obtain four-fold orthogonal decompositions of regular (non-deterministic) homogeneous random fields. In Francos *et. al.* [5] this decomposition of random fields on $\mathbb{Z}^2$ was further extended. This is done by considering *all* the rational nonsymmetrical half plane linear orders (RNSHP), each inducing a different partitioning of the two-dimensional lattice into two sets by a broken straight line of rational slope. Clearly, there are countably many such linear orders. The Wold decomposition of a regular random field into purely-indeterministic and deterministic components is the same for all RNSHP orders. The decomposition in [5] asserts that we can represent the deterministic component of the field as a mutually orthogonal sum of a "half-plane deterministic" field and a countable number of evanescent fields. The half-plane deterministic field has no innovations, nor column-to-column innovations, with respect to any RNSHP linear order. Each evanescent field spans a Hilbert space identical to the one spanned by its column-to-column innovations, where the column-to-column innovation at each lattice point is defined as the difference between the actual value of the deterministic field and its projection on the Hilbert space spanned by the deterministic field samples in all previous columns. (Clearly, the term "column" is redefined for each definition of the linear order). Each of the evanescent fields can be revealed only by using the corresponding linear order. This decomposition yields a corresponding spectral decomposition, *i.e.*, we can decompose the spectral measure of the deterministic part into a countable sum of mutually singular spectral measures, such that the spectral measure of each evanescent component is concentrated on a line with a rational slope. Based on these results, a parametric model for the evanescent field is derived in [5]. Finally, [3] provides a detailed analysis of the spectral and ergodic properties of evanescent fields.

Evanescent random fields are of great practical importance. Such fields arise quite naturally in problems of texture modeling, estimation, and coding of images (see, *e.g.*, [6] and the references therein), and in space-time adaptive processing of airborne radar data, (see [7] and the references therein). In [7] it is shown that the same parametric model that results from the 2-D Wold-like decomposition naturally arises as the physical model in the problem of space-time processing of airborne radar data: In the space-time domain the target model is that of a 2-D harmonic component. The sum of the white noise field due to the internally generated receiver amplifier noise, and the sky noise contribution, is the purely-indeterministic component of the space-time field decomposition. The presence of a *jammer* (an undesired interference source, transmitting high power noise aimed at "blinding" the radar system) results in a barrage of noise localized in angle, determined by the angle of the jammer with respect to the radar, and uniformly distributed over all Doppler frequencies (since the transmitted noise is white). Hence, in the space-time domain each jammer is modeled as an horizontal evanescent component such that its 1-D modulating process is the random process of the jammer amplitudes. The jammer samples from different pulses are uncorrelated. In the angle-Doppler domain each jammer contributes a 1-D delta function, parallel to the Doppler axis and located at a specific angle. The ground clutter, caused by returns from the ground of the transmitted radar pulse, results in an additional evanescent component of the observed 2-D space-time field.

The correspondence between the model arising from the 2-D Wold like decomposition and the STAP physical model can be exploited to derive computationally efficient fully adaptive and partially adaptive detection algorithms. These detection schemes require the estimation of the noise and interference components of the field, which are then substituted into the expression of the interference-plus-noise covariance matrix. Thus, an estimate of the fully-adaptive weight vector is obtained, and a corresponding test is derived. In order to analyze the performance of such detectors, and in order to determine their operational parameters, analysis of the properties of the sample mean and covariances of the field is an essential building block.

To the best of our knowledge, the problem of analyzing the asymptotic properties of the sample mean and sample covariance sequence of evanescent random fields, or of regular homogenous random fields in general, is an open problem. In this paper we consider the asymptotic properties of the sample mean and sample covariance sequence of a field composed of the sum of evanescent components and a purely-indeterministic component.

There is a fairly rich literature concerning the asymptotic normality of the (weighted) sample mean (in the form of central limit theorems), and sample covariance sequence for 1-D stationary processes. Asymptotic normality of the sample covariances of linear processes was first introduced by Bartlett (see Brockwell and Davis [1], Propositions 7.3.1-7.3.4). The asymptotic normality of the sample covariances for real multivariate time series with continuous spectra was established by Hannan ([8], pp. 209-212). Li *et. al.* [14] proved the asymptotic normality of the sample covariances for a time series with mixed spectra, where the observed mixed-spectrum process is the sum of a stationary process with a continuous spectra, and a finite number of real sinusoids. In, Li [16] completed the generalization of Bartlett's result to the case of complex multivariate time series with mixed spectra. In [9], Ivanov and Leonenko deal with the statistical analysis of random fields. In this framework they also consider the asymptotic normality of various estimators of 2-D models, and in particular the asymptotic normality of the weighted sample mean and sample covariances of homogeneous random fields satisfying constraints on the field moments, and mixing rate. These mixing conditions are not satisfied by random fields having evanescent components, due to the singularity of the spectral measure of the evanescent components (which implies that the field correlation function does not decay fast enough for the mixing conditions to hold).

The asymptotic normality of the sample covariance sequence of a homogenous field composed of only harmonic and purely-indeterministic components with no evanescent components, can be shown based on the results of [14], subject to some restricting assumptions on the properties of the purely-indeterministic component. In this paper we prove, subject to some restricting conditions, a central limit theorem (CLT) establishing the asymptotic normality of the sample mean of a *horizontal evanescent* field, observed in the presence of a purely-indeterministic component. We then derive a Bartlett-type formula for the covariances of the sample covariance sequence of the observed field. Finally, we establish the asymptotic normality of the sample covariance function of the observed field.

2 Notations and Definitions

We begin by recalling the basic definitions, [3],[5]. A homogeneous random field $\{y(n,m)\}$ is called *regular* with respect to the *usual lexicographic order* if for every (n,m), $E[y(n,m) - \hat{y}(n,m)]^2 = \sigma^2 > 0$ where $\hat{y}(n,m)$ is the projection of $y(n,m)$ on the *c.l.m.* $\Big[\{y(k,l) : k < n, l \in \mathbb{Z}\} \cup$

$\{y(n,l) : l < m\}\Big]$. A "rotation" of the usual lexicographic order, such that the resulting NSHP is delimited by a line with rational slope, leads to a generalization of the above order definition. More specifically, for a Rational Non Symmetrical Half Plane (RNSHP) induced by any two coprime integers (a, b), we define the past $P_{a,b}$ by

$$P_{a,b} = \{(n,m) \in \mathbb{Z}^2 : na + mb < 0 \text{ , or } na + mb = 0 \text{ and } m \leq 0\}. \tag{1}$$

Then $P = P_{a,b}$ satisfies

$$(i)\ P \cap (-P) = \{0\}, \quad (ii)\ P \cup (-P) = \mathbb{Z}^2, \quad (iii)\ P + P \subset P\ (\textit{usual addition}).$$

By (i)-(iii), P induces on $\mathbb{Z}^2$ a linear order, which is defined by $(p,q) \preceq (n,m)$ if and only if $(p-n, q-m) \in P$. Let O denote the set of all possible RNSHP definitions on the 2-D lattice. It has been shown in [3],[5] that the model of the evanescent field corresponding to the RNSHP defined by $(a,b) \in O$ is given by

$$\begin{aligned} e_{(a,b)}(n,m) &= \sum_{i=1}^{I^{(a,b)}} e_i^{(a,b)}(n,m) \\ &= \sum_{i=1}^{I^{(a,b)}} s_i^{(a,b)}(na+mb)\cos\Big(\nu_i^{(a,b)}(nc+md)\Big) + t_i^{(a,b)}(na+mb)\sin\Big(\nu_i^{(a,b)}(nc+md)\Big) \end{aligned} \tag{2}$$

where c and d are coprime integers such that $|ad - bc| = 1$. For the case where $(a,b) = (0,1)$ we set $(c,d) = (1,0)$, (due to the lack of uniqueness) and for $(a,b) = (1,0)$ we similarly set $(c,d) = (0,1)$. The processes $\{s_i^{(a,b)}(p)\}$, $\{s_j^{(a,b)}(p)\}$, $\{t_k^{(a,b)}(p)\}$, $\{t_\ell^{(a,b)}(p)\}$ are purely indeterministic and mutually orthogonal for all $i, j, k, \ell, i \neq j, k \neq \ell$. For all i the processes $\{s_i^{(a,b)}(p)\}$ and $\{t_i^{(a,b)}(p)\}$ have an identical autocorrelation function. The number of evanescent components, $I^{(a,b)} \in \mathbb{Z}$, as well as the frequency parameters, $\nu_i^{(a,b)} \in [0, 2\pi)$, are deterministic constants, such that $\nu_i^{(a,b)} \neq \nu_j^{(a,b)}$ and $\nu_i^{(a,b)} + \nu_j^{(a,b)} \neq 2\pi$ for all $i, j, i \neq j$. An evanescent component with parameters $(a,b) = (1,0)$ and $(c,d) = (0,1)$ is usually called *horizontal evanescent*.

In this paper we consider the case where the observed 2-D field is real valued, regular, and homogeneous such that its 2-D Wold decomposition contains a horizontal evanescent component and a purely-indeterministic component. More specifically, let $\{y(n,m)\}$, $(n,m) \in D$ where $D = \{(i,j) \,|\, 0 \leq i \leq N-1,\ 0 \leq j \leq M-1\}$ denote this observed field, such that its 2-D Wold decomposition is given by

$$y(n,m) = e_{(1,0)}(n,m) + w(n,m). \tag{3}$$

Assumption 1: The purely-indeterministic component, $\{w(n,m)\}$ admits the following 2-D infinite order MA representation,

$$w(n,m) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} b(i,j)u(n-i,m-j) \ , \tag{4}$$

where $\{u(n,m)\}$ is an i.i.d. 2-D zero mean field, with variance σ_u^2,

$$\mathrm{E}\Big[u^4(n,m)\Big] = \lambda\sigma_u^4 < \infty \tag{5}$$

and $\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |b(i,j)| < \infty$.

It is not difficult to verify that

$$C_{ww}(\tau,\rho) = \mathrm{E}[w(n+\tau,m+\rho)w(n,m)] = \sigma_u^2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} b(i,j)b(i+\tau,j+\rho) \tag{6}$$

Moreover, one can easily show that $C_{ww}(\tau,\rho)$ is absolutely summable with respect to τ and ρ.

The field $\{e_{(1,0)}(n,m)\}$ is an evanescent random field, *i.e.*,

$$e_{(1,0)}(n,m) \quad = \quad \sum_{i=1}^{I^{(1,0)}} \left[s_i^{(1,0)}(n) \cos\left(\nu_i^{(1,0)} m\right) + t_i^{(1,0)}(n) \sin\left(\nu_i^{(1,0)} m\right) \right]. \tag{7}$$

In the following, for simplicity, we omit the index (1,0). The processes $\{s_i(n)\}$ and $\{t_i(n)\}$ have an identical autocorrelation function given by

$$C_i(\tau) = \mathrm{E}[s_i(n+\tau)s_i(n)] = \mathrm{E}[t_i(n+\tau)t_i(n)]. \tag{8}$$

Assumption 2: The modulating 1-D purely-indeterministic processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field are linear processes admitting infinite order MA representations such that,

$$s_i(n) = \sum_{j=-\infty}^{\infty} a_i(j)\xi_i(n-j) \ , \tag{9}$$

and

$$t_i(n) = \sum_{j=-\infty}^{\infty} a_i(j)\zeta_i(n-j) \ , \tag{10}$$

where $\{\xi_i(n)\}$, $\{\zeta_i(n)\}$ are i.i.d. 1-D zero mean processes, independent of each other and of $\{w(n,m)\}$. Both have an identical variance σ_i^2 such that

$$\mathrm{E}\Big[\xi_i^4(n)\Big] = \mathrm{E}\Big[\zeta_i^4(n)\Big] = \eta\sigma_i^4 < \infty \tag{11}$$

and $\sum_{j=-\infty}^{\infty} |a_i(j)| < \infty$.

Remark: The 2-D Wold decomposition implies that $\{w(n,m)\}$, the purely indeterministic component of the regular field, admits an innovations driven 2-D NSHP MA representation, such that the sequence of MA model coefficients is square summable. Similarly, the modulating 1-D purely-indeterministic processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field admit innovations driven causal MA representations, such that the sequence of MA model coefficients is square summable. Also, the innovations sequences are orthogonal, and not necessarily i.i.d. Thus, the family of random fields satisfying Assumption 1 and 2 is a subset of the set of fields modeled by (3). As shown throughout the proofs below, the added restrictions are needed in order to achieve the boundedness conditions and the convergence rates required for the asymptotic results of this paper to hold. We analyze this point in a greater detail in Section 5.

For any random field $\{u(n,m)\}$ the *sample mean* is defined by

$$\bar{u}_{N,M} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u(n,m). \tag{12}$$

The sample mean of 1-D processes is similarly defined. Clearly the sample mean is an unbiased estimator of the field's mean.

For any two jointly homogeneous random fields $\{u(n,m)\}$ and $\{v(n,m)\}$ the *sample covariance function* of any two such fields is defined by

$$\tilde{C}_{uv}^{N,M}(\tau,\rho) = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u(n+\tau, m+\rho) v(n,m). \tag{13}$$

Similarly, the *sample covariance function* of the stationary 1-D processes $\{u(n)\}$ and $\{v(n)\}$ is defined by

$$\tilde{C}_{uv}^{N}(\tau) = \frac{1}{N} \sum_{n=0}^{N-1} u(n+\tau) v(n). \tag{14}$$

In the following, for simplicity, we will omit the superscript notation N, M and the *sample covariance function* will be denoted by $\tilde{C}_{uv}(\tau,\rho)$ (and similarly for $\tilde{C}_{uv}(\tau)$).

Using (8), the covariance function of $\{e(n,m)\}$ is given by

$$C_{ee}(\tau,\rho) = \mathrm{E}[e(n+\tau, m+\rho) e(n,m)] = \sum_{i=1}^{I} C_i(\tau) \cos(\nu_i \rho). \tag{15}$$

The spectral measure of the evanescent field is not absolutely continuous with respect to the Lebesgue measure on the torus, and therefore its covariance function is not absolutely summable with respect to τ and ρ. Note that Assumption 2 implies the absolute summability of the covariance function $C_i(\tau)$ with respect to τ.

The mutual orthogonality of the evanescent and noise fields ensures that,

$$C_{yy}(\tau,\rho) = C_{ee}(\tau,\rho) + C_{ww}(\tau,\rho). \tag{16}$$

The sample covariance function of $\{y(n,m)\}$ is a function of the sample covariance functions of $\{e(n,m)\}$ and $\{w(n,m)\}$:

$$\begin{aligned}\tilde{C}_{yy}(\tau,\rho) &= \tfrac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\left(e(n+\tau,m+\rho)+w(n+\tau,m+\rho)\right)\left(e(n,m)+w(n,m)\right)\\ &= \tilde{C}_{ee}(\tau,\rho)+\tilde{C}_{ew}(\tau,\rho)+\tilde{C}_{we}(\tau,\rho)+\tilde{C}_{ww}(\tau,\rho).\end{aligned} \tag{17}$$

It can be easily verified that $\tilde{C}_{yy}(\tau,\rho)$ is an unbiased estimator of $C_{yy}(\tau,\rho)$, *i.e.*

$$\mathrm{E}\left[\tilde{C}_{yy}(\tau,\rho)\right] = C_{yy}(\tau,\rho). \tag{18}$$

Let $\{\Psi_i\}$ be a sequence of rectangles such that $\Psi_i = \{(n,m)\in\mathbb{Z}^2 \mid 0\le n\le N_i-1, 0\le m\le M_i-1\}$.

Definition 1: The sequence of subsets $\{\Psi_i\}$ is said to tend to infinity (we adopt the notation $\Psi_i\to\infty$) as $i\to\infty$ if $\lim_{i\to\infty}\min(N_i,M_i)=\infty$ and $\liminf N_i/M_i>0$ and $\limsup N_i/M_i<\infty$. To simplify notations, we shall omit in the following the subscript i. Thus, the notation $\Psi(N,M)\to\infty$ implies that both N and M tend to infinity as functions of i, and at roughly the same rate.

3 Asymptotic normality of the sample mean

In this section we establish the asymptotic normality of the sample mean of the observed field.

From (3) and (7) we have

$$\begin{aligned}\bar{y}_{N,M} &= \bar{e}_{N,M}+\bar{w}_{N,M}\\ &= \sum_{i=1}^{I}\left[\bar{s}_{iN}\frac{1}{M}\sum_{m=0}^{M-1}\cos\left(\nu_i m\right)+\bar{t}_{iN}\frac{1}{M}\sum_{m=0}^{M-1}\sin\left(\nu_i m\right)\right]+\bar{w}_{N,M}\end{aligned} \tag{19}$$

Theorem 1. *Let $\{y(n,m)\}$ be given by (3) and (7), such that Assumption 1 and Assumption 2 are satisfied. Then:*

1. *If there exists an index k, $1 \le k \le I$, such that $\nu_k = 0$, then as $\Psi(N,M) \to \infty$, the random variable $N^{\frac{1}{2}}\bar{y}_{N,M}$ is asymptotically normal with zero asymptotic mean and asymptotic variance $\tilde{v}$ given by*
$$\tilde{v} = \sigma_k^2 \left(\sum_{j=-\infty}^{\infty} a_k(j) \right)^2 . \tag{20}$$

2. *If $\nu_i \neq 0$ for all $1 \le i \le I$, then as $\Psi(N,M) \to \infty$, the random variable $N^{\frac{1}{2}}M^{\frac{1}{2}}\bar{y}_{N,M}$ is asymptotically normal with zero asymptotic mean and asymptotic variance v given by*
$$v = \sigma_u^2 \left(\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} b(i,j) \right)^2 . \tag{21}$$

Proof: By the Central Limit Theorem for 1-D linear processes, [1], Theorem 7.1.2, we know that for all $1 \le i \le I$ the random variable $N^{\frac{1}{2}}\bar{s}_{iN}$ is asymptotically normal with a zero asymptotic mean and asymptotic variance $\sigma_i^2 \left(\sum_{j=-\infty}^{\infty} a_i(j) \right)^2$. Using exactly the same arguments for sample mean of 2-D moving average field $w(n,m)$ one can easily show that the random variable $N^{\frac{1}{2}}M^{\frac{1}{2}}\bar{w}_{N,M}$is asymptotically normal with zero asymptotic mean and asymptotic variance $\sigma_u^2 \left(\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} b(i,j) \right)^2$.

If condition 1 of the theorem is satisfied, then
$$\begin{aligned} N^{\frac{1}{2}}\bar{y}_{N,M} &= N^{\frac{1}{2}}\bar{s}_{kN} \\ &+ N^{\frac{1}{2}} \sum_{\substack{i=1 \\ i \neq k}}^{I} \left[\bar{s}_{iN} \frac{1}{M} \sum_{m=0}^{M-1} \cos(\nu_i m) + \bar{t}_{iN} \frac{1}{M} \sum_{m=0}^{M-1} \sin(\nu_i m) \right] + N^{\frac{1}{2}}\bar{w}_{N,M}. \end{aligned} \tag{22}$$

The asymptotic normality of $N^{\frac{1}{2}}\bar{s}_{iN}$ implies that it converges in distribution to a zero-mean, finite-variance, normal random variable. Hence this random sequence is bounded in probability, (we say that the sequence $\{X_N\}$ is *bounded in probability* if for every $\epsilon > 0$ there exist $\delta(\epsilon)$ such that $P(|X_N| > \delta(\epsilon)) < \epsilon$ for all N), *i.e.*, $N^{\frac{1}{2}}\bar{s}_{iN} = O_P(1)$. For similar arguments $N^{\frac{1}{2}}M^{\frac{1}{2}}\bar{w}_{N,M} = O_P(1)$. Thus, since for $\nu \in [0, 2\pi)$ we have
$$\begin{aligned} \frac{1}{M} \sum_{m=0}^{M-1} \cos(\nu m) &= \frac{\sin\left([M - \frac{1}{2}]\nu\right) + \sin\left(\frac{\nu}{2}\right)}{2M \sin\left(\frac{\nu}{2}\right)} = \begin{cases} O(M^{-1}), & \nu \neq 0; \\ 1, & \nu = 0. \end{cases} \\ \frac{1}{M} \sum_{m=0}^{M-1} \sin(\nu m) &= \frac{\cos\left(\frac{\nu}{2}\right) - \cos\left([M - \frac{1}{2}]\nu\right)}{2M \sin\left(\frac{\nu}{2}\right)} = \begin{cases} O(M^{-1}), & \nu \neq 0; \\ 0, & \nu = 0. \end{cases} \end{aligned} \tag{23}$$

we conclude based on Definition 1 that
$$N^{\frac{1}{2}}\bar{y}_{N,M} = N^{\frac{1}{2}}\bar{s}_{kN} + o_P(1), \tag{24}$$

(we say that the sequence $\{X_N\}$ *convergence in probability to zero*, and use the notation $X_N = o_P(1)$, if for every $\epsilon > 0$, $P(|X_N| > \epsilon) \to 0$ as $N \to \infty$). Hence, by the CLT for 1-D processes, we have that $N^{\frac{1}{2}}\bar{y}_{N,M}$ is asymptotically normal with zero asymptotic mean and asymptotic variance $\sigma_k^2 \left(\sum_{j=-\infty}^{\infty} a_k(j)\right)^2$, which proves the first part of the theorem.

For the case where $\nu_i \neq 0$ for all $1 \leq i \leq I$, we have using (23),

$$\begin{aligned} N^{\frac{1}{2}}M^{\frac{1}{2}}\bar{y}_{N,M} &= N^{\frac{1}{2}}\sum_{i=1}^{I}\left[\bar{s}_{iN}\frac{1}{M^{\frac{1}{2}}}\sum_{m=0}^{M-1}\cos(\nu_i m) + \bar{t}_{iN}\frac{1}{M^{\frac{1}{2}}}\sum_{m=0}^{M-1}\sin(\nu_i m)\right] + N^{\frac{1}{2}}M^{\frac{1}{2}}\bar{w}_{N,M} \\ &= N^{\frac{1}{2}}M^{\frac{1}{2}}\bar{w}_{N,M} + o_P(1) \end{aligned} \tag{25}$$

Therefore if condition 2 is satisfied we have following the same considerations as in [1], Theorem 7.1.2, that $N^{\frac{1}{2}}M^{\frac{1}{2}}\bar{y}_{N,M}$ is asymptotically normal with zero asymptotic mean and asymptotic variance $\sigma_u^2 \left(\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} b(i,j)\right)^2$. ■

Using similar arguments to those in the proof, an immediate consequence of Theorem 1 is that as $\Psi(N, M) \to \infty$

$$\bar{y}_{N,M} = o_P(1) \tag{26}$$

4 Asymptotic normality of the sample covariances

In this section we establish the asymptotic normality of the sample covariances of the observed field.

Let

$$\Delta\tilde{C}_{uv}(\tau, \rho) = \tilde{C}_{uv}(\tau, \rho) - C_{uv}(\tau, \rho), \tag{27}$$

and similarly for 1-D processes

$$\Delta\tilde{C}_{uv}(\tau) = \tilde{C}_{uv}(\tau) - C_{uv}(\tau). \tag{28}$$

We shall need the following proposition:

Proposition 1. *Let* $\Psi(N, M) \to \infty$. *Then,*

$$N^{\frac{1}{2}}\Delta\tilde{C}_{yy}(\tau, \rho) = N^{\frac{1}{2}}X(\tau, \rho) + o_P(1) \tag{29}$$

where

$$\begin{aligned} X(\tau, \rho) = \sum_{i=1}^{I}\Big[&\delta_{\{\nu_i \neq k\pi\}}\{\frac{1}{2}[\Delta\tilde{C}_{s_i s_i}(\tau) + \Delta\tilde{C}_{t_i t_i}(\tau)]\cos(\nu_i\rho) + \frac{1}{2}[\tilde{C}_{t_i s_i}(\tau) - \tilde{C}_{s_i t_i}(\tau)]\sin(\nu_i\rho)\} \\ &+\delta_{\{\nu_i = k\pi\}}\Delta\tilde{C}_{s_i s_i}(\tau)\cos(\nu_i\rho)\Big] \end{aligned} \tag{30}$$

and where $\delta_{\{\cdot\}}$ is the indicator function and $k = \{0, 1\}$.

Proof: See Appendix A for a detailed proof. ■

Let g and h be arbitrary non-negative finite integers. Denote by $\Delta\tilde{\mathbf{C}}_{yy}$ the $(2h+1) \times (2g+1)$ matrix of sample covariances

$$\Delta\tilde{\mathbf{C}}_{yy} = \begin{bmatrix} \Delta\tilde{C}_{yy}(-h,-g) & \Delta\tilde{C}_{yy}(-h,-g+1) & \dots & \dots & \Delta\tilde{C}_{yy}(-h,g-1) & \Delta\tilde{C}_{yy}(-h,g) \\ \Delta\tilde{C}_{yy}(-h+1,-g) & \Delta\tilde{C}_{yy}(-h+1,-g+1) & \dots & \dots & \Delta\tilde{C}_{yy}(-h+1,g-1) & \Delta\tilde{C}_{yy}(-h+1,g) \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \Delta\tilde{C}_{yy}(h,-g) & \Delta\tilde{C}_{yy}(h,-g+1) & \dots & \dots & \Delta\tilde{C}_{yy}(h,g-1) & \Delta\tilde{C}_{yy}(h,g) \end{bmatrix} \quad (31)$$

The main objective of this section is to establish the limiting distribution of the $(2g+1)(2h+1) \times 1$ vector $N^{\frac{1}{2}}\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{yy})$ as $\Psi(N, M) \to \infty$.

Define a matrix $\mathbf{X}$ in a similar way to (31). It is clear from (29) that as $\Psi(N, M) \to \infty$

$$N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{yy} = N^{\frac{1}{2}}\mathbf{X} + \mathbf{o}_P(1) \quad (32)$$

where $\mathbf{o}_P(1)$ is a $(2h+1) \times (2g+1)$ matrix where every entry decays to zero in probability. Define the $(h+1) \times 1$ vector

$$\Delta\tilde{\mathbf{C}}_{s_is_i} = \left[\Delta\tilde{C}_{s_is_i}(0), \Delta\tilde{C}_{s_is_i}(1), \dots, \Delta\tilde{C}_{s_is_i}(h)\right]^T \quad (33)$$

and define the vector $\Delta\tilde{\mathbf{C}}_{t_it_i}$ in the similar manner. Let also define the $h \times 1$ vector

$$\tilde{\mathbf{C}}_{s_it_i} = \left[\tilde{C}_{s_it_i}(1), \tilde{C}_{s_it_i}(2), \dots, \tilde{C}_{s_it_i}(h)\right]^T \quad (34)$$

and define the vector $\tilde{\mathbf{C}}_{t_is_i}$ in the similar manner. From the definition of the 1-D sample covariance function and the definitions of the processes $\{s_i(n)\}, \{t_i(n)\}$ it may be shown (see Appendix B) that for any $0 < \tau \leq h$

$$\begin{aligned} &N^{\frac{1}{2}}\Delta\tilde{C}_{s_is_i}(\tau) = N^{\frac{1}{2}}\Delta\tilde{C}_{s_is_i}(-\tau) + o_p(1) \\ &N^{\frac{1}{2}}\Delta\tilde{C}_{t_it_i}(\tau) = N^{\frac{1}{2}}\Delta\tilde{C}_{t_it_i}(-\tau) + o_p(1) \\ &N^{\frac{1}{2}}[\tilde{C}_{t_is_i}(\tau) - \tilde{C}_{s_it_i}(\tau)] = -N^{\frac{1}{2}}[\tilde{C}_{t_is_i}(-\tau) - \tilde{C}_{s_it_i}(-\tau)] + o_p(1) \end{aligned} \quad (35)$$

Hence, $N^{\frac{1}{2}}\Delta\tilde{C}_{s_is_i}(\tau)$ and $N^{\frac{1}{2}}\Delta\tilde{C}_{t_it_i}(\tau)$ are *"even in probability"* functions of τ, while $N^{\frac{1}{2}}[\tilde{C}_{t_is_i}(\tau)-\tilde{C}_{s_it_i}(\tau)]$ is an *"odd in probability"* function of τ.

Define the $(2g+1)\times 1$ vector

$$\mathbf{COS}_i = \left[\begin{array}{cccc} \cos(-g\nu_i), & \dots & \dots, & \cos(g\nu_i) \end{array}\right]^T \tag{36}$$

Similarly we will define the vector $\mathbf{SIN}_i$. Finally, define the *even mirror duplication* $(2h+1)\times(h+1)$ matrix $\mathbf{D}_+$ by

$$\mathbf{D}_+ = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \tag{37}$$

and the *odd mirror duplication* $(2h+1)\times h$ matrix $\mathbf{D}_-$ by

$$\mathbf{D}_- = \begin{bmatrix} 0 & \dots & 0 & -1 \\ 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \tag{38}$$

Hence we can write $N^{\frac{1}{2}}\mathbf{X}$ as:

$$\begin{aligned} N^{\frac{1}{2}}\mathbf{X} &= N^{\frac{1}{2}}\sum_{i=1}^{I}\Bigg[\delta_{\{\nu_i\neq k\pi\}}\frac{1}{2}\Big\{\mathbf{D}_+(\Delta\tilde{\mathbf{C}}_{s_is_i}+\Delta\tilde{\mathbf{C}}_{t_it_i})\mathbf{COS}_i^T+\mathbf{D}_-(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})\mathbf{SIN}_i^T\Big\} \\ &\quad +\delta_{\{\nu_i=k\pi\}}\mathbf{D}_+\Delta\tilde{\mathbf{C}}_{s_is_i}\mathbf{COS}_i^T\Bigg]+\mathbf{o}_P(1) \end{aligned} \tag{39}$$

Using simple matrix manipulations (see [12], p. 254) we have

$$\begin{aligned}
N^{\frac{1}{2}}\mathrm{vec}(\mathbf{X}) &= N^{\frac{1}{2}}\sum_{i=1}^{I}\Bigg[\delta_{\{\nu_i\neq k\pi\}}\frac{1}{2}\Big\{(\mathbf{COS}_i\otimes\mathbf{D}_+)(\Delta\tilde{\mathbf{C}}_{s_is_i}+\Delta\tilde{\mathbf{C}}_{t_it_i})+(\mathbf{SIN}_i\otimes\mathbf{D}_-)(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})\Big\}\\
&+ \delta_{\{\nu_i=k\pi\}}(\mathbf{COS}_i\otimes\mathbf{D}_+)\Delta\tilde{\mathbf{C}}_{s_is_i}\Bigg]+\mathrm{vec}(\mathbf{o}_P(1)) \qquad (40)
\end{aligned}$$

Now we are in position to prove the main result of this section.

Theorem 2. *Let $\{y(n,m)\}$ be given by (3) and (7), such that Assumption 1 and Assumption 2 are satisfied. Then, as $\Psi(N,M)\to\infty$, the vector $N^{\frac{1}{2}}vec(\Delta\tilde{\mathbf{C}}_{yy})$ is asymptotically normal with zero asymptotic mean and asymptotic covariance matrix $\mathbf{\Gamma}$, given by*

$$\begin{aligned}
\mathbf{\Gamma} &= \sum_{i=1}^{I}\Bigg[\delta_{\{\nu_i\neq k\pi\}}\frac{1}{4}\left\{(\mathbf{COS}_i\otimes\mathbf{D}_+)2\mathbf{V}_i(\mathbf{COS}_i\otimes\mathbf{D}_+)^T+(\mathbf{SIN}_i\otimes\mathbf{D}_-)\mathbf{U}_i(\mathbf{SIN}_i\otimes\mathbf{D}_-)^T\right\}\\
&+\delta_{\{\nu_i=k\pi\}}(\mathbf{COS}_i\otimes\mathbf{D}_+)\mathbf{V}_i(\mathbf{COS}_i\otimes\mathbf{D}_+)^T\Bigg] \qquad (41)
\end{aligned}$$

where $\mathbf{V}_i=[v_i]_{\tau\kappa}$, $(\tau,\kappa=0,\ldots,h)$, is given by

$$v_{i_{\tau\kappa}} = (\eta-3)C_i(\tau)C_i(\kappa)+\sum_{z=-\infty}^{\infty}\Big[C_i(z)C_i(z+\tau-\kappa)+C_i(z+\tau)C_i(z-\kappa)\Big] \qquad (42)$$

and $\mathbf{U}_i=[u_i]_{\tau\kappa}$, $(\tau,\kappa=1,\ldots,h)$, is given by

$$u_{i_{\tau\kappa}} = 2\sum_{z=-\infty}^{\infty}\Big[C_i(z)C_i(z+\tau-\kappa)-C_i(z+\tau)C_i(z-\kappa)\Big]. \qquad (43)$$

Proof: We begin by showing that the vectors $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{s_is_i}$, $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{t_it_i}$ and $N^{\frac{1}{2}}(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})$ are asymptotically jointly normal. Since the proof follows the technique of [1], Propositions 7.3.2 and 7.3.3, we only provide its outlines.

First, we assume that the modulating 1-D processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field are **finite** order MA processes, *i.e.*,

$$s_i(n)=\sum_{j=-m}^{m}a_i(j)\xi_i(n-j)\ , \qquad (44)$$

$$t_i(n)=\sum_{j=-m}^{m}a_i(j)\zeta_i(n-j)\ , \qquad (45)$$

where $\xi_i(n)$, $\zeta_i(n)$ satisfy the same conditions as in Assumption 2.

Define a sequence of random $[3(h+1)-1]\times 1$ vectors $R(n)$ by

$$R(n) = \begin{bmatrix} s_i(n)s_i(n) - C_i(0) \\ \vdots \\ s_i(n)s_i(n+h) - C_i(h) \\ t_i(n)t_i(n) - C_i(0) \\ \vdots \\ t_i(n)t_i(n+h) - C_i(h) \\ t_i(n)s_i(n+1) - s_i(n)t_i(n+1) \\ \vdots \\ t_i(n)s_i(n+h) - s_i(n)t_i(n+h) \end{bmatrix} \tag{46}$$

Thus, $\{R(n)\}$ is a strictly stationary $(2m+h)$-dependent sequence (for a definition of m-dependence, see [1], Definition 6.4.3) and

$$N^{-1}\sum_{n=0}^{N-1} R(n) = \begin{bmatrix} \Delta\tilde{\mathbf{C}}_{s_i s_i} \\ \Delta\tilde{\mathbf{C}}_{t_i t_i} \\ \tilde{\mathbf{C}}_{t_i s_i} - \tilde{\mathbf{C}}_{s_i t_i} \end{bmatrix} \tag{47}$$

Therefore, we wish to prove the asymptotic normality of $N^{-\frac{1}{2}}\sum_{n=0}^{N-1} R(n)$. By definition (see, *e.g.*, [1], Definition 6.4.2) we need to show that (i) the sequence $N^{-1}\lambda^T\sum_{n=0}^{N-1} R(n)$ is asymptotically normal with zero mean and covariance $N^{-1}\lambda^T\mathbf{\Lambda}\lambda$ for every $\lambda \in \mathcal{R}^{(3h+2)}$ such that $\lambda^T\mathbf{\Lambda}\lambda > 0$, where $\mathbf{\Lambda} = N^{-1}var(\sum_{n=0}^{N-1} R(n))$; and (ii) that $\mathbf{\Lambda}$ has no zero diagonal elements.

Since $s_i(n)$ and $t_i(n)$ are independent with zero mean, it can be easily verified that the vector $(\tilde{\mathbf{C}}_{t_i s_i} - \tilde{\mathbf{C}}_{s_i t_i})$ is uncorrelated with the vectors $\Delta\tilde{\mathbf{C}}_{s_i s_i}$ and $\Delta\tilde{\mathbf{C}}_{t_i t_i}$. Moreover, the latter two are independent. Hence,

$$\mathbf{\Lambda} = \lim_{N\to\infty} Nvar\left(\begin{bmatrix} \Delta\tilde{\mathbf{C}}_{s_i s_i} \\ \Delta\tilde{\mathbf{C}}_{t_i t_i} \\ \tilde{\mathbf{C}}_{t_i s_i} - \tilde{\mathbf{C}}_{s_i t_i} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{V}_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{U}_i \end{bmatrix} \tag{48}$$

where $\mathbf{V}_i$ and $\mathbf{U}_i$ are given by (42), (43), respectively. (Evaluation of the elements of $\mathbf{V}_i$ follows from a direct application of the classical Bartlett formula for 1-D linear processes, [1], Propositions 7.3.1 - 7.3.3. See Appendix C for the evaluation of the entries of $\mathbf{U}_i$).

To establish the positivity of the diagonal elements of $\mathbf{\Lambda}$ we consider first the diagonal elements of $\mathbf{U}_i$. Let $S_i(e^{j\omega})$ denote the spectral density function of the processes $\{s_i(n)\}$ and $\{t_i(n)\}$. Clearly, $S_i(e^{j\omega})$ is real, even, and continuous. Thus,

$$u_{i_{\kappa\kappa}} = 2\sum_{z=-\infty}^{\infty}\Big[C_i^2(z) - C_i(z+\kappa)C_i(z-\kappa)\Big] = 2\int_{-\pi}^{\pi} S_i^2(e^{j\omega})[1-e^{j2\kappa\omega}]d\omega \tag{49}$$

where we have used Parseval's theorem to establish the last equality. Since the imaginary part of the integral expression vanishes, while $[1 - \cos(2\kappa\omega)] \geq 0$ and vanishes on a finite number of points, the continuity and non-negativity of $S_i^2(e^{j\omega})$ imply the positivity of the diagonal elements of $\mathbf{U}_i$. Similar arguments establish the positivity of the diagonal elements of $\mathbf{V}_i$.

For any of the above vectors $\lambda \in \mathcal{R}^{(3h+2)}$, the sequence $\lambda^T R(n)$ is a strictly stationary, zero mean, $(2m+h)$-dependent process. Hence, applying the central limit theorem for strictly stationary m-dependent sequences ([1], Theorem 6.4.2), the asymptotic normality of $N^{-1}\lambda^T \sum_{n=0}^{N-1} R(n)$ is established. Extension to the case where the modulating processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field are **infinite** order MA processes (by letting $m \to \infty$) follows from *exactly* the same reasons as in [1], Proposition 7.3.3. We have thus established the asymptotic normality of (47).

Since $N^{\frac{1}{2}}(\tilde{\mathbf{C}}_{t_i s_i} - \tilde{\mathbf{C}}_{s_i t_i})$, $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{s_i s_i}$ and $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{t_i t_i}$ are jointly asymptotically normal, by [1], Proposition 6.4.2, $N^{\frac{1}{2}}\mathrm{vec}(\mathbf{X})$ is asymptotically normal with a zero asymptotic mean and an asymptotic covariance matrix $\mathbf{\Gamma}$ given by (41). Therefore, by Proposition 1 above, and by [1], Proposition 6.3.3, $N^{\frac{1}{2}}\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{yy})$ has the same limiting distribution as $N^{\frac{1}{2}}\mathrm{vec}(\mathbf{X})$, i.e. $N^{\frac{1}{2}}\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{yy})$ is asymptotically normal with a zero asymptotic mean and an asymptotic covariance matrix $\mathbf{\Gamma}$ given by (41). ■

As a result of above theorem, equation (41), combined with the 1-D Bartlett formulae (42) and (43), provides a Bartlett-type formula for the asymptotic covariance matrix of the sample covariances.

Finally we note that in Appendix D it is proven that the results on the asymptotic normality we have proven for $N^{\frac{1}{2}}\tilde{C}_{yy}(\tau,\rho)$ hold for $N^{\frac{1}{2}}\hat{C}_{yy}(\tau,\rho)$, as well, where

$$\hat{C}_{uv}(\tau,\rho) = \frac{1}{NM} \sum_{n=0}^{N-1-\tau} \sum_{m=0}^{M-1-\rho} (u(n+\tau, m+\rho) - \overline{u}_{N,M})(v(n,m) - \overline{v}_{N,M}). \tag{50}$$

5 Discussion

In the last section we have established the limiting distribution of the sample covariances of a horizontal evanescent field observed in the presence of a purely-indeterministic field. Using a "naive" analogy with the case of a 1-D mixed-spectrum regular process, one could expect an $(MN)^{\frac{1}{2}}$ convergence rate to the limiting distribution. However, it turns out that the rate is $N^{\frac{1}{2}}$. This fact could be explained by observing that the stochastic properties of the evanescent field are determined by those of its modulating 1-D processes. Since the rate of convergence of the sample covariances of the modulating processes is $N^{\frac{1}{2}}$, so is the rate for the evanescent field itself. As a result of this slow convergence rate, one can expect that in estimating the covariances of a 2-D homogeneous random field, the dominant estimation error would be due to the evanescent components.

Finally, as mentioned in Section 2, the 2-D Wold decomposition implies that $\{w(n,m)\}$, the purely indeterministic component of the regular field, admits an innovations driven 2-D NSHP MA representation, such that the sequence of MA model coefficients is square summable. Similarly, the modulating 1-D purely-indeterministic processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field admit innovations driven causal MA representations, such that the sequence of MA model coefficients is square summable. Thus, the set of random fields considered here is a subset of the set of fields modeled by (3). Yet, the absolute summability of the sequences of MA model coefficients, assumed in Assumptions 1 and 2 guaranties that the asymptotic variances in (20), (21) are bounded. On the other hand, if the sequences $\{a_k(j)\}$ and $\{b(i,j)\}$ are only square summable, the desired boundedness cannot be guarantied. Furthermore, in establishing the asymptotic normality of the sample covariances, one has to evaluate the convergence rates of various quantities to their limits. Such rates can be obtained assuming absolute summability of the sequences of MA model coefficients, yet square summability is too weak to enable the evaluation of these rates. (See (52) for a typical example.) In addition, orthogonality of the innovations sequence by itself, is in general too weak for establishing central limit theorem type of results. In order to establish the asymptotic normality for the sample mean and sample covariances of the field modeled by (3), we assume that the driving noise sequences are i.i.d. The question of wether these conditions can be relaxed and replaced by other constraints, such as constraints on the moments and mixing rates of the field components, is beyond the scope of this paper.

6 Conclusions

In the paper we have considered the asymptotic properties of the sample mean and of the sample covariance sequence of a field composed of a sum of horizontal evanescent components and a purely-indeterministic component. We have proved that the sample mean and the sample covariances of the field are asymptotically normal, and derived a Bartlett-type asymptotic formula for the covariances of the sample covariances.

Acknowledgment

The authors would like to thanks the two referees and the Associate Editor, for their valuable comments and suggestions.

Appendix A

Proof of the Proposition 1:

Proof: From (16) and (17), it follows that,

$$\begin{aligned}\Delta\tilde{C}_{yy}(\tau,\rho) &= \tilde{C}_{ee}(\tau,\rho)+\tilde{C}_{ew}(\tau,\rho)+\tilde{C}_{we}(\tau,\rho)+\\ &\qquad \tilde{C}_{ww}(\tau,\rho)-C_{ee}(\tau,\rho)-C_{ww}(\tau,\rho)\\ &= \tilde{C}_{ew}(\tau,\rho)+\tilde{C}_{we}(\tau,\rho)+\Delta\tilde{C}_{ee}(\tau,\rho)+\Delta\tilde{C}_{ww}(\tau,\rho).\end{aligned} \tag{51}$$

First we consider convergence in probability of $N^{\frac{1}{2}}\tilde{C}_{ew}(\tau,\rho)$ as $\Psi(N,M)\to\infty$, where we let $n-p=u, m-r=z$:

$$\begin{aligned}\mathrm{E}\left[|N^{\frac{1}{2}}\tilde{C}_{ew}(\tau,\rho)|^2\right] &= \frac{N}{(NM)^2}\,\mathrm{E}\left[\left(\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} e(n+\tau,m+\rho)\,w(n,m)\right)^2\right]\\ &= \frac{1}{NM^2}\sum_{n,p=0}^{N-1}\sum_{m,r=0}^{M-1} C_{ee}(n-p,m-r)C_{ww}(n-p,m-r)\\ &\le \frac{1}{NM^2}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\sum_{p,r\in\mathbb{Z}} |C_{ee}(n-p,m-r)C_{ww}(n-p,m-r)|\\ &= \frac{1}{NM^2}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\sum_{u,v\in\mathbb{Z}} |C_{ee}(u,v)C_{ww}(u,v)|\\ &= \frac{1}{M}\sum_{u,v\in\mathbb{Z}}\left|\sum_{i=1}^{I} C_i(u)\cos(\nu_i v)C_{ww}(u,v)\right|\\ &\le \frac{1}{M}\sum_{u,v\in\mathbb{Z}}\sum_{i=1}^{I}\left|C_i(u)C_{ww}(u,v)\right|\\ &\le \frac{1}{M}\sum_{u,v\in\mathbb{Z}}\sum_{i=1}^{I}\sum_{u'\in\mathbb{Z}}\left|C_i(u')\right|\cdot\left|C_{ww}(u,v)\right|\\ &= \frac{1}{M}\sum_{i=1}^{I}\sum_{u'\in\mathbb{Z}}\left|C_i(u')\right|\cdot\sum_{u,v\in\mathbb{Z}}\left|C_{ww}(u,v)\right| = O(M^{-1}).\end{aligned} \tag{52}$$

where the last equality is due to Assumption 1 and Assumption 2 that guarantee the absolute summability of $\{C_{ww}(u,z)\}$ with respect to u and z, while $\{C_i(u)\}$ is absolutely summable with respect to u. Since convergence of the sequence $N^{\frac{1}{2}}\tilde{C}_{ew}(\tau,\rho)$ to zero in L^2 implies its convergence to zero in probability, we have

$$N^{\frac{1}{2}}\tilde{C}_{ew}(\tau,\rho) = o_P(1) \tag{53}$$

as $\Psi(N,M) \to \infty$. Similarly,

$$N^{\frac{1}{2}}\tilde{C}_{we}(\tau,\rho) = o_P(1) \tag{54}$$

as $\Psi(N,M) \to \infty$.

Next we consider convergence in probability of $N^{\frac{1}{2}}\Delta\tilde{C}_{ww}(\tau,\rho)$ as $\Psi(N,M) \to \infty$. Note that since $\tilde{C}_{ww}(\tau,\rho)$ is an unbiased estimator of $C_{ww}(\tau,\rho)$,

$$\begin{aligned} &\mathrm{E}\left[|N^{\frac{1}{2}}\Delta\tilde{C}_{ww}(\tau,\rho)|^2\right] \\ &= N\mathrm{E}\left[\tilde{C}^2_{ww}(\tau,\rho)\right] - NC^2_{ww}(\tau,\rho) \\ &= N\left\{\tfrac{1}{N^2M^2}\sum_{n,p=0}^{N-1}\sum_{m,r=0}^{M-1}\mathrm{E}\left[w(n+\tau,m+\rho)w(n,m)w(p+\tau,r+\rho)w(p,r)\right] - C^2_{ww}(\tau,\rho)\right\}. \end{aligned} \tag{55}$$

Similarly to 1-D case one can evaluate 4th order moment,

$$\begin{aligned} &\mathrm{E}\left[w(n+\tau,m+\rho)w(n,m)w(p+\tau,r+\rho)w(p,r)\right] = \\ &C^2_{ww}(\tau,\rho) + C_{ww}(n-p,m-r)^2 + C_{ww}(n-p+\tau,m-r+\rho)C_{ww}(n-p-\tau,m-r-\rho) + \\ &(\lambda-3)\sigma_u^4\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} b(i,j)b(i+\tau,j+\rho)b(i+n-p,j+m-r)b(i+n-p+\tau,j+m-r+\rho) \end{aligned} \tag{56}$$

Substituting (56) into (55) and using similar arguments to those employed in the derivation of (52), one can rewrite (55),

$$\begin{aligned} &\mathrm{E}\left[|N^{\frac{1}{2}}\Delta\tilde{C}_{ww}(\tau,\rho)|^2\right] = \\ &\frac{1}{M}\sum_{u=-\infty}^{\infty}\sum_{z=-\infty}^{\infty}\Big\{C_{ww}(u,z)^2 + C_{ww}(u+\tau,z+\rho)C_{ww}(u-\tau,z-\rho) + \\ &\quad (\lambda-3)\sigma_u^4\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} b(i,j)b(i+\tau,j+\rho)b(i+u,j+z)b(i+u+\tau,j+z+\rho)\Big\} \underset{\Psi(N,M)\to\infty}{\longrightarrow} 0 \end{aligned} \tag{57}$$

which is due to Assumption 1 that guarantees that $\{b(u,z)\}$ as well as $\{C_{ww}(u,z)\}$ are absolutely summable, and hence the sequences $\{C^2_{ww}(u,z)\}$, $\{C_{ww}(u+\tau,z+\rho)C_{ww}(u-\tau,z-\rho)\}$, and $\{b(i,j)b(i+\tau,j+\rho)b(i+u,j+z)b(i+u+\tau,j+z+\rho)\}$ are absolutely summable for any τ, ρ.

Since convergence of a sequence to zero in L^2 implies its convergence to zero in probability, we have as $\Psi(N,M) \to \infty$

$$N^{\frac{1}{2}} \Delta \tilde{C}_{ww}(\tau, \rho) = o_P(1) \tag{58}$$

Based on (7), and in similarly to (17)-(27), we have

$$\Delta \tilde{C}_{ee}(\tau, \rho) = \sum_{i,j=1}^{I} \Delta \tilde{C}_{e_i e_j}(\tau, \rho) \tag{59}$$

where if $i = j$

$$\begin{aligned} \Delta \tilde{C}_{e_i e_i}(\tau, \rho) = \quad & \frac{1}{2}[\Delta \tilde{C}_{s_i s_i}(\tau) + \Delta \tilde{C}_{t_i t_i}(\tau)] \cos(\nu_i \rho) + \\ & \frac{1}{2}[\tilde{C}_{t_i s_i}(\tau) - \tilde{C}_{s_i t_i}(\tau)] \sin(\nu_i \rho) + \\ & \frac{1}{2}[\tilde{C}_{s_i s_i}(\tau) - \tilde{C}_{t_i t_i}(\tau)] \frac{1}{M} \sum_{m=0}^{M-1} \cos(\nu_i[\rho + 2m]) + \\ & \frac{1}{2}[\tilde{C}_{t_i s_i}(\tau) + \tilde{C}_{s_i t_i}(\tau)] \frac{1}{M} \sum_{m=0}^{M-1} \sin(\nu_i[\rho + 2m]) \end{aligned} \tag{60}$$

and if $i \neq j$

$$\begin{aligned} \Delta \tilde{C}_{e_i e_j}(\tau, \rho) = \quad & \frac{1}{2}[\tilde{C}_{s_i s_j}(\tau) + \tilde{C}_{t_i t_j}(\tau)] \frac{1}{M} \sum_{m=0}^{M-1} \cos([\nu_i \rho + (\nu_i - \nu_j)m]) + \\ & \frac{1}{2}[\tilde{C}_{t_i s_j}(\tau) - \tilde{C}_{s_i t_j}(\tau)] \frac{1}{M} \sum_{m=0}^{M-1} \sin([\nu_i \rho + (\nu_i - \nu_j)m]) + \\ & \frac{1}{2}[\tilde{C}_{s_i s_j}(\tau) - \tilde{C}_{t_i t_j}(\tau)] \frac{1}{M} \sum_{m=0}^{M-1} \cos([\nu_i \rho + (\nu_i + \nu_j)m]) + \\ & \frac{1}{2}[\tilde{C}_{t_i s_j}(\tau) + \tilde{C}_{s_i t_j}(\tau)] \frac{1}{M} \sum_{m=0}^{M-1} \sin([\nu_i \rho + (\nu_i + \nu_j)m]). \end{aligned} \tag{61}$$

We consider convergence in probability of the terms in (60) and (61). Using (7) and the basic definitions in (13)- (14), we have that for $\nu_i = k\pi$, $k = \{0, 1\}$,

$$\Delta \tilde{C}_{e_i e_i}(\tau, \rho) = \Delta \tilde{C}_{s_i s_i}(\tau) \cos(\nu_i \rho) = (-1)^{k\rho} \Delta \tilde{C}_{s_i s_i} \tag{62}$$

For the case where $\nu_i \neq k\pi$ we begin by evaluating the last two terms of (60). Similarly to (23), one can show that for $\nu_i \neq k\pi$,

$$
\begin{aligned}
\frac{1}{M^2}\left(\sum_{m=0}^{M-1}\cos(\nu_i[\rho+2m])\right)^2 &= o(1)\\
\frac{1}{M^2}\left(\sum_{m=0}^{M-1}\sin(\nu_i[\rho+2m])\right)^2 &= o(1)
\end{aligned}
\tag{63}
$$

Using the definitions of the processes $\{s_i(n)\}, \{t_i(n)\}$ and Assumption 2 one can write (see *e.g.*, [1], pp. 226-227)

$$
\begin{aligned}
&N\mathrm{E}\left[\{\tilde{C}_{s_is_i}(\tau)-\tilde{C}_{t_it_i}(\tau)\}^2\right] = \frac{1}{N}\sum_{n,p=0}^{N-1}\Big(\mathrm{E}\left[s_i(n+\tau)s_i(n)s_i(p+\tau)s_i(p)\right]+\\
&\mathrm{E}\left[t_i(n+\tau)t_i(n)t_i(p+\tau)t_i(p)\right]-2\mathrm{E}\left[s_i(n+\tau)s_i(n)t_i(p+\tau)t_i(p)\right]\Big) =\\
&\frac{1}{N}\sum_{n,p=0}^{N-1}\Big\{2(\eta-3)\sigma_i^4\sum_{j=-\infty}^{\infty}a_i(j)a_i(j+\tau)a_i(j+n-p)a_i(j+n-p+\tau)+\\
&2C_i^2(\tau)+2C_i^2(n-p)+2C_i(n-p+\tau)C_i(n-p-\tau)-2C_i^2(\tau)\Big\} \leq\\
&\sum_{z=-\infty}^{\infty}\Big\{2(\eta-3)\sigma_i^4\sum_{j=-\infty}^{\infty}a_i(j)a_i(j+\tau)a_i(j+z)a_i(j+z+\tau)+\\
&2C_i^2(z)+2C_i(z+\tau)C_i(z-\tau)\Big\} = O(1)
\end{aligned}
\tag{64}
$$

where the absolute summability of $\{a_i(j)\}$ implies the boundedness of the last sum. However, since $\mathrm{E}\left[\left|\frac{N^{\frac{1}{2}}}{M}[\tilde{C}_{s_is_i}(\tau)-\tilde{C}_{t_it_i}(\tau)]\sum_{m=0}^{M-1}\cos(\nu_i[\rho+2m])\right|^2\right]$
$= N\mathrm{E}\left[\{\tilde{C}_{s_is_i}(\tau)-C_{t_it_i}(\tau)\}^2\right]\frac{1}{M^2}\left(\sum_{m=0}^{M-1}\cos(\nu_i[\rho+2m])\right)^2$ we have by combining (63), (64), and Definition 1

$$
\mathrm{E}\left[\left|\frac{N^{\frac{1}{2}}}{M}[\tilde{C}_{s_is_i}(\tau)-\tilde{C}_{t_it_i}(\tau)]\sum_{m=0}^{M-1}\cos(\nu_i[\rho+2m])\right|^2\right] = o(1) \tag{65}
$$

Since convergence of a sequence to zero in L^2 implies its convergence to zero in probability,

$$
\frac{N^{\frac{1}{2}}}{M}[\tilde{C}_{s_is_i}(\tau)-\tilde{C}_{t_it_i}(\tau)]\sum_{m=0}^{M-1}\cos(\nu_i[\rho+2m]) = o_P(1) \tag{66}
$$

Similarly to (64),

$$N\mathrm{E}\left[\{\tilde{C}_{t_is_i}(\tau)+\tilde{C}_{s_it_i}(\tau)\}^2\right]=\frac{1}{N}\sum_{n,p=0}^{N-1}\Big(\mathrm{E}\left[t_i(n+\tau)s_i(n)t_i(p+\tau)s_i(p)\right]+$$

$$\mathrm{E}\left[s_i(n+\tau)t_i(n)s_i(p+\tau)t_i(p)\right]+2\mathrm{E}\left[t_i(n+\tau)s_i(n)s_i(p+\tau)t_i(p)\right]\Big)=$$

$$\frac{1}{N}\sum_{n,p=0}^{N-1}\left\{2C_i^2(n-p)+2C_i(n-p+\tau)C_i(n-p-\tau)\right\}\leq$$

$$\sum_{z=-\infty}^{\infty}\left\{2C_i^2(z)+2C_i(z+\tau)C_i(z-\tau)\right\}=O(1) \tag{67}$$

Hence, combining (63), (67), and Definition 1 we have

$$\frac{N^{\frac{1}{2}}}{M}[\tilde{C}_{t_is_i}(\tau)+\tilde{C}_{s_it_i}(\tau)]\sum_{m=0}^{M-1}\sin(\nu_i[\rho+2m])=o_P(1) \tag{68}$$

Employing the same considerations to summands in (61) one can show that

$$N^{\frac{1}{2}}\Delta\tilde{C}_{e_ie_j}(\tau,\rho)=o_P(1) \tag{69}$$

Substituting (62),(66) and (68) into the (60), and together with (69) one can write

$$\begin{aligned}N^{\frac{1}{2}}\Delta\tilde{C}_{ee}(\tau,\rho)= \quad & N^{\frac{1}{2}}\sum_{i=1}^{I}\Big[\delta_{\{\nu_i\neq k\pi\}}\frac{1}{2}\{[\Delta\tilde{C}_{s_is_i}(\tau)+\Delta\tilde{C}_{t_it_i}(\tau)]\cos(\nu_i\rho)+\\ & \frac{1}{2}[\tilde{C}_{t_is_i}(\tau)-\tilde{C}_{s_it_i}(\tau)]\sin(\nu_i\rho)\}+\delta_{\{\nu_i=k\pi\}}\Delta\tilde{C}_{s_is_i}(\tau)\cos(\nu_i\rho)\Big]+o_P(1)\end{aligned} \tag{70}$$

Collecting (51),(53),(54),(58) and (70) proves the Proposition.

■

Appendix B

In the following we prove the asymptotic equalities stated in (35):

$$N^{\frac{1}{2}}[\Delta\tilde{C}_{s_is_i}(\tau)-\Delta\tilde{C}_{s_is_i}(-\tau)]=N^{-\frac{1}{2}}\sum_{n=N-\tau}^{N-1}s_i(n+\tau)s_i(n)-N^{-\frac{1}{2}}\sum_{n=-\tau}^{-1}s_i(n+\tau)s_i(n) \tag{71}$$

Using the triangle and the Cauchy-Schwartz inequalities, we have,

$$
\begin{aligned}
&\mathrm{E}\left[\left|\sum_{n=N-\tau}^{N-1} s_i(n+\tau)s_i(n) - \sum_{n=-\tau}^{-1} s_i(n+\tau)s_i(n)\right|\right] \leq \\
&\sum_{n=-\tau}^{-1} \mathrm{E}\left[\left|s_i(n+\tau)s_i(n)\right|\right] + \sum_{n=N-\tau}^{N-1} \mathrm{E}\left[\left|s_i(n+\tau)s_i(n)\right|\right] \leq \\
&\sum_{n=-\tau}^{-1} \sqrt{\mathrm{E}\left[s_i(n+\tau)^2\right]\mathrm{E}\left[s_i(n)^2\right]} + \sum_{n=N-\tau}^{N-1} \sqrt{\mathrm{E}\left[s_i(n+\tau)^2\right]\mathrm{E}\left[s_i(n)^2\right]} = \\
&2\tau C_{s_is_i}(0) < \infty.
\end{aligned} \tag{72}
$$

By Markov's inequality, this implies that

$$
N^{\frac{1}{2}}[\Delta\tilde{C}_{s_is_i}(\tau) - \Delta\tilde{C}_{s_is_i}(-\tau)] = o_p(1) \tag{73}
$$

The proofs of the remaining two asymptotic equalities in (35) follows along identical lines.

Appendix C

In this appendix we derive the expression for the general element of the matrix $\mathbf{U}_i$, defined in (43)

$$
\begin{aligned}
&\mathbf{U}_i = [u_i]_{\tau\kappa} = N\mathrm{E}\Big[\{\tilde{C}_{t_is_i}(\tau) - \tilde{C}_{s_it_i}(\tau)\}\{\tilde{C}_{t_is_i}(\kappa) - \tilde{C}_{s_it_i}(\kappa)\}\Big] = \\
&\frac{1}{N}\sum_{n,p=0}^{N-1}\Big\{\mathrm{E}\Big[t_i(n+\tau)s_i(n)t_i(p+\kappa)s_i(p)\Big] - \mathrm{E}\Big[t_i(n+\tau)s_i(n)s_i(p+\kappa)t_i(p)\Big] - \\
&\mathrm{E}\Big[s_i(n+\tau)t_i(n)t_i(p+\kappa)s_i(p)\Big] + \mathrm{E}\Big[s_i(n+\tau)t_i(n)s_i(p+\kappa)t_i(p)\Big]\Big\} = \\
&2\sum_{|z|<N}(1-\frac{|z|}{N})\Big[C_i(z)C_i(z+\tau-\kappa) - C_i(z+\tau)C_i(z-\kappa)\Big]
\end{aligned} \tag{74}
$$

where in the last equality we have used the independence of the processes $\{s_i(n)\}, \{t_i(n)\}$. Since by Assumption 2 all summands in the last line of (74) are absolutely summable, we obtain

$$
\begin{aligned}
&[u_i]_{\tau\kappa} = \lim_{N\to\infty} N\mathrm{E}\Big[\{\tilde{C}_{t_is_i}(\tau) - \tilde{C}_{s_it_i}(\tau)\}\{\tilde{C}_{t_is_i}(\kappa) - \tilde{C}_{s_it_i}(\kappa)\}\Big] = \\
&2\sum_{z=-\infty}^{\infty}\Big[C_i(z)C_i(z+\tau-\kappa) - C_i(z+\tau)C_i(z-\kappa)\Big]
\end{aligned} \tag{75}
$$

as stated in (43).

Appendix D

Asymptotic Normality for the Centered Sample Covariances: For any two homogeneous random fields $\{u(n,m)\}$ and $\{v(n,m)\}$, with finite second order moments, the *centered sample covariance function* between $\{u(n,m)\}$ and $\{v(n,m)\}$ can be defined as:

$$\hat{C}_{uv}(\tau,\rho) = \frac{1}{NM} \sum_{n=0}^{N-1-\tau} \sum_{m=0}^{M-1-\rho} (u(n+\tau, m+\rho) - \overline{u}_{N,M})(v(n,m) - \overline{v}_{N,M}). \tag{76}$$

Let us begin by showing that the difference between $N^{\frac{1}{2}}\tilde{C}_{yy}(\tau,\rho)$ and $N^{\frac{1}{2}}\hat{C}_{yy}(\tau,\rho)$ convergence to zero in probability.

$$\begin{aligned}\tilde{C}_{yy}(\tau,\rho) - \hat{C}_{yy}(\tau,\rho) &= N^{-1}M^{-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} y(n+\tau, m+\rho) y(n,m) \\ -N^{-1}M^{-1} \sum_{n=0}^{N-\tau-1} \sum_{m=0}^{M-\rho-1} &(y(n+\tau,m+\rho) - \overline{y}_{N,M})(y(n,m) - \overline{y}_{N,M}) = I_1 + I_2\end{aligned} \tag{77}$$

where

$$\begin{aligned} I_1 &= N^{-1}M^{-1} \Bigg[\sum_{n=N-\tau}^{N-1} \sum_{m=0}^{M-1} y(n+\tau,m+\rho)y(n,m) + \sum_{n=0}^{N-1} \sum_{m=M-\rho}^{M-1} y(n+\tau,m+\rho)y(n,m) \\ &- \sum_{n=N-\tau}^{N-1} \sum_{m=M-\rho}^{M-1} y(n+\tau,m+\rho)y(n,m) \Bigg]. \end{aligned} \tag{78}$$

and

$$\begin{aligned} I_2 &= \overline{y}_{N,M} N^{-1}M^{-1} \Bigg[\sum_{n=0}^{N-\tau-1} \sum_{m=0}^{M-\rho-1} y(n,m) + \sum_{n=0}^{N-\tau-1} \sum_{m=0}^{M-\rho-1} y(n+\tau,m+\rho) \\ &-\overline{y}_{NM}(M-\rho)(N-\tau) \Bigg] \end{aligned} \tag{79}$$

For the first term of I_1, using the triangle and the Cauchy-Schwartz inequalities, we have,

$$\begin{aligned} &N^{-1}M^{-1} \mathrm{E} \left[\left| \sum_{n=N-\tau}^{N-1} \sum_{m=0}^{M-1} y(n+\tau,m+\rho)y(n,m) \right| \right] \leq \\ &N^{-1}M^{-1} \sum_{n=N-\tau}^{N-1} \sum_{m=0}^{M-1} \mathrm{E} \left[\left| y(n+\tau,m+\rho)y(n,m) \right| \right] \leq \\ &N^{-1}M^{-1} \sum_{n=N-\tau}^{N-1} \sum_{m=0}^{M-1} \sqrt{\mathrm{E}\left[y(n+\tau,m+\rho)^2\right] \mathrm{E}\left[y(n,m)^2\right]} = \\ &N^{-1}M^{-1}\tau M \sqrt{C_{yy}(0,0)C_{yy}(0,0)} \end{aligned} \tag{80}$$

Similarly one can show that

$$\mathrm{E}\left[|I_1|\right] = O(N^{-1}) + O(M^{-1}) + O(N^{-1}M^{-1}) \tag{81}$$

Since we always assume that N and M tend to infinity at the same rate, we have

$$\mathrm{E}\left[N^{\frac{1}{2}}|I_1|\right] = o(1) \tag{82}$$

By Markov's inequality, this implies that

$$N^{\frac{1}{2}} I_1 = o_P(1). \tag{83}$$

For the second term I_2 we know from Theorem 1 that $N^{\frac{1}{2}}\overline{y}_{N,M}$ is either $O_P(1)$, or $o_P(1)$. Moreover, using (26) it is not difficult to show

$$\begin{aligned} &N^{-1}M^{-1}\sum_{n=0}^{N-\tau-1}\sum_{m=0}^{M-\rho-1} y(n,m) + N^{-1}M^{-1}\sum_{n=0}^{N-\tau-1}\sum_{m=0}^{M-\rho-1} y(n+\tau,m+\rho) \\ &-\overline{y}_{NM}(1-\rho M^{-1})(1-\tau N^{-1}) = o_P(1) \end{aligned} \tag{84}$$

Hence,

$$N^{\frac{1}{2}} I_2 = o_P(1). \tag{85}$$

and finally

$$N^{\frac{1}{2}}\tilde{C}_{yy}(\tau,\rho) - N^{\frac{1}{2}}\hat{C}_{yy}(\tau,\rho) = o_P(1). \tag{86}$$

Since convergence *in probability* implies convergence *in distribution* we conclude based on [1], Proposition 6.3.3, that the results of the asymptotic normality we have proven for $N^{\frac{1}{2}}\tilde{C}_{yy}(\tau,\rho)$ hold also for $N^{\frac{1}{2}}\hat{C}_{yy}(\tau,\rho)$.

References

[1] P.J. Brockwell and R.A. Davis, *Time Series: Theory and Methods*, Springer, 1991.

[2] T. P. Chiang, "The Prediction Theory of Stationary Random Fields III, Fourfold Wold Decompositions," *Jou. Multivariate Anal.*, vol. 37, pp. 46-65, 1991.

[3] G. Cohen and J. M. Francos, "Spectral Representation and Asymptotic Properties of Certain Deterministic Fields with Innovation Components," *Probab. Theory Relat. Fields*, vol. 127, pp. 277-304, August 2003.

[4] J. L. Doob, *Stochastic Processes*, John Wiley, 1953.

[5] J. M. Francos, A. Z. Meiri, and B. Porat, "A Wold-like decomposition of 2-D discrete homogenous random fields," *Ann. Appl. Prob.*, vol. 5, pp. 248-260, 1995.

[6] J. M. Francos, A. Narashimhan, and J. W. Woods, "Maximum-Likelihood Estimation of Textures Using a Wold Decomposition Model," *IEEE Trans. Image Process.*, vol. 4, pp. 1655-1666, 1995.

[7] J. M. Francos and A. Nehorai, "Interference Mitigation in STAP Using the Two-Dimensional Wold Decomposition Model," *IEEE Tran. Signal Proces.*, vol. 51, pp. 2461-2470, 2003.

[8] E. J. Hannan, *Multiple Time Series*, John Wiley & sons, 1970.

[9] A. V. Ivanov and N. N. Leonenko, *Statistical Analysis of Random Fields*, Kluwer Academic Press, 1989.

[10] H. Helson and D. Lowdenslager, "Prediction Theory and Fourier Series in Several Variables," *Acta Mathematica.*, 99, pp. 165-201, 1958.

[11] H. Helson and D. Lowdenslager, "Prediction Theory and Fourier Series in Several Variables II," *Acta Mathematica*, vol. 106, pp. 175-213, 1961.

[12] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[13] G. Kallianpur , A. G. Miamee and H. Niemi, "On the Prediction Theory of Two-Parameter Stationary Random Fields," *Jou. Multivariate Anal.*, vol. 32, pp. 120-149, 1990.

[14] T.-H. Li, B. Kedem and S. Yakowitz, "Asymptotic normality of sample autocovariance with an application in frequency estimation," *Stochastic Processes appl.*, vol. 52, pp. 329-349, 1994.

[15] H. Korezlioglu and P. Loubaton, "Spectral Factorization of Wide Sense Stationary Processes on $\mathbb{Z}^2$," *Jou. Multivariate Anal.*, vol. 19, pp. 24-47, 1986.

[16] T.-H. Li, "Bartlett-type formulas for complex multivariate time series of mixed spectra," *Stat. Prob. Let.*, vol. 28, pp. 259-268, 1996.