Maximum-Likelihood Parameter Estimation of Discrete Homogeneous Random Fields with Mixed Spectral Distributions

Joseph M. Francos, *Member, IEEE*, Anand Narasimhan, and John W. Woods, *Fellow, IEEE*

***Abstract*—This paper presents a maximum-likelihood solution to the general problem of fitting a parametric model to observations from a single realization of a real valued, 2-D, homogeneous random field with mixed spectral distribution. On the basis of a 2-D Wold-like decomposition, the field is represented as a sum of mutually orthogonal components of three types: purely indeterministic, harmonic, and evanescent. The proposed algorithm provides a complete solution to the joint estimation problem of the random field components. By introducing appropriate parameter transformations, the highly nonlinear least-squares problem that results from the maximization of the likelihood function is transformed into a separable least-squares problem. In this new problem, the solution for the unknown spectral supports of the harmonic and evanescent components reduces the problem of solving for the transformed parameters of the field to linear least squares. Solution of the transformation equations provides a complete solution of the field model parameter estimation problem.**

I. Introduction

In this paper, we consider the problem of fitting a parametric model to observations from a single realization of a two-dimensional (2-D) real valued discrete and homogeneous random field with mixed spectral distribution. This fundamental problem is of great theoretical and practical importance. It arises quite naturally in terms of the texture estimation of images [10], [11], as well as in several areas of radar, sonar, and seismic signal processing.

The general problem of random fields' parameter estimation has received considerable attention. Most approaches reported to date fall into one of two categories. They either try to fit noise-driven linear models (2-D autoregressive (AR), moving average (MA), or autoregressive moving average (ARMA)) to the observed field, *or* they treat the special case of estimation of the parameters of sinusoidal signals in white noise. Noise-driven linear models have absolutely continuous spectral distribution functions and, hence, are inappropriate for the general problem considered here. Parameter estimation techniques of sinusoidal signals in additive white noise include the periodogram-based approximation (applicable for widely spaced sinusoids) to the maximum likelihood (ML) solution [7], extensions to the Pisarenko harmonic decomposition [4], or the singular value decomposition (SVD) [5]. These methods rely heavily on the white noise assumption and are therefore not applicable here since in our more general setting, the noise is colored and *a priori* unknown.

An early discussion on the problem of analyzing 2-D homogeneous random fields with discontinuous spectral distribution functions can be found in [8]. There, harmonic analysis is employed to analyze the long-lag sample covariances since for such lags, the contribution of the purely indeterministic component is assumed to be insignificant. In this framework, the detection problem for a special class of evanescent fields is also discussed. The idea in [8] is to first test for the existence of the deterministic components. If such components are detected, their parameters are estimated, and their contribution to the sample covariances is removed. Next, the spectral density function of the purely indeterministic component can be estimated from the "corrected" sample covariances. In [10], a similar periodogram-based approach was used. Note that covariance-based estimation procedures must assume knowledge of the true covariances. If these are unknown, substituting them with the sample covariances is incorrect since it is well known [13] that even under the Gaussian assumption, the sample covariances are not consistent estimates of the covariance function if the spectral distribution function has discontinuities.

The 2-D Wold-like decomposition [1] implies that any 2-D regular and homogeneous discrete random field can be represented as a sum of two mutually orthogonal components: a *purely indeterministic* field and a *deterministic* one. The deterministic component is further orthogonally decomposed into a *harmonic* field and a countable number of mutually orthogonal *evanescent* fields. This decomposition results in a corresponding decomposition of the spectral measure of the regular random field into a countable sum of mutually singular spectral measures. The spectral distribution function of the purely indeterministic component is absolutely continuous, whereas the spectral measure of the deterministic component is singular with respect to the Lebesgue measure, and therefore, it is concentrated on a set of Lebesgue measure zero in the frequency plane. For practical applications, the "spectral density function" of the regular field's deterministic component can be

Manuscript received March 21, 1993; revised September 21, 1995. This work was supported, in part, by NSF Grant MIP-9120377. The associate editor coordinating the review of this paper and approving it for publication was Prof. Russell M. Mersereau.

J. M. Francos is with the Electrical and Computer Engineering Department, Ben-Gurion University, Beer-Sheva, Israel.

A. Narasimhan is with the I.B.M. Thomas J. Watson Research Center, Yorktown Heights, NY 10598 USA.

J. W. Woods is with the Electrical Computer and Systems Engineering Department, Rensselaer Polytechnic Institute, Troy NY 12180 USA.

Publisher Item Identifier S 1053-587X(96)03067-X.

assumed to have the form of a countable sum of 1-D and 2-D delta functions. The 1-D delta functions are singular functions that are supported on curves in the 2-D spectral domain. The 2-D delta functions are singular functions that are supported on discrete points in the spectral domain.

In this paper, we consider the problem of estimating the parameters of the different components of the decomposition from a single realization of the field. In general, an unbiased estimator of the field parameters will require *joint* estimation of the parameters of the harmonic, evanescent, and purely indeterministic components. We present a conditional ML solution to this simultaneous parameter estimation problem for the case in which the purely indeterministic component is a real-valued Gaussian random field. The algorithm is a two-stage procedure. In the first stage, we obtain suboptimal initial estimates for the parameters of the spectral support of the evanescent and harmonic components. The initial estimates are obtained by solving the set of 2-D overdetermined normal equations for the parameters of a high-order linear predictor of the observed data. In the second stage, we refine these initial estimates by iterative maximization of the conditional likelihood of the observed data. This maximization requires the solution of a highly nonlinear least-squares problem. By introducing appropriate parameter transformations, the nonlinear least-squares problem is transformed into a *separable* least-squares problem [15], [16]. In this new problem, the solution for the unknown spectral supports of the harmonic and evanescent components reduces the problem of solving for the transformed parameters of the field to linear least squares. Hence, the solution of the original least squares problem becomes much simpler. Solution of the transformation equations provides a complete solution of the field model parameter estimation problem. The proposed method is useful even when the separation between the spectral supports of any two deterministic components is less than $1/N$ in each dimension (for an $N \times N$ observed field).

The paper is organized as follows. In Section II, we briefly summarize the results of the 2-D Wold-like decomposition, which establishes the theoretical basis for the suggested solution. In Section III, we derive the solution for the conditional ML estimation problem. We first present the solution assuming the presence of only a single evanescent component with partially known spectral support parameters. This solution is then generalized to include all possible evanescent fields. In Section IV, an algorithm for estimating the unknown spectral support parameters of the evanescent and harmonic fields is presented. Section V describes a solution to the problem of estimating the parameters of the evanescent random fields. In Section VI, we present some numerical examples to illustrate the performance of the suggested algorithm.

II. The Homogeneous Random Field Model

The presented random field model is derived based on the results of the Wold-type decomposition of 2-D regular and homogeneous random fields, [1]. In this section, we briefly summarize the results of [1]. Let $\{y(n,m), (n,m) \in \mathcal{Z}^2\}$, be a real valued, regular, and homogeneous random field.

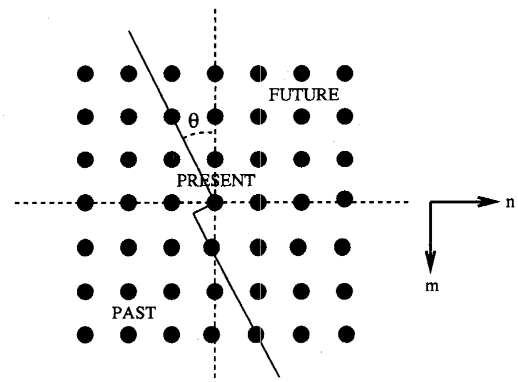


Fig. 1. RNSHP support.

Then, $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + v(n,m). \quad (1)$$

The field $\{w(n,m)\}$ is purely indeterministic and has a unique white innovations driven moving average representation. The field $\{v(n,m)\}$ is a deterministic random field.

We call a 2-D deterministic random field $\{e_o(n,m)\}$ *evanescent w.r.t. the NSHP total-order* o if it spans a Hilbert space identical to the one spanned by its *column-to-column innovations* at each coordinate (n,m) (w.r.t. the total-order o). The deterministic field column-to-column innovation at each coordinate $(n,m) \in \mathcal{Z}^2$ is defined as the difference between the actual value of the field and its projection on the Hilbert space spanned by the deterministic field samples in all previous columns.

It can be shown that it is possible to define a family of NSHP total-order definitions such that the boundary line of the NSHP has rational slope. Let α, β be two coprime integers such that $\alpha \neq 0$. The angle θ of the slope is given by $\tan\ \theta = \beta/\alpha$ (see, for example, Fig. 1). Each of these supports is called a rational nonsymmetrical half-plane (RNSHP). We denote by O the set of all possible RNSHP definitions on the 2-D lattice (i.e., the set of all NSHP definitions in which the boundary line of the NSHP has rational slope). The introduction of the family of RNSHP total-ordering definitions results in the following countably infinite orthogonal decomposition of the deterministic component of the random field:

$$v(n,m) = p(n,m) + \sum_{(\alpha,\beta) \in O} e_{(\alpha,\beta)}(n,m). \quad (2)$$

The random field $\{p(n,m)\}$ is *half-plane deterministic*, i.e., it has no column-to-column innovations w.r.t. any RNSHP total-ordering definition. The field $\{e_{(\alpha,\beta)}(n,m)\}$ is the evanescent component that generates the column-to-column innovations of the deterministic field w.r.t. the RNSHP total-ordering definition $(\alpha,\beta) \in O$.

Hence, if $\{y(n,m)\}$ is a 2-D regular and homogeneous random field, then $y(n,m)$ can be uniquely represented by the orthogonal decomposition

$$y(n,m) = w(n,m) + p(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m). \quad (3)$$

In this paper, all spectral measures are defined on the square region $K = [-1/2, 1/2] \times [-1/2, 1/2]$. It is shown in [1] that the spectral measures of the decomposition components in (3) are mutually singular. The spectral distribution function of the purely indeterministic component is absolutely continuous, whereas the spectral measures of the half-plane deterministic component and all the evanescent components are concentrated on a set L of Lebesgue measure zero in K. Since, for practical applications, we can exclude singular-continuous spectral distribution functions from the framework of our treatment, a model for the evanescent field that corresponds to the RNSHP defined by $(\alpha, \beta) \in O$ is given by

$$\begin{aligned} e_{(\alpha,\beta)}&(n,m) \\ &= \sum_{i=1}^{I^{(\alpha,\beta)}} s_i^{(\alpha,\beta)}(n\alpha - m\beta)\cos\left(2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha)\right) \\ &\quad + t_i^{(\alpha,\beta)}(n\alpha - m\beta)\sin\left(2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha)\right) \quad (4)\end{aligned}$$

where the 1-D purely indeterministic processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}, \{s_j^{(\alpha,\beta)}(n\alpha - m\beta)\}, \{t_k^{(\alpha,\beta)}(n\alpha - m\beta)\}, \{t_\ell^{(\alpha,\beta)}(n\alpha - m\beta)\}$ are mutually orthogonal for all $i, j, k, \ell, i \neq j, k \neq \ell$, and for all i, the processes $\{s_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ and $\{t_i^{(\alpha,\beta)}(n\alpha - m\beta)\}$ have an identical autocorrelation function. Hence, the "spectral density function" of each evanescent field has the form of a countable sum of 1-D delta functions that are supported on lines of rational slope in the 2-D spectral domain.

Let $n^{(\alpha,\beta)} = n\alpha - m\beta$. In the following, we assume that the modulating 1-D processes $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ of each evanescent field can be modeled by a finite order AR model, i.e.

$$\begin{aligned} s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = &-\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau) s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - \tau) \\ &+ \xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}), \quad (5)\end{aligned}$$

and

$$\begin{aligned} t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) = &-\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau) t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)} - \tau) \\ &+ \zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}) \quad (6)\end{aligned}$$

where $\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}), \zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})$ are independent 1-D white innovation processes of identical variance.

One of the half-plane deterministic field components, which is often found in physical problems, is the harmonic random field

$$\begin{aligned} h(n,m) = \sum_{p=1}^{P} (&C_p \cos 2\pi(n\omega_p + m\nu_p) \\ &+ D_p \sin 2\pi(n\omega_p + m\nu_p)) \quad (7)\end{aligned}$$

where the C_p's and D_p's are mutually orthogonal random variables, $E[C_p]^2 = E[D_p]^2 = \sigma_p^2$, and (ω_p, ν_p) are the spatial frequencies of the pth harmonic. In general, P is infinite. This component generates the 2-D delta functions of the "spectral density." The parametric modeling of deterministic random fields whose spectral measures are concentrated on curves, other than lines of rational slope or discrete points in the frequency plane, is still an open question to the best of our knowledge.

As stated earlier, the most general model for the purely indeterministic component $w(n,m)$ is the MA model. However, if its spectral density function is strictly positive on the unit bicircle and analytic in some neighborhood of it, a 2-D AR representation for the purely indeterministic field exists as well [11]. In the following, we assume that the above requirements are satisfied. Hence, the purely indeterministic component *AR model* is given by

$$w(n,m) = -\sum_{(0,0)\prec(k,\ell)} b(k,\ell) w(n-k, m-\ell) + u(n,m) \quad (8)$$

where $\{u(n,m)\}$ is the 2-D white innovations field, whose variance is σ^2.

III. The Conditional ML Estimator

A. Problem Definition and Assumptions

The orthogonal decompositions of the previous section imply that if we exclude from the framework of our model those 2-D random fields whose spectral measures are concentrated on curves other than lines of rational slope, $y(n,m)$ is uniquely represented by $y(n,m) = w(n,m) + h(n,m) + \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n,m)$. Hence, in this paper, we concentrate on a joint solution to the problem of estimating the parameters of the harmonic and evanescent components of the field in the presence of an unknown colored noise generated by the purely indeterministic component and estimating the purely indeterministic component parameters.

When expressed in the general form (7), the coefficients $\{C_p, D_p\}$ of the harmonic component are real-valued, mutually orthogonal random variables. However, since, in general, only a single realization of the random field is observed, we cannot infer anything about the variation of these coefficients over different realizations. The best we can do is to estimate the particular values that the C_p's and D_p's take for the given realization; in other words, we might as well treat the C_p's and D_p's as unknown constants.

We next state our assumptions and introduce some necessary notations. Let $\{y(n,m)\}, (n,m) \in D$ where $D = \{(i,j) \mid 0 \leq i \leq S-1, 0 \leq j \leq T-1\}$ be the observed random field. Note, however, that the observed field just as well could have any *arbitrary* shape.

Assumption 1: The purely indeterministic component is a real-valued Gaussian AR field whose model is given by (8) with $(k,\ell) \in S_{N,M} \backslash \{(0,0)\}$, where $S_{N,M} = \{(i,j) \mid i = 0, 0 \leq j \leq M\} \cup \{(i,j) \mid 1 \leq i \leq N, -M \leq j \leq M\}$, and

N, M are *a priori* known. The driving noise of the AR model is a white Gaussian field with zero mean and variance σ^2.

Assumption 2: The number P of harmonic components in (7) is *a priori* known. For all (α, β), the number $I^{(\alpha,\beta)}$ of evanescent components in (4), is *a priori* known.

In the proposed algorithm, we take the approach of first estimating a *nonparametric* representation of the 1-D purely indeterministic processes $\{s_i^{(\alpha,\beta)}\}, \{t_i^{(\alpha,\beta)}\}$, and only in a second stage, the AR models of these processes are estimated. Hence, in the first stage, we estimate the particular values that the processes take for the given realization, i.e., we treat these as unknown constants.

Let us define the following matrix notations:

$$\begin{aligned}\mathbf{u} \triangleq [&u(N,M),\ldots,u(N,T-1-M),\\ &u(N+1,M),\ldots,u(N+1,T-1-M),\ldots,\\ &u(S-1,T-1-M)]^T. \end{aligned} \tag{9}$$

The vector $\mathbf{y}$ is similarly defined. (See (10) at the bottom of the page.) In addition, we set (11) at the bottom of the page, and $\mathbf{E}_h^R = \mathrm{Re}\{\mathbf{E}_h\}, \mathbf{E}_h^I = \mathrm{Im}\{\mathbf{E}_h\}$.

$$\begin{aligned}\mathbf{b} = -[&b(0,1),\ldots,b(0,M),b(1,-M),\ldots,b(1,M),\ldots,\\ &b(N,-M),\ldots,b(N,M)]^T. \end{aligned} \tag{12}$$

In order to simplify the presentation and keep the notations as simple as possible, we first restrict our attention to the case in which it is *a priori* known that $(\alpha, \beta) = (1, 0)$. The more general problem of estimating the field parameters in the presence of evanescent fields that are characterized by unknown (α, β) parameters is discussed subsequently.

B. The Conditional MLE in the Presence of a Single Evanescent Component with Known Spectral Support Orientation Parameters

The problem faced in this section is the parameter estimation of the harmonic and evanescent components (those of $(\alpha, \beta) = (1, 0)$) of the field in the presence of an unknown colored noise generated by the purely indeterministic component, jointly with the estimation of the purely indeterministic component parameters.

To further simplify the presentation of this section, we shall describe the solution for $I^{(1,0)} = 1$, i.e., in (4), $i \equiv 1$. Hence, in the following, we omit all the subindices i. Thus, the parameters to be estimated are $\{C_p, \omega_p, \nu_p\}_{p=1}^P$, $\nu^{(1,0)}, \{s^{(1,0)}(n)\}_{n=0}^{S-1}, \{t^{(1,0)}(n)\}_{n=0}^{S-1}, \{b(k,\ell)\}_{(k,\ell)\in S_{N,M}}, \sigma^2$. We denote this vector of unknown parameters by $\boldsymbol{\theta}$.

Since $u(n, m)$ is assumed to be Gaussian

$$p(\mathbf{Y}; \boldsymbol{\theta}, D\backslash D_1) = p(U)$$

$$\mathbf{Y} \triangleq \begin{bmatrix} y(N,M-1) & \cdots & y(N,0) & y(N-1,2M) & \cdots & y(N-1,0) & \cdots & y(0,2M) & \cdots & y(0,0)\\ y(N,M) & \cdots & y(N,1) & y(N-1,2M+1) & \cdots & & \cdots & & \cdots & y(0,1)\\ \vdots & & & & \vdots & & & & \vdots & \\ y(N,T-M-2) & \cdots & y(N,T-1-2M) & y(N-1,T-1) & \cdots & y(N-1,T-1-2M) & \cdots & y(0,T-1) & \cdots & y(0,T-1-2M)\\ y(N+1,M-1) & \cdots & y(N+1,0) & y(N,2M) & \cdots & y(N,0) & \cdots & y(1,2M) & \cdots & y(1,0)\\ \vdots & & & & \vdots & & & & \vdots & \\ y(S-1,T-M-2) & \cdots & y(S-1,T-1-2M) & & & & \cdots & y(S-1-N,T-1) & \cdots & y(S-1-N,T-1-2M) \end{bmatrix} \tag{10}$$

$$\mathbf{E}_h \triangleq \begin{bmatrix} e^{j2\pi[N\omega_1+M\nu_1]} & e^{j2\pi[N\omega_2+M\nu_2]} & \cdots & e^{j2\pi[N\omega_P+M\nu_P]}\\ \vdots & \vdots & \vdots & \vdots\\ e^{j2\pi[N\omega_1+(T-1-M)\nu_1]} & e^{j2\pi[N\omega_2+(T-1-M)\nu_2]} & \cdots & \\ e^{j2\pi[(N+1)\omega_1+M\nu_1]} & e^{j2\pi[(N+1)\omega_2+M\nu_2]} & \cdots & \\ \vdots & \vdots & \vdots & \vdots\\ e^{j2\pi[(S-1)\omega_1+(T-1-M)\nu_1]} & \cdots & \cdots & e^{j2\pi[(S-1)\omega_P+(T-1-M)\nu_P]} \end{bmatrix} \tag{11}$$

$$= \frac{1}{(\sqrt{2\pi}\sigma)^{|D_1|}} \times \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=N}^{S-1} \sum_{m=M}^{T-1-M} u^2(n,m) \right\}. \quad (13)$$

The conditional MLE of $\boldsymbol{\theta}$ is found by maximizing (13) or, equivalently, by minimizing $J(\boldsymbol{\theta}) = \sum_{(n,m)\in D_1} u^2(n,m)$, where $D_1 = \{(i,j) \mid N \le i \le S-1, M \le j \le T-1-M\}$, and $D \backslash D_1$ is the set of required initial conditions. Thus, only actually occurring values of the observed field are used in the estimation procedure. Using this method, we sum the squares of only $|D_1|$ values of $u(n,m)$, but this slight loss of information will be unimportant if the size of the observed field $|D|$ is large enough.

Let $S'_{N,M} = S_{N,M} \backslash \{(0,0)\}$. Using the derivation presented in Appendix A, $u(n,m)$ is given by

$$\begin{aligned} u(n,m) = y(n,m) &+ \sum_{(k,\ell)\in S'_{N,M}} b(k,\ell) y(n-k, m-\ell) \\ &- \sum_{p=1}^{P} \mu_p^1 \cos 2\pi(n\omega_p + m\nu_p) \\ &- \sum_{p=1}^{P} \mu_p^2 \sin 2\pi(n\omega_p + m\nu_p) \\ &- \eta^1(n) \cos 2\pi\nu^{(1,0)} m \\ &- \eta^2(n) \sin 2\pi\nu^{(1,0)} m \qquad (n,m) \in D_1 \end{aligned} \quad (14)$$

where we define the following systems of transformations:

$$\begin{aligned} \mu_p^1 \triangleq C_p &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \cos 2\pi(k\omega_p + \ell\nu_p) \\ - D_p &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \sin 2\pi(k\omega_p + \ell\nu_p) \end{aligned} \quad (15a)$$

$$\begin{aligned} \mu_p^2 \triangleq C_p &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \sin 2\pi(k\omega_p + \ell\nu_p) \\ + D_p &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \cos 2\pi(k\omega_p + \ell\nu_p) \end{aligned} \quad (15b)$$

$$\begin{aligned} \eta^1(n) \triangleq &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) s^{(1,0)}(n-k) \cos 2\pi\ell\nu^{(1,0)} \\ - &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) t^{(1,0)}(n-k) \sin 2\pi\ell\nu^{(1,0)} \end{aligned} \quad (16a)$$

$$\begin{aligned} \eta^2(n) \triangleq &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) s^{(1,0)}(n-k) \sin 2\pi\ell\nu^{(1,0)} \\ + &\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) t^{(1,0)}(n-k) \cos 2\pi\ell\nu^{(1,0)}. \end{aligned} \quad (16b)$$

We next show that transformations (15) and (16) are one to one. Let $B(e^{j2\pi\omega}, e^{j2\pi\nu}) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) e^{-j2\pi(\omega k + \nu\ell)}$. The assumptions made in the previous section as to the properties of the spectral density function of the purely indeterministic field imply that the field AR model is such that $B(z_1, z_2)$ is minimum phase. (We assume that the finite support $B(z_1, z_2)$ defined here retains this property of the infinite support filter.) Thus, $B(e^{j2\pi\omega}, e^{j2\pi\nu})$ is nonzero on the unit bicircle and, in particular, at the frequencies of the harmonic components. Since the AR model coefficients are real, we conclude that $\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \cos 2\pi(k\omega + \ell\nu)$ and $\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \sin 2\pi(k\omega + \ell\nu)$, which are the real and imaginary parts of $B(e^{j2\pi\omega}, e^{j2\pi\nu})$, are both nonzero at the frequencies of the harmonic components. Rewriting the system (15) in a matrix form for each p, we get (17), which appears at the bottom of the page. It is easily verified that the determinant of the transformation matrix in (17) is strictly positive. Hence, the transformation (15) is one-to-one.

Define

$$G^{(1,0)}(k) \triangleq \begin{cases} \sum_{\ell=0}^{M} b(0,\ell) \cos 2\pi\ell\nu^{(1,0)} & k = 0 \\ \sum_{\ell=-M}^{M} b(k,\ell) \cos 2\pi\ell\nu^{(1,0)} & 1 \le k \le N \end{cases}, \quad (18a)$$

$$H^{(1,0)}(k) \triangleq \begin{cases} \sum_{\ell=0}^{M} b(0,\ell) \sin 2\pi\ell\nu^{(1,0)} & k = 0 \\ \sum_{\ell=-M}^{M} b(k,\ell) \sin 2\pi\ell\nu^{(1,0)} & 1 \le k \le N. \end{cases} \quad (18b)$$

Let us rewrite (16a) as

$$\begin{aligned} \eta^1(n) &= \left[\sum_{\ell=0}^{M} b(0,\ell) s^{(1,0)}(n) \cos 2\pi\ell\nu^{(1,0)} + \sum_{k=1}^{N} \sum_{\ell=-M}^{M} b(k,\ell) s^{(1,0)}(n-k) \cos 2\pi\ell\nu^{(1,0)} \right] \\ &\quad - \left[\sum_{\ell=0}^{M} b(0,\ell) t^{(1,0)}(n) \sin 2\pi\ell\nu^{(1,0)} + \sum_{k=1}^{N} \sum_{\ell=-M}^{M} b(k,\ell) t^{(1,0)}(n-k) \sin 2\pi\ell\nu^{(1,0)} \right] \\ &= \left[s^{(1,0)}(n) G^{(1,0)}(0) + \sum_{k=1}^{N} G^{(1,0)}(k) s^{(1,0)}(n-k) \right] \\ &\quad - \left[t^{(1,0)}(n) H^{(1,0)}(0) + \sum_{k=1}^{N} H^{(1,0)}(k) t^{(1,0)}(n-k) \right] \\ &= \sum_{k=0}^{N} G^{(1,0)}(k) s^{(1,0)}(n-k) - \sum_{k=0}^{N} H^{(1,0)}(k) t^{(1,0)}(n-k), \\ &\qquad n = N, \ldots, S-1. \end{aligned} \quad (19)$$

$$\begin{pmatrix} \mu_p^1 \\ \mu_p^2 \end{pmatrix} = \begin{pmatrix} \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \cos 2\pi(k\omega_p + \ell\nu_p) & -\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \sin 2\pi(k\omega_p + \ell\nu_p) \\ \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \sin 2\pi(k\omega_p + \ell\nu_p) & \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \cos 2\pi(k\omega_p + \ell\nu_p) \end{pmatrix} \begin{pmatrix} C_p \\ D_p \end{pmatrix} \quad (17)$$

In a similar way, we obtain

$$\eta^2(n) = \sum_{k=0}^{N} H^{(1,0)}(k)s^{(1,0)}(n-k) + \sum_{k=0}^{N} G^{(1,0)}(k)t^{(1,0)}(n-k), \quad n = N, \dots, S-1. \tag{20}$$

Using (19) and (20), it is clear that the transformation system (16) is one-to-one, given N initial values of the processes $\{s^{(1,0)}(n)\}$ and $\{t^{(1,0)}(n)\}$, since each newly introduced pair $s^{(1,0)}(n), t^{(1,0)}(n), \quad n = N, \dots, S-1$ results in a unique pair $\eta^1(n), \eta^2(n)$. Conversely, given these initial values, the solution for the values of $s^{(1,0)}(n),\ t^{(1,0)}(n)\ \ n = N, \dots, S-1$, using the transformed values $\eta^1(n), \eta^2(n)$, is reduced to a recursive solution (in n) of systems of two equations with two unknowns.

Let

$$\mathbf{W} = \begin{bmatrix} e^{j2\pi M\nu^{(1,0)}} \\ e^{j2\pi(M+1)\nu^{(1,0)}} \\ \vdots \\ e^{j2\pi(T-1-M)\nu^{(1,0)}} \end{bmatrix} \tag{21}$$

$$\mathbf{E}_{e_{(1,0)}} = \begin{bmatrix} \mathbf{W} & & & & \\ & \mathbf{W} & & \mathbf{0} & \\ & & \mathbf{W} & & \\ & \mathbf{0} & & \ddots & \\ & & & & \mathbf{W} \end{bmatrix} \tag{22}$$

and $\mathbf{E}^R_{e_{(1,0)}} = \mathrm{Re}\{\mathbf{E}_{e_{(1,0)}}\}, \mathbf{E}^I_{e_{(1,0)}} = \mathrm{Im}\{\mathbf{E}_{e_{(1,0)}}\}$.

$$\boldsymbol{\mu}^i = [\mu^i_1, \mu^i_2, \dots, \mu^i_p]^T \qquad i = 1, 2 \tag{23}$$

$$\boldsymbol{\eta}^j_{(1,0)} = [\eta^j(N), \eta^j(N+1), \dots, \eta^j(S-1)]^T \quad j = 1, 2. \tag{24}$$

Since $J(\boldsymbol{\theta}) = \mathbf{u}^T\mathbf{u}$, we obtain, by writing (14) for all $(n, m) \in D_1$, the following matrix representation for $J(\boldsymbol{\theta})$:

$$J(\boldsymbol{\theta}) = \big\|\mathbf{y} - \mathbf{Y}\mathbf{b} - \mathbf{E}^R_h\boldsymbol{\mu}^1 - \mathbf{E}^I_h\boldsymbol{\mu}^2 - \mathbf{E}^R_{e_{(1,0)}}\boldsymbol{\eta}^1_{(1,0)} - \mathbf{E}^I_{e_{(1,0)}}\boldsymbol{\eta}^2_{(1,0)}\big\|^2. \tag{25}$$

Thus, the transformations (15) and (16) allow us to minimize the objective function $J(\boldsymbol{\theta})$ with respect to $\mathbf{b}, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \boldsymbol{\eta}^1_{(1,0)}, \boldsymbol{\eta}^2_{(1,0)}$ and the deterministic component spectral support parameters $\{\omega_p, \nu_p\}^P_{p=1}, \nu^{(1,0)}$ instead of minimizing it w.r.t. the original problem parameters. The properties of the above transformations guarantee that both minimizations will result in the same minima for J. Define $\mathbf{D} \triangleq [\mathbf{Y}\mathbf{E}^R_h\mathbf{E}^I_h\mathbf{E}^R_{e_{(1,0)}}\mathbf{E}^I_{e_{(1,0)}}]$ and $\boldsymbol{\theta}_1 \triangleq [\mathbf{b}^T(\boldsymbol{\mu}^1)^T(\boldsymbol{\mu}^2)^T(\boldsymbol{\eta}^1_{(1,0)})^T(\boldsymbol{\eta}^2_{(1,0)})^T]^T$. Then, we can rewrite (25) as

$$J(\boldsymbol{\theta}) = \|\mathbf{y} - \mathbf{D}\boldsymbol{\theta}_1\|^2. \tag{26}$$

Because the objective function is a quadratic function of $\boldsymbol{\theta}_1$, the minimization over $\boldsymbol{\theta}_1$ can be carried out analytically for any given value of $\mathbf{D}$. Using the well-known solution to the least-squares problem, we have that

$$\hat{\boldsymbol{\theta}}_1 = \left(\mathbf{D}^T\mathbf{D}\right)^{-1}\mathbf{D}^T\mathbf{y} \tag{27}$$

will minimize $J(\boldsymbol{\theta})$ over $\boldsymbol{\theta}_1$. By inserting (27) into (26), we find that the minimum value of $J(\boldsymbol{\theta})$ is given by

$$J_{\min}(\{\omega_p, \nu_p\}^P_{p=1}, \nu^{(1,0)}) = \mathbf{y}^T(\mathbf{I} - \mathbf{D}(\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T)\mathbf{y}. \tag{28}$$

Here, $\mathbf{D}$ is assumed to be full rank so that $(\mathbf{D}^T\mathbf{D})^{-1}$ exists.

Thus, maximization of the likelihood function is achieved by minimizing the new objective function $J_{\min}(\{\omega_p, \nu_p\}^P_{p=1}, \nu^{(1,0)})$, which is a function only of the deterministic component spectral support parameters. We have thus shown that the minimization problem (25), which is obtained after taking the transformations (15) and (16), is *separable* since its solution can be reduced to a minimization problem in the nonlinear deterministic component's spectral support parameters $\{\omega_p, \nu_p\}^P_{p=1}, \nu^{(1,0)}$ only, whereas $\mathbf{b}, \boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \boldsymbol{\eta}^1_{(1,0)}, \boldsymbol{\eta}^2_{(1,0)}$ can then be determined by solving a *linear* least-squares problem. This new minimization problem is of a considerably lower complexity. A broad discussion on the subject of separable, nonlinear, least-squares minimization problems can be found in [15] and [16]. Since $J_{\min}\big(\{\omega_p, \nu_p\}^P_{p=1}, \nu^{(1,0)}\big)$ is a nonlinear function of $\{\omega_p, \nu_p\}^P_{p=1}, \nu^{(1,0)}$, this optimization problem cannot be solved analytically, and we must resort to numerical methods. In order to avoid the enormous computational burden of an exhaustive search, we use the two-step procedure described in Section IV.

In the discussion above, we assumed that the AR model driving noise variance σ^2 is known. If it is not known, it can be estimated. The ML estimate of σ^2 is derived by maximizing (13) with respect to σ^2. Using the estimated frequencies and (28), we have that

$$\hat{\sigma}^2 = \frac{\hat{J}_{\min}}{(S-N)(T-2M)}. \tag{29}$$

Thus, (27) and (29) establish the estimate for the AR model of the purely indeterministic component of the field. Using the estimated frequencies of the harmonic component and the transformation (15), a complete estimate for the parameters of the harmonic component is obtained. The solution for the parameters of the evanescent field is more involved, and it is given in Section V.

C. The Conditional MLE in the Presence of Multiple Evanescent Components

In this section, we consider the general problem of simultaneously estimating the parameters of the harmonic, purely indeterministic, and evanescent components of a regular and homogeneous random field using a finite sample from a single observed realization of the field. In the following, we make no assumptions with respect to the parameters of the spectral supports of the evanescent components. The complete solution for the estimation problem is derived using the method developed for the special case of $(\alpha, \beta) = (1, 0)$ presented in

Section III-B. The detailed derivation is presented in Appendix B.

Using the notation $n^{(\alpha,\beta)} = n\alpha - m\beta$, we can generalize the transformation introduced in (16)

$$\eta_i^1(n^{(\alpha,\beta)}) \triangleq \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)s_i^{(\alpha,\beta)}[(n-k)\alpha-(m-\ell)\beta] \times \cos 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha) - \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)t_i^{(\alpha,\beta)}[(n-k)\alpha-(m-\ell)\beta] \times \sin 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha) \quad (30a)$$

$$\eta_i^2(n^{(\alpha,\beta)}) \triangleq \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)s_i^{(\alpha,\beta)}[(n-k)\alpha-(m-\ell)\beta] \times \sin 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha) + \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)t_i^{(\alpha,\beta)}[(n-k)\alpha-(m-\ell)\beta] \times \cos 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta+\ell\alpha). \quad (30b)$$

We can thus generalize (14)

$$u(n,m) = y(n,m) + \sum_{(k,\ell)\in S'_{N,M}} b(k,\ell)y(n-k,m-\ell) - \sum_{p=1}^{P}\mu_p^1\cos 2\pi(n\omega_p+m\nu_p) - \sum_{p=1}^{P}\mu_p^2\sin 2\pi(n\omega_p+m\nu_p) - \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}}\left\{\eta_i^1(n^{(\alpha,\beta)})\cos 2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta+m\alpha) + \eta_i^2(n^{(\alpha,\beta)})\sin 2\pi\frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta+m\alpha)\right\}, \quad (n,m)\in D_1. \quad (31)$$

The matrix representation of J is given by

$$J = \left\| \mathbf{y} - \mathbf{Yb} - \mathbf{E}_h^R\boldsymbol{\mu}^1 - \mathbf{E}_h^I\boldsymbol{\mu}^2 - \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}}\{\mathbf{E}^R_{e_{(\alpha,\beta)_i}}\boldsymbol{\eta}^1_{(\alpha,\beta)_i} + \mathbf{E}^I_{e_{(\alpha,\beta)_i}}\boldsymbol{\eta}^2_{(\alpha,\beta)_i}\}\right\|^2. \quad (32)$$

In the present case, for each $i = 1,\ldots,I^{(\alpha,\beta)}$ and $j = 1,2$, the $\eta_i^j(n^{(\alpha,\beta)})$'s corresponding to all $n\alpha - m\beta$ such that $(n,m)\in D_1$ are assembled into the vector $\boldsymbol{\eta}^j_{(\alpha,\beta)_i}$. Hence, the structure of the matrices $\mathbf{E}^R_{e_{(\alpha,\beta)_i}}$ and $\mathbf{E}^I_{e_{(\alpha,\beta)_i}}$ is determined by the structure of the vector $\boldsymbol{\eta}^j_{(\alpha,\beta)_i}$. In general, $\mathbf{E}^R_{e_{(\alpha,\beta)_i}}$ and $\mathbf{E}^I_{e_{(\alpha,\beta)_i}}$ are sparse, but they do not posses the block diagonal structure obtained for $(\alpha,\beta) = (1,0)$.

Following Section III-B, we conclude that in the general case, $J_{\min}$ is a function of the parameters of the spectral support of the field deterministic component: $\{\omega_p,\nu_p\}_{p=1}^P$, the $(\alpha,\beta)\in O$, and the $\{\nu_i^{(\alpha,\beta)}\}_{i=1}^{I^{(\alpha,\beta)}}$ for each (α,β). Hence, $J_{\min}$ has to be minimized w.r.t. the unknown real frequencies and the unknown coprime integer pairs (α,β). This optimization problem is generally difficult since it requires mixed programming solutions. However, this difficulty can be eliminated by defining the following parameter transformation:

$$\gamma \triangleq \frac{\alpha}{\alpha^2+\beta^2} \quad (33a)$$

$$\delta \triangleq \frac{\beta}{\alpha^2+\beta^2}. \quad (33b)$$

The transformed parameters γ,δ are rational numbers. Hence, we can jointly optimize the objective function $J_{\min}$ with respect to the unknown frequencies $\{\omega_p,\nu_p\}_{p=1}^P$, $\{\nu_i^{(\alpha,\beta)}\}_{i=1}^{I^{(\alpha,\beta)}}$ and with respect to γ,δ using some standard programming method. This procedure is described in Section IV. Using (33), we can rewrite (31) in the following form:

$$u(n,m) = y(n,m) + \sum_{(k,\ell)\in S'_{N,M}} b(k,\ell)y(n-k,m-\ell) - \sum_{p=1}^{P}\mu_p^1\cos 2\pi(n\omega_p+m\nu_p) - \sum_{p=1}^{P}\mu_p^2\sin 2\pi(n\omega_p+m\nu_p) - \sum_{(\alpha,\beta)\in O}\sum_{i=1}^{I^{(\alpha,\beta)}}\{\eta_i^1(n^{(\alpha,\beta)})\cos 2\pi\nu_i^{(\alpha,\beta)}(n\delta+m\gamma) + \eta_i^2(n^{(\alpha,\beta)})\sin 2\pi\nu_i^{(\alpha,\beta)}(n\delta+m\gamma)\}, \quad (n,m)\in D_1. \quad (34)$$

The matrix representation of J remains as in (32), except that the matrices $\mathbf{E}^R_{e_{(\alpha,\beta)_i}}$ and $\mathbf{E}^I_{e_{(\alpha,\beta)_i}}$ are now expressed in terms of γ and δ instead of α and β (which is clearly the same). In Section IV, we describe the algorithm for evaluating $\hat{\alpha},\hat{\beta}$ from the estimated values of γ,δ for which the minimum of the cost function is achieved.

IV. The Solution for the Spectral Support Parameters of the Deterministic Component

In Section III, we concluded that the problem of maximizing (13) is made *separable* by taking the transformations (15) and (30) since its solution is reduced to a minimization problem in the nonlinear deterministic component's spectral support parameters ($\{\omega_p,\nu_p\}_{p=1}^P$, the $(\alpha,\beta)\in O$, and the $\{\nu_i^{(\alpha,\beta)}\}_{i=1}^{I^{(\alpha,\beta)}}$ for each (α,β)) only, whereas $\mathbf{b},\boldsymbol{\mu}^1,\boldsymbol{\mu}^2,\{\boldsymbol{\eta}^1_{(\alpha,\beta)_i}\}_{i=1}^{I^{(\alpha,\beta)}},\{\boldsymbol{\eta}^2_{(\alpha,\beta)_i}\}_{i=1}^{I^{(\alpha,\beta)}}$ can then be determined by solving a linear least squares problem. Hence, the first step in solving the presented estimation problem is the minimization of $J_{\min}$ w.r.t. the unknown spectral support parameters of the deterministic component. Since $J_{\min}$ is a nonlinear function of the deterministic component's spectral support parameters, this optimization problem cannot be solved analytically, and we must resort to numerical methods.

In general, $J_{\min}$ has a complicated multimodal shape. In order to avoid the enormous computational burden of an exhaustive search, we used the following two-step procedure. In the first stage, we obtain a suboptimal initial estimate for the parameters of the spectral support of the deterministic component. This stage is implemented by solving the system of overdetermined 2-D normal equations for the parameters of a high-order linear predictor of the observed data. This is followed by a search for the peaks of the magnitude of the predictor transfer function inverse. The harmonic components result in isolated peaks, whereas the evanescent components result in peaks that form continuous lines. In the second stage, we refine these initial estimates by an iterative numerical minimization of the objective function $J_{\min}$. In our experiments, we used the conjugate gradient method of Fletcher and Reeves [18] (p. 253). Note that only for the case of a quadratic objective function, the conjugate gradient procedure is guaranteed to converge in at most N steps. For our problem, we simply restart the algorithm using new gradients until the objective function becomes appreciably small. As is well known, this type of iterative optimization procedure converges to a local minimum and does not guarantee global optimality unless the initial estimate is sufficiently close to the global optimum.

TABLE I
ESTIMATION ALGORITHM

0. Let Q be the total number of evanescent components in the field. Let $\mathbf{x} = \left\{\{(\omega_p, \nu_p)\}_{p=1}^{P}, \{\gamma_q, \delta_q, \nu_q\}_{q=1}^{Q}\right\}$.
1. Find the minimum of J_{min} with respect to $\mathbf{x}$.
2. If for some evanescent component(s) $\hat{\delta}_q << \epsilon$ then for these component(s) $\hat{\beta}_q = 0, \hat{\alpha}_q = 1$.
3. If for some evanescent component(s) $\hat{\gamma}_q << \epsilon$ then for these component(s) $\hat{\beta}_q = 1, \hat{\alpha}_q = 0$.
4. For each one of the remaining evanescent components, find all coprime integer pairs (k_q, ℓ_q) such that $0 < |k_q|, |\ell_q| < min(S,T)/C$ for which $(\hat{\delta}_q/\hat{\gamma}_q) - \epsilon < \ell_q/k_q < (\hat{\delta}_q/\hat{\gamma}_q) + \epsilon$. For all q's for which only a single pair results, set $(\hat{\alpha}_q, \hat{\beta}_q) = (k_q, \ell_q)$.
5. If for one (or more) q's, more than one pair (k_q, ℓ_q) results from step 4, then
   a) For each resolved evanescent component, set $\gamma_q = \frac{\hat{\alpha}_q}{\hat{\alpha}_q^2+\hat{\beta}_q^2}, \delta_q = \frac{\hat{\beta}_q}{\hat{\alpha}_q^2+\hat{\beta}_q^2}$.
   b) For *each* possible combination of (k_q, ℓ_q)'s, where each (k_q, ℓ_q) is associated with a different unresolved evanescent component: Set for each unresolved evanescent component $(\gamma_q, \delta_q) = (\frac{k_q}{k_q^2+\ell_q^2}, \frac{\ell_q}{k_q^2+\ell_q^2})$. Minimize J_{min} w.r.t. the remaining unknown parameters.
6. For each unresolved q from step 5, set $(\hat{\alpha}_q, \hat{\beta}_q) = (k_q, \ell_q)$ where (k_q, ℓ_q) is the pair for which the minimal value of J_{min} was achieved.
7. Set the $\{(\hat{\omega}_p, \hat{\nu}_p)\}_{p=1}^{P}$ and $\{\hat{\nu}_q\}_{q=1}^{Q}$, to their values obtained by the minimization procedure for which the minimal value of J_{min} was achieved.

The overdetermined Yule–Walker method is a modification of the basic Yule–Walker method, which was reported in [9] to lead to a considerable increase in the estimation accuracy of the frequencies of harmonic signals in white noise for 1-D signals. It is further concluded in [9] that the asymptotic accuracy of the estimates will increase with the number of Yule–Walker equations used and with the model support. Intuitively, it can be expected that increasing the predictor support will improve the accuracy of the estimates of the deterministic components' spectral support since the covariances for large lags contain "useful information" about the deterministic components. Based on the conclusions in the 1-D case, it is clear, however, that the initial estimates provided by the solution of the overdetermined high-order normal equations system provide a good initial starting point (i.e., one that leads to convergence to the global minimum) only as long as the local signal to noise ratio is sufficiently high, and the frequencies of the different deterministic components are not too close.

Since the spectral support parameters of each evanescent field are defined by the ratio of the two coprime integers α and β, we need to estimate these parameters using the estimated $\hat{\gamma}$ and $\hat{\delta}$, which have been obtained from the above numerical minimization procedure. Let us first describe the algorithm for the case in which only a single evanescent component exists in the observed field. It is clear from the definition of the transformation (33) that $\delta/\gamma = \beta/\alpha$. We thus search for all coprime integer pairs (k, ℓ) such that $|k|, |\ell| < \min(S,T)/C$ for which $(\hat{\delta}/\hat{\gamma}) - \epsilon < \ell/k < (\hat{\delta}/\hat{\gamma}) + \epsilon$. C is some predetermined constant that guarantees that we consider only order definitions for which there is a "sufficiently" large number of samples per column (row), and ϵ is a small predetermined constant. If more than one pair (k, ℓ) results from the above procedure, we minimize the objective function $J_{\min}$ for each pair using the numerical procedure described above, while assuming that (α, β) is *known* to be equal (k, ℓ). We then set $(\hat{\alpha}, \hat{\beta}) = (k, \ell)$ for the (k, ℓ) pair that achieved the minimal value of the cost function among all the possible candidates. In addition, $\{(\hat{\omega}_p, \hat{\nu}_p)\}_{p=1}^{P}$ and the $\{\hat{\nu}_i^{(\alpha,\beta)}\}_{i=1}^{I^{(\alpha,\beta)}}$ are given by their values, which correspond to the minimal value of the cost function. The detailed algorithm for the case in which more than one evanescent component exists in the field is given in Table I.

Using the estimated parameters, we can now return to the parameter transformation (15) to obtain estimates for the amplitude parameters C_p, D_p of each harmonic component. The estimates are obtained by solving the simultaneous equation (15) for each p. The solution for the parameters of the 1-D modulating purely indeterministic processes associated with each evanescent field is given next.

V. THE PARAMETER ESTIMATION OF THE MODULATING 1-D PURELY INDETERMINISTIC PROCESSES OF THE EVANESCENT FIELDS

Substituting the AR models of $\{s^{(1,0)}(n)\}$ and $\{t^{(1,0)}(n)\}$ (see (5) and (6), respectively) into (19), we obtain

$$\eta^1(n) = \sum_{k=0}^{N} G^{(1,0)}(k)$$

$$\times\left[-\sum_{\tau=1}^{V^{(1,0)}} a^{(1,0)}(\tau)s^{(1,0)}(n-k-\tau)+\xi^{(1,0)}(n-k)\right]$$
$$-\sum_{k=0}^{N} H^{(1,0)}(k)$$
$$\times\left[-\sum_{\tau=1}^{V^{(1,0)}} a^{(1,0)}(\tau)t^{(1,0)}(n-k-\tau)+\zeta^{(1,0)}(n-k)\right]$$
$$=-\sum_{\tau=1}^{V^{(1,0)}} a^{(1,0)}(\tau)\sum_{k=0}^{N} G^{(1,0)}(k)s^{(1,0)}(n-k-\tau)$$
$$+\sum_{k=0}^{N} G^{(1,0)}(k)\xi^{(1,0)}(n-k)$$
$$+\sum_{\tau=1}^{V^{(1,0)}} a^{(1,0)}(\tau)\sum_{k=0}^{N} H^{(1,0)}(k)t^{(1,0)}(n-k-\tau)$$
$$-\sum_{k=0}^{N} H^{(1,0)}(k)\zeta^{(1,0)}(n-k)$$
$$=-\sum_{\tau=1}^{V^{(1,0)}} a^{(1,0)}(\tau)\eta^{1}(n-\tau)$$
$$+\sum_{k=0}^{N}\left[G^{(1,0)}(k)\xi^{(1,0)}(n-k)-H^{(1,0)}(k)\zeta^{(1,0)}(n-k)\right]$$
$$n=N,\ldots,S-1. \quad (35)$$

In a similar way, by substituting the AR models of $\{s^{(1,0)}(n)\}$ and $\{t^{(1,0)}(n)\}$ into (20), we obtain

$$\eta^2(n)=-\sum_{\tau=1}^{V^{(1,0)}} a^{(1,0)}(\tau)\eta^2(n-\tau)+\sum_{k=0}^{N}\left[H^{(1,0)}(k)\xi^{(1,0)}(n-k)+G^{(1,0)}(k)\zeta^{(1,0)}(n-k)\right]$$
$$n=N,\ldots,S-1. \quad (36)$$

Hence, (35) and (36) imply that solving the problem of estimating the unknown parameters of the 1-D purely indeterministic processes associated with the evanescent component is equivalent to solving the above 1-D two-channel ARMA problem, where the $\{G^{(1,0)}(k)\}_{k=0}^{N}$ and $\{H^{(1,0)}(k)\}_{k=0}^{N}$ have previously been estimated. The "observations" are the $\{\eta^1(n)\}$ and $\{\eta^2(n)\}$ for $n=N,\ldots,S-1$.

This two-channel ARMA problem can be solved using any standard estimation procedure for vector ARMA processes like the modified Yule–Walker method of [20]. Note, however, that since the AR parameters are identical in both channels, this estimation procedure can be significantly simplified. If we further assume that $\{\xi^{(1,0)}(n)\}$ and $\{\zeta^{(1,0)}(n)\}$ are independent, zero mean, white Gaussian processes with variance $(\sigma^{(1,0)})^2$ each and that $V^{(1,0)}$ is known, it becomes possible to obtain ML estimates of the unknown parameters by maximizing the log likelihood function of the "observations" $\{\eta^1(n)\}$ and $\{\eta^2(n)\}$ w.r.t. the unknown parameters $\{a^{(1,0)}(\tau)\}_{\tau=1}^{V^{(1,0)}}$ and $(\sigma^{(1,0)})^2$. In [19, pp. 205–208], it is shown how the exact likelihood function of an ARMA process can be computed from the ARMA model parameters by using the Kalman filter.

The above solution can be extended to any (α,β). Define the transformations $k^{(\alpha,\beta)}=k\alpha-\ell\beta, \ell^{(\alpha,\beta)}=\frac{k\beta+\ell\alpha}{\alpha^2+\beta^2}$. Let us also denote by $b^{(\alpha,\beta)}(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})$ the coefficient $b(k,\ell)$ for $(k,\ell)\in S_{N,M}$ under the total-order definition $(\alpha,\beta)\in O$. Following the definitions of $G^{(1,0)}(k)$ and $H^{(1,0)}(k)$, we now define

$$G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)}) \triangleq \sum_{\ell^{(\alpha,\beta)}} b^{(\alpha,\beta)}(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})\cos 2\pi\nu_i^{(\alpha,\beta)}\ell^{(\alpha,\beta)} \quad (37a)$$
$$H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)}) \triangleq \sum_{\ell^{(\alpha,\beta)}} b^{(\alpha,\beta)}(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})\sin 2\pi\nu_i^{(\alpha,\beta)}\ell^{(\alpha,\beta)} \quad (37b)$$

where the summation w.r.t. $\ell^{(\alpha,\beta)}$ is taken over all pairs $(k^{(\alpha,\beta)},\ell^{(\alpha,\beta)})$ that result from the mapping of $(k,\ell)\in S_{N,M}$ by the above transformation while holding $k^{(\alpha,\beta)}$ fixed. Parallel to (19) and (20), we get, by substituting these definitions into (30)

$$\eta_i^1(n^{(\alpha,\beta)})=\sum_{k^{(\alpha,\beta)}} G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})-\sum_{k^{(\alpha,\beta)}} H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)}) \quad (38a)$$
$$\eta_i^2(n^{(\alpha,\beta)})=\sum_{k^{(\alpha,\beta)}} H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})+\sum_{k^{(\alpha,\beta)}} G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)}) \quad (38b)$$

where the summations are taken over all $k^{(\alpha,\beta)}$ such that $(k,\ell)\in S_{N,M}$. Substituting the AR models of $\{s_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ and $\{t_i^{(\alpha,\beta)}(n^{(\alpha,\beta)})\}$ (see (5) and (6), respectively) into (38) and applying steps similar to those in (35) and (36), we obtain

$$\eta_i^1(n^{(\alpha,\beta)})=-\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)\eta_i^1(n^{(\alpha,\beta)}-\tau)+\sum_{k^{(\alpha,\beta)}}\left[G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})-H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})\right], \quad (39)$$

and

$$\eta_i^2(n^{(\alpha,\beta)})=-\sum_{\tau=1}^{V_i^{(\alpha,\beta)}} a_i^{(\alpha,\beta)}(\tau)\eta_i^2(n^{(\alpha,\beta)}-\tau)+\sum_{k^{(\alpha,\beta)}}\left[H_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\zeta_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})+G_i^{(\alpha,\beta)}(k^{(\alpha,\beta)})\xi_i^{(\alpha,\beta)}(n^{(\alpha,\beta)}-k^{(\alpha,\beta)})\right]. \quad (40)$$

TABLE II
PARAMETERS AND ESTIMATION RESULTS FOR EXAMPLES 1 AND 2

Parameters		*Example 1*			*Example 2*		
		Orig.	*Bias*	*Var*	*Orig.*	*Bias*	*Var*
First harmonic component	ω_1	0.15	2.0018e-05	4.7602e-06	0.15	3.5025e-05	5.3371e-07
	ν_1	-0.25	2.3221e-05	5.0118e-06	0.25	3.2844e-05	5.9912e-07
	C_1	0.05	5.1213e-03	2.7661e-03	0.05	2.3178e-03	3.9001e-03
	D_1	0.05	7.2030e-03	3.0199e-03	0.05	3.7821e-03	4.0300e-03
Second harmonic component	ω_2	0.16	2.7221e-05	5.1228e-06	-	-	-
	ν_2	-0.26	2.8780e-05	5.3129e-06	-	-	-
	C_2	0.09	6.2000e-03	3.1069e-03	-	-	-
	D_2	0.09	7.3221e-03	3.2145e-03	-	-	-
First Evanescent component	α_1	1	-	-	0	-	-
	β_1	0	-	-	1	-	-
	$\nu^{(\alpha_1,\beta_1)}$	0.3	5.8781e-05	2.7177e-06	0.2	3.8000e-06	5.0101e-07
Second Evanescent component	α_2	1	-	-	2	-	-
	β_2	2	-	-	5	-	-
	$\nu^{(\alpha_2,\beta_2)}$	0	3.0101e-05	5.2771e-07	0.3	6.0001e-06	5.2320e-08
Third Evanescent component	α_3	-	-	-	2	-	-
	β_3	-	-	-	5	-	-
	$\nu^{(\alpha_3,\beta_3)}$	-	-	-	0.29	6.7800e-06	7.0199e-07
Purely Indeterministic component	$b(0,1)$	0.1	2.2290e-03	7.8900e-04	0.2	1.0005e-04	4.9010e-04
	$b(1,-1)$	0.2	5.0178e-03	9.1222e-04	0.105	1.8933e-03	2.0105e-04
	$b(1,0)$	0.03	3.1000e-03	1.0091e-03	0.046	3.4445e-03	9.0202e-04
	$b(1,1)$	0.018	5.2099e-03	3.9996e-03	0.021	5.9809e-03	3.3460e-03
	σ^2	1	4.0104e-04	6.8819e-04	1	3.6777e-04	5.0007e-04

TABLE III
ESTIMATION RESULTS OF THE SPECTRAL SLOPE PARAMETERS FOR EXAMPLES 1 AND 2

	Example 1				*Example 2*			
	Orig.	*Bias*	*Var*	(α,β) *error rate*	*Orig.*	*Bias*	*Var*	(α,β) *error rate*
γ_1	1	7.9899e-05	5.8788e-04	0	0	5.0001e-06	1.2003e-04	0
δ_1	0	8.0102e-05	6.2900e-04		1	5.9945e-06	2.2297e-04	
γ_2	$\frac{1}{5}$	2.6551e-05	3.0909e-04	0	$\frac{2}{29}$	1.0047e-05	9.8777e-04	0.02
δ_2	$\frac{2}{5}$	3.3000e-05	4.9011e-04		$\frac{5}{29}$	1.5573e-05	1.0232e-03	
γ_3	-	-	-	-	$\frac{2}{29}$	2.3109e-05	1.0004e-03	0.01
δ_3	-	-	-		$\frac{5}{29}$	3.0017e-05	1.0193e-03	

Note that the resulting ARMA model has, in general, a noncausal MA part. Nevertheless, since $\xi_i^{(\alpha,\beta)}$ and $\zeta_i^{(\alpha,\beta)}$ are stationary white noise processes, the noncausal MA part can be replaced by its shifted, and hence causal, version (i.e., the white input sequence is replaced with its shifted version, which has the same statistics). Estimates for $\{a_i^{(\alpha,\beta)}(\tau)\}_{\tau=1}^{V_i^{(\alpha,\beta)}}$ and $(\sigma_i^{(\alpha,\beta)})^2$ for $i = 1 \cdots I^{(\alpha,\beta)}$ can now be obtained using the procedures described earlier in this section.

VI. NUMERICAL EXAMPLES

In this section, we investigate the performance of the suggested ML algorithm using some specific examples. The algorithm performance is illustrated by estimating the bias and the variance of the estimation errors through Monte Carlo simulations. The experimental results are based on 100 independent realizations of the purely indeterministic component and of the modulating 1-D purely indeterministic processes of each evanescent field. We consider two sets of test data that are represented as 64×64 realizations of the fields. The parameters of these examples and the experimental results are listed in Tables II and III.

In Table III, we present the estimation results of the evanescent components' spectral support parameters. It is clear that an incorrect estimate of an (α, β) pair would result in incorrect estimates of the other parameters of that evanescent component. Since the probability of such an event is very small, as indicated by the results in Table III, we consider such events to be outliers. Hence, we ignore the results of these experiments in the computation of the bias and variance of the parameter estimates that are tabulated in Table II.

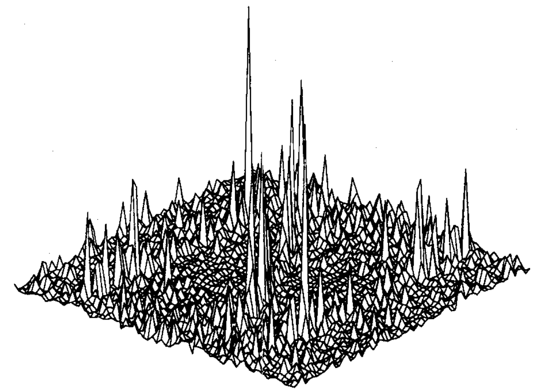


Fig. 2. Periodogram of a 64×64 realization of the field considered in Example 1.

Example 1: Consider a field that consists of the sum of a purely indeterministic component modeled by a 2-D AR model with support $S_{1,1}$ whose parameters are listed in Table II, two sinusoids of different amplitudes, and two evanescent components. The periodogram of one realization of this field is shown in Fig. 2. To improve the clarity of the presentation, the spectral supports of the harmonic and the evanescent components are shown in Fig. 3. The frequencies of the two harmonic components are $(\omega_1, \nu_1) = (0.15, -0.25)$ and $(\omega_2, \nu_2) = (0.16, -0.26)$. The frequencies of the two evanescent components are $\nu^{(1,0)} = 0.3$ and $\nu^{(1,2)} = 0$. The modulating 1-D purely indeterministic processes $s^{(1,0)}$ and $t^{(1,0)}$ are independent second-order AR processes, each with parameters $a_1 = 0.3$, $a_2 = 0.4$ and a unit variance Gaussian white noise input process. The modulating 1-D purely indeterministic processes $s^{(1,2)}$ and $t^{(1,2)}$ are also independent second-order AR processes with the same parameters as $s^{(1,0)}$ and $t^{(1,0)}$. Note from Table III that the slope parameters of the evanescent components spectral supports were accurately estimated in all of the 100 experiments.

Example 2: In this example, we consider a field that is the sum of a purely indeterministic component with AR model support $S_{1,1}$, a single harmonic component with frequency (0.15, 0.25), and three evanescent components. The first evanescent component has a frequency parameter of 0.2, and the slope of its spectral support is specified by $\alpha_1 = 0, \beta_1 = 1$. The modulating 1-D purely indeterministic processes $s^{(0,1)}$ and $t^{(0,1)}$ of this component are independent second-order AR processes, each with parameters $a_1 = 0.3$, $a_2 = 0.4$ and a unit variance Gaussian white noise input process. The second and third evanescent components have spectral supports of identical slopes: $\alpha_2 = \alpha_3 = 2$ and $\beta_2 = \beta_3 = 5$. Their frequency parameters are 0.30 and 0.29, respectively. The modulating 1-D purely indeterministic

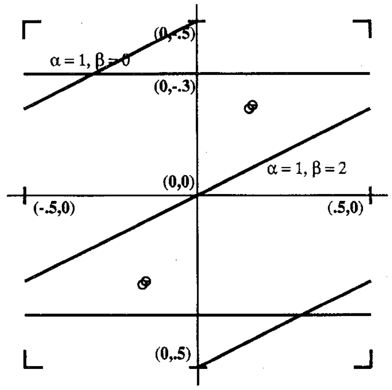


Fig. 3. Locations of harmonic and evanescent components in the spectral domain for Example 1. The two harmonic components are represented by circles and the two evanescent components by lines. Observe that the second evanescent component support wraps around the boundary of the spectral domain.

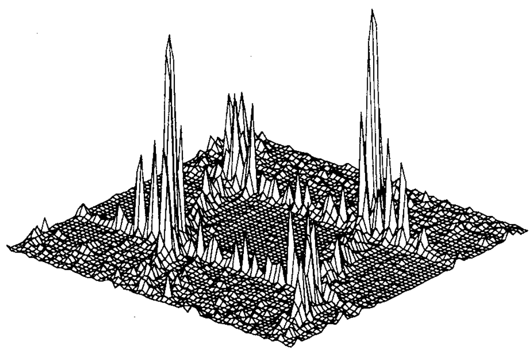


Fig. 4. Periodogram of a 64 × 64 realization of the field considered in Example 2.

processes $s^{(2,5)}$ and $t^{(2,5)}$ of the second component are independent second-order AR processes, each with parameters $a_1 = 0.3$, $a_2 = 0.4$ and a unit variance Gaussian white noise input process, whereas the modulating 1-D purely indeterministic processes $s^{(2,5)}$ and $t^{(2,5)}$ of the third component are independent second-order AR processes, each with parameters $a_1 = 0.2$, $a_2 = 0.4$ and a unit variance Gaussian white noise input process. The periodogram of a single realization of the field considered in this example, and a plot of the spectral supports of the deterministic components, are shown in Figs. 4 and 5, respectively. The results are summarized in Tables II and III.

The experimental results show that the estimates obtained by the proposed conditional ML algorithm are essentially unbiased because the experimental bias is much smaller than the standard deviation of the experimental results. Moreover, the proposed estimation algorithm of the (α, β) pairs of the different evanescent components seems to be quite effective. Note that the errors in estimating (α_2, β_2) and (α_3, β_3) of Example 2 occurred for the case in which the two components have parallel and closely spaced spectral supports. In all other cases, no errors occurred in the estimation procedure of the different (α, β) pairs.

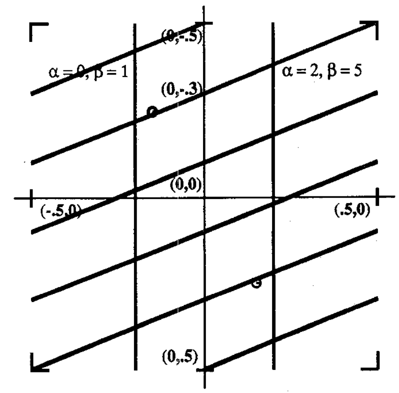


Fig. 5. Locations of harmonic and evanescent components in the spectral domain for Example 2. The two evanescent components with $\alpha_2 = \alpha_3 = 2$ and $\beta_2 = \beta_3 = 5$ and frequencies of 0.30 and 0.29 are too close to be visually distinct. For the second and third evanescent components, the wrapping around the boundary of the spectral domain is also shown.

Computationally, the iterative search is the most expensive part of the proposed algorithm. However, if the local ratio of the power of each deterministic component to the power of the purely indeterministic component in the neighborhood of the deterministic component spectral support is not too low, our experiments indicate that the initial guesses are quite accurate, and hence, the minimization of the objective function is achieved quite rapidly.

VII. Conclusion

Homogeneous random fields are characterized, in general, by mixed spectral distributions. In this paper, we have presented a conditional maximum-likelihood solution to the general problem of fitting a parametric model to observations from a single realization of a real-valued homogeneous random field. The proposed algorithm provides a complete solution to the joint parameter estimation problem of the harmonic, purely indeterministic, and evanescent components of the field.

The solution to this general problem has many applications in the areas of processing and estimation of 2-D signals. In [11], we describe one such application for the parameter estimation and synthesis of natural textures. The synthesis procedure is based only on the estimated parameters. This application results in a complete and more accurate parameter estimation (and, hence, in improved synthesis results) than a previous approach presented in [10], although both rely on the same theoretical basis.

Appendix A

Using (8), $u(n,m)$ is given by $u(n,m) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)w(n-k, m-\ell)$ with $b(0,0) = 1$. Since $w = y - h - e_{1,0}$, we have

$$u(n,m) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\{y(n-k,m-\ell) - h(n-k,m-\ell) - e_{1,0}(n-k,m-\ell)\}$$

$$= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \times \Big\{ y(n-k,m-\ell) - \sum_{p=1}^{P} (C_p \cos 2\pi[(n-k)\omega_p + (m-\ell)\nu_p] + D_p \sin 2\pi[(n-k)\omega_p + (m-\ell)\nu_p]) - s^{(1,0)}(n-k)\cos 2\pi\nu^{(1,0)}(m-\ell) - t^{(1,0)}(n-k)\sin 2\pi\nu^{(1,0)}(m-\ell) \Big\}$$

$$= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) y(n-k,m-\ell) - \sum_{p=1}^{P} C_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \times \cos 2\pi[(n\omega_p + m\nu_p) - (k\omega_p + \ell\nu_p)] - \sum_{p=1}^{P} D_p \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \times \sin 2\pi[(n\omega_p + m\nu_p) - (k\omega_p + \ell\nu_p)] - \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) s^{(1,0)}(n-k) \times \cos 2\pi\nu^{(1,0)}(m-\ell) - \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) t^{(1,0)}(n-k) \times \sin 2\pi\nu^{(1,0)}(m-\ell)$$

$$= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) y(n-k,m-\ell) - \sum_{p=1}^{P} C_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\cos 2\pi(k\omega_p + \ell\nu_p) \right) \times \cos 2\pi(n\omega_p + m\nu_p) - \sum_{p=1}^{P} C_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\sin 2\pi(k\omega_p + \ell\nu_p) \right) \times \sin 2\pi(n\omega_p + m\nu_p) - \sum_{p=1}^{P} D_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\cos 2\pi(k\omega_p + \ell\nu_p) \right) \times \sin 2\pi(n\omega_p + m\nu_p) + \sum_{p=1}^{P} D_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\sin 2\pi(k\omega_p + \ell\nu_p) \right) \times \cos 2\pi(n\omega_p + m\nu_p) - \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) s^{(1,0)}(n-k)\cos 2\pi\ell\nu^{(1,0)} \right) \times \cos 2\pi m\nu^{(1,0)} - \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) s^{(1,0)}(n-k)\sin 2\pi\ell\nu^{(1,0)} \right) \times \sin 2\pi m\nu^{(1,0)} - \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) t^{(1,0)}(n-k)\cos 2\pi\ell\nu^{(1,0)} \right) \times \sin 2\pi m\nu^{(1,0)} + \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) t^{(1,0)}(n-k)\sin 2\pi\ell\nu^{(1,0)} \right) \times \cos 2\pi m\nu^{(1,0)}. \quad (41)$$

APPENDIX B

Following the derivation in Section III-B and Appendix A

$$u(n,m) = \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \Big\{ y(n-k,m-\ell) - h(n-k,m-\ell) - \sum_{(\alpha,\beta)\in O} e_{(\alpha,\beta)}(n-k,m-\ell) \Big\}$$

$$= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \times \Big\{ y(n-k,m-\ell) - \sum_{p=1}^{P} (C_p \cos 2\pi[(n-k)w_p + (m-\ell)\nu_p] + D_p \sin 2\pi[(n-k)w_p + (m-\ell)\nu_p]) - \sum_{(\alpha,\beta)\in O} \Big\{ \sum_{i=1}^{I^{(\alpha,\beta)}} s_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta] \times \cos 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}[(n-k)\beta + (m-\ell)\alpha] + \sum_{i=1}^{I^{(\alpha,\beta)}} t_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta] \times \sin 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}[(n-k)\beta + (m-\ell)\alpha] \Big\} \Big\}$$

$$= \sum_{(k,\ell)\in S_{N,M}} b(k,\ell) y(n-k,m-\ell) - \sum_{p=1}^{P} C_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\cos 2\pi(k\omega_p + \ell\nu_p) \right) \times \cos 2\pi(n\omega_p + m\nu_p) - \sum_{p=1}^{P} C_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell)\sin 2\pi(k\omega_p + \ell\nu_p) \right) \times \sin 2\pi(n\omega_p + m\nu_p)$$

$$
\begin{aligned}
&-\sum_{p=1}^{P} D_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \cos 2\pi(k\omega_p + \ell\nu_p) \right) \\
&\times \sin 2\pi(n\omega_p + m\nu_p) \\
&+\sum_{p=1}^{P} D_p \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \sin 2\pi(k\omega_p + \ell\nu_p) \right) \\
&\times \cos 2\pi(n\omega_p + m\nu_p) \\
&-\sum_{(\alpha,\beta)\in O} \left\{ \sum_{i=1}^{I^{(\alpha,\beta)}} \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \right.\right. \\
&\qquad \times s_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta] \\
&\qquad \left. \times \cos 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta + \ell\alpha) \right) \\
&\qquad \left. \times \cos 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha) \right\} \\
&-\sum_{(\alpha,\beta)\in O} \left\{ \sum_{i=1}^{I^{(\alpha,\beta)}} \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \right.\right. \\
&\qquad \times s_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta] \\
&\qquad \left. \times \sin 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta + \ell\alpha) \right) \\
&\qquad \left. \times \sin 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha) \right\} \\
&-\sum_{(\alpha,\beta)\in O} \left\{ \sum_{i=1}^{I^{(\alpha,\beta)}} \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \right.\right. \\
&\qquad \times t_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta] \\
&\qquad \left. \times \cos 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta + \ell\alpha) \right) \\
&\qquad \left. \times \sin 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha) \right\} \\
&+\sum_{(\alpha,\beta)\in O} \left\{ \sum_{i=1}^{I^{(\alpha,\beta)}} \left(\sum_{(k,\ell)\in S_{N,M}} b(k,\ell) \right.\right. \\
&\qquad \times t_i^{(\alpha,\beta)}[(n-k)\alpha - (m-\ell)\beta] \\
&\qquad \left. \times \sin 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(k\beta + \ell\alpha) \right) \\
&\qquad \left. \times \cos 2\pi \frac{\nu_i^{(\alpha,\beta)}}{\alpha^2+\beta^2}(n\beta + m\alpha) \right\}. \qquad (42)
\end{aligned}
$$

References

[1] J. M. Francos, A. Z. Meiri, and B. Porat, "A wold-like decomposition of 2-D discrete homogeneous random fields," *Ann. Appl. Prob.*, vol. 5, pp. 248–260, 1995.

[2] R. R. Hansen and R. Chellappa, "Two-dimensional robust spectrum estimation," *IEEE Trans. Acoust., Speech Signal Processing*, vol. 36, pp. 1051–1066, 1988.

[3] X.-D. Zhang, "Two dimensional harmonic retrieval and its time-domain analysis technique," *IEEE Trans. Inform. Theory*, vol. 37, pp. 1185–1188, 1991.

[4] S. W. Lang and J. H. McClellan, "The extension of Pisarenko's method to multiple dimensions," in *Proc. Int. Conf. Acoust., Speech, Signal Processing*, 1982, pp. 125–128.

[5] R. Kumaresan and D. W. Tufts, "A two-dimensional technique for frequency-wavenumber estimation," *Proc. IEEE*, vol. 69, pp. 1515–1517, 1981.

[6] C. Chatterjee, R. L. Kashyap, and G. Boray, "Estimation of close sinusoids in colored noise and model discrimination," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 328–337, 1987.

[7] A. M. Walker, "On the estimation of a harmonic component in a time series with stationary independent residuals," *Biometrika*, vol. 58, pp. 21–36, 1971.

[8] M. B. Priestley, "The analysis of two-dimensional stationary processes with discontinuous spectra," *Biometrika*, vol. 51, pp. 195–217, 1964.

[9] P. Stoica, R. L. Moses, B. Friedlander, and T. Söderström, "Maximum likelihood estimation of the parameters of multiple sinusoids from noisy measurements," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 378–391, 1989.

[10] J. M. Francos, A. Z. Meiri, and B. Porat, "A unified texture model based on a 2-D Wold like decomposition," *IEEE Trans. Signal Processing*, vol. 41, pp. 2665–2678, 1993.

[11] J. M. Francos, A. Narasimhan, and J. W. Woods, "Maximum likelihood parameter estimation of textures using a Wold decomposition based model," *IEEE Trans. Image Processing*, vol. 4, no. 12, pp. 1655–1666, Dec. 1995.

[12] M. P. Ekstrom and J. W. Woods, "Two-dimensional spectral factorization with applications in recursive digital filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 115–128, 1976.

[13] J. L. Doob, *Stochastic Processes*. New York: Wiley, 1953.

[14] M. B. Priestley, *Spectral Analysis and Time Series*. New York: Academic, 1981.

[15] G. H. Golub and V. Pereyra, "The differentiation of pseudo-inverse and nonlinear least square problems whose variables separate," *SIAM J. Numer. Anal.*, vol. 10, pp. 413–432, Apr. 1973.

[16] M. R. Osborne, "Some special nonlinear least square problems," *SIAM J. Numer. Anal.*, vol. 12, pp. 571–592, Sept. 1975.

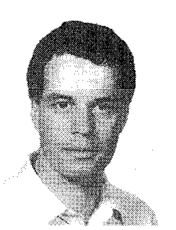
[17] S. M. Kay, *Modern Spectral Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1988.

[18] D. G. Luenberger, *Linear and Nonlinear Programming*, 2nd ed. Reading, MA: Addison-Wesley, 1984.

[19] B. Porat, *Digital Processing of Random Signals*. Englewood Cliffs, NJ: Prentice-Hall, 1994.

[20] B. Friedlander and B. Porat, "Multichannel ARMA spectral estimation by the modified Yule-Walker method," *Signal Processing*, vol. 10, no. 1, pp. 49–59, Jan. 1986.

[21] T. W. Anderson, *The Statistical Analysis of Time Series*. New York: Wiley, 1971.



Joseph M. Francos (M'90) was born on November 6, 1959 in Tel-Aviv, Israel. He received the B.Sc. degree in computer engineering in 1982 and the D.Sc. degree in electrical engineering in 1990, both from the Technion—Israel Institute of Technology.

From 1982 to 1987, he was with the Signal Corps Research Laboratories, Israeli Defense Forces. From 1991 to 1992, he was with the Department of Electrical Computer and Systems Engineering at the Rensselaer Polytechnic Institute, Troy, NY, USA, as a Visiting Assistant Professor. During 1993, he was with Signal Processing Technology, Palo Alto, CA, USA. In 1993, he joined the Department of Electrical and Computer Engineering, Ben-Gurion University, Beer-Sheva, Israel, where he is now a Lecturer. He also held visiting positions at the Media Laboratory of the Massachusetts Institute of Technology, Cambridge, USA, and at the Electrical and Computer Engineering Department, University of California, Davis, USA. His current research interests are in parametric modeling and estimation of 2-D random fields, random fields theory, parametric modeling and estimation of nonstationary signals, image modeling, and texture analysis and synthesis.

Anand Narasimhan was born in Madras, India. He received the B.E. degree in electrical and electronics engineering from the College of Engineering, Madras, India, in 1986, the M.S. degree in electrical engineering from the University of Rhode Island, Kingston, USA, in 1988, and the M.S. degree in mathematics and the Ph.D. degree in electrical engineering from the Rensselaer Polytechnic Institute, Troy, NY, in 1991 and 1992, respectively.

He has been with the Personal Systems Group at the I.B.M Thomas J. Watson Research Center, Yorktown Heights, NY,USA, since July 1992. His current research interests include wireless communication systems architecture, problems in spectrum estimation, and computer telephony.

Dr. Narasimhan was a recipient of the I.B.M. Graduate Fellowship (1990–1992) and is a member of Sigma Xi.

John W. Woods (F'88) received the Ph.D. degree from the Massachusetts Institute of Technology, Cambridge, USA, in 1970. Since 1976, he has been with the Electrical Computer and Systems Engineering Department at Rensselaer Polytechnic Institute, Troy, NY, USA, where he is currently Professor and Acting Director of the Center for Image Processing Research.

Dr. Woods was corecipient of 1976 and 1987 Senior Paper Awards of the now IEEE Signal Processing (SP) Society. He served on the editorial board of *Graphical Models and Image Processing* and was chairman of the Seventh Workshop on Multidimensional Signal Processing in 1991. He served as Technical Program Co-chairman for the First IEEE International Conference on Image Processing (ICIP) held in Austin, TX, USA, in November 1994. He received the Signal Processing Society Meritorious Service Award in 1990. He is currently on the editorial board of the IEEE TRANSACTIONS ON VIDEO TECHNOLOGY. He received a Technical Achievement Award from the IEEE Signal Processing Society in 1994. He is a member of Sigma Xi, Tau Beta Pi, Eta Kappa Nu, and the AAAS.