TWO-DIMENSIONAL POLYNOMIAL PHASE SIGNALS: PARAMETER ESTIMATION AND BOUNDS

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ABSTRACT

This paper considers the problem of parametric modeling and estimation of nonhomogeneous two-dimensional (2-D) signals. In particular, we focus our study on the class of constant modulus polynomial-phase 2-D nonhomogeneous signals. We present two different phase models and develop computationally efficient estimation algorithms for the parameters of these models. Both algorithms are based on phase differencing operators. The basic properties of the operators are analyzed and used to develop the estimation algorithms. The Cramer-Rao lower bound on the accuracy of jointly estimating the model parameters is derived, for both models. To get further insight on the problem we also derive the asymptotic Cramer-Rao bounds. The performance of the algorithms in the presence of additive white Gaussian noise is illustrated by numerical examples, and compared with the corresponding exact and asymptotic Cramer-Rao bounds. The algorithms are shown to be robust in the presence of noise, and their performance close to the CRB, even at moderate signal to noise ratios.

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Index terms: Two-Dimensional Nonhomogeneous Signals, Parameter Estimation of Nonhomogeneous Signals, Two-Dimensional Phase Differencing Operator, High-Order Ambiguity Function, Cramer-Rao bound.

1 Introduction

Many two-dimensional (2-D) signal processing applications require modeling and analysis of non-homogeneous signals. In fact, almost any application where image interpretation is required, has to face the challenge of analyzing non-homogeneities in the observed image. This is due to the fact that, essentially in any image taken by a camera perspective exists. Hence, even if in its own coordinate system the surface of an observed object was homogeneous, its projection on the 2-D image plan produces a 2-D non-homogeneous image. Moreover, applications requiring the processing of 2-D non-homogeneous signals are not restricted to image processing problems alone. For example, the problem of modeling and analyzing Synthetic Aperture Radar (SAR) data, and in particular Interferometric SAR (INSAR) images, involves the analysis of complex valued 2-D non-homogeneous signals.

Perspective estimation is a key problem in many image modeling and understanding applications. A problem closely related to that of perspective estimation is the problem estimating the shape of a 3-D rigid body based on its surface texture information. This problem is usually referred to as *shape from texture*. Based on the 2-D Wold decomposition it was shown in [7] that the deterministic component of a homogeneous texture field can be approximated by a sum of 2-D sinusoids. Since continuous functions can be approximated by polynomials, a natural choice for modeling any continuous coordinate transformation of such deterministic component is the multi-component model, where each component is of a constant amplitude times a sine of a polynomial function of the field coordinates. A solution to the problem of parametric modeling and estimation of homogeneous textures that appear nonhomogeneous due to the perspective projection, is essential for solving problems such as camera calibration and the computation of shape from texture.

Existing solutions to problems where perspective estimation is involved are traditionally based on local analysis of the image, using its edge information.(See, *e.g.*, [15].) Recently, a nonparametric approach for finding the three-dimensional orientation of a planar surface from its texture information, has been suggested in [17]. The algorithm evaluates the dominant frequency at each image point using the wavelet transform, and then employs the spatial dependence of this frequency component to estimate the surface orientation. In [19], we present a parametric solution to the problem of estimating the orientation of a planar surface using an algorithm developed in the next sections. A different method for estimating and canceling, the effects of perspective based on the 1-D Chirplet transform was suggested in

[18]. In [16] the problem of estimating the shape of a smooth curved surface from its texture information is investigated.

In this paper we study a special case of the foregoing general model, namely, the single component constant amplitude sine of a polynomial function of the field coordinates. This model belongs to the general class of AM-FM signals, [3], [4].

As we have previously indicated, applications requiring the processing of 2-D non-homogeneous signals are not restricted to image processing problems alone. In [8] we present a model based, 2-D phase unwrapping algorithm for complex valued 2-D signals with continuous phase functions. Since the signal phase is a function of the coordinates, the signal is necessarily non-homogeneous. The basic building block of this phase unwrapping algorithm is a parameter estimation algorithm which is investigated in this paper. Since 2-D continuous functions can be approximated by 2-D polynomials, the first step of the phase unwrapping algorithm is to fit a 2-D polynomial model to the observed phase. The estimated phase is then used as a reference information that directs the actual phase unwrapping process: The phase of each sample of the observed field is unwrapped by increasing (decreasing) it by the multiple of 2π, which is the nearest to the difference between the principle value of the phase and the estimated phase value at this coordinate.

The estimation algorithms presented in this paper are designed to work with complex valued constant amplitude polynomial phase signals. In applications where the 2-D signal is real, it can be converted into complex form through the Hilbert Transform [5], [6]. However, as we show in [19] this procedure causes some degradation in the algorithm performance.

We present suboptimal (relative to the maximum likelihood estimator), but computationally efficient algorithms (since no multi-dimensional search in the parameter space is required), for estimating the signal parameters, given noisy observations of it. The algorithms proposed in this paper are based on the properties of a 2-D polynomial phase difference operator. The derivation of this operator extends the derivation of the 1-D high-order ambiguity function, [1], [20], to the case of 2-D signals. More specifically, we propose in this paper two models of polynomial phase signals, one with triangular support polynomial phase and the second with rectangular support polynomial phase. As we show later, the estimation algorithm of the rectangular support polynomial phase signal is computationally simpler than the algorithm for estimating the parameters of a triangular support polynomial phase signal. However, this saving in computations comes at a cost of lower accuracy of the estimation results (in terms of error variance).

In this paper we introduce the phase difference operator and show how it can be applied to parametric modeling of 2-D signals. In section 2 we first define the parametric model of the observed signal in general terms. We then define the 2-D phase difference operator, and analyze its properties. In sections 3 and 4 we present the triangular support polynomial phase model and the rectangular support model. The specific properties of each model are investigated and employed to derive parameter estimation algorithms for the two models. We note here that part of the results presented in sections 2 and 4 were stated earlier in a less general form, and without proofs in [2] and [8]. In section 5 we derive the exact Cramer-Rao Lower Bound (CRLB) on the accuracy of estimating the model parameters, for both models. Section 6 presents a derivation of the asymptotic CRLB, which gives more insight into the behavior of the bound. In section 7 we illustrate the operation of the proposed algorithms in the presence of noise using some numerical examples and Monte-Carlo simulations.

2 The Phase Difference Operator

2.1 The Signal Model and the Phase Difference Operator

Let $\{y(n,m)\}$ be a discrete 2-D random field consisting of the sum of a deterministic complex valued signal, and additive white Gaussian noise. More specifically

$$y(n,m) = v(n,m) + u(n,m) \ , \quad n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1 \ , \tag{1}$$

where

$$v(n,m) = A\exp\{j\phi(n,m)\} \ . \tag{2}$$

The phase $\phi(n,m)$ is a function of the coordinates n and m. At the moment we will not specify the functional dependence of $\phi(n,m)$ on n and m. The amplitude A is a real valued positive constant. The observation noise $u(n,m)$, is assumed to be complex valued, zero mean white Gaussian noise.

We now define the phase difference operator in a general form, which is valid for any complex valued 2-D signal. We then derive and prove its basic properties.

Definition 1, [2]: Let τ_m and τ_n be some positive integers, and let $v(n,m)$ be a complex valued signal. Define

$$\mathrm{PD}_{m^{(0)}}[v(n,m)] \stackrel{\Delta}{=} v(n,m) \ , n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1 \ , \tag{3}$$

$$\mathrm{PD}_{m^{(1)}}[v(n,m)] \stackrel{\Delta}{=} v(n,m)v(n,m+\tau_m)^* \ , \tag{4}$$

where the resulting 2-D signal $\mathrm{PD}_{m^{(1)}}[v(n,m)]$ exists for $n = 0, 1, \ldots, N-1$, $m = 0, 1, \ldots, M-1-\tau_m$. In the following we keep the same type of notation to indicate the indices for which the left hand-side of the equation exists.

In general we have

$$\mathrm{PD}_{m^{(q)}}[v(n,m)] \stackrel{\Delta}{=} \mathrm{PD}_{m^{(q-1)}}[v(n,m)]\Big(\mathrm{PD}_{m^{(q-1)}}[v(n,m+\tau_m)]\Big)^* \ , \\ n = 0, 1, \ldots, N-1 \ , m = 0, 1, \ldots, M-1-q\tau_m \ . \tag{5}$$

Similarly

$$\mathrm{PD}_{n^{(0)}}[v(n,m)] \stackrel{\Delta}{=} v(n,m) \ , n = 0, 1, \ldots, N-1 \ , m = 0, 1, \ldots, M-1 \ , \tag{6}$$

and in general

$$\mathrm{PD}_{n^{(p)}}[v(n,m)] \stackrel{\Delta}{=} \mathrm{PD}_{n^{(p-1)}}[v(n,m)]\Big(\mathrm{PD}_{n^{(p-1)}}[v(n+\tau_n,m)]\Big)^* \ , \\ n = 0, 1, \ldots, N-1-p\tau_n \ , m = 0, 1, \ldots, M-1 \ . \tag{7}$$

The operators are called phase differencing operators since they perform on the phase of the observed 2-D discrete signal, an operation which is equivalent to phase differentiation of a continuous parameter 2-D phase. Later in this section we provide an alternative representation and interpretation of the operators $\mathrm{PD}_{m^{(q)}}[\cdot]$, and $\mathrm{PD}_{n^{(p)}}[\cdot]$.

2.2 The ∇_m and ∇_n Difference Operators

In principle, phase differencing can be accomplished either by applying the PD operator to the 2-D signal directly, or by first extracting the phase of the signal, and then differencing the phase. In practice, the first approach is to be preferred, since it can be accomplished *without* phase unwrapping. Hence we adopt this approach in this paper. Nevertheless, some of the properties of the PD operator are more easily proven by investigating the latter approach. Therefore, we introduce here the difference operator which operates on the phase function, and investigate some of its properties. These properties are later used in the proofs of Theorem 1 and Theorem 2. To distinguish this operator from the PD operator, we will call it the "$\nabla_m, (\nabla_n)$ Difference Operator".

Definition 2: Let τ_m and τ_n be some positive integers. The ∇_m-*difference operator* of a 2-D function $\phi(n, m)$ is a linear operator defined by

$$\nabla_m[\phi(n,m)] = \phi(n,m) - \phi(n, m+\tau_m) \ , \tag{8}$$

i.e., ∇_m is a difference operator along the m-axis. Similarly, the ∇_n-*difference operator* is defined by

$$\nabla_n[\phi(n,m)] = \phi(n,m) - \phi(n+\tau_n, m) \ . \tag{9}$$

It is easy to show using the definitions, and the linearity of the operators, that the difference operations are commutative *i.e.*,

$$\nabla_n\Big[\nabla_m[\phi(n,m)]\Big] = \nabla_m\Big[\nabla_n[\phi(n,m)]\Big] \ . \tag{10}$$

Hence for example

$$\nabla_m\Big[\nabla_n\Big[\nabla_m[\phi(n,m)]\Big]\Big] = \nabla_m\Big[\nabla_m\Big[\nabla_n[\phi(n,m)]\Big]\Big] \ . \tag{11}$$

Assume we have applied the ∇_n difference operator P times, and the ∇_m difference operator $S-P$ times, to $\phi(n,m)$. In the following we denote the resulting signal by $\nabla_{n^{(P)},m^{(S-P)}}[\phi(n,m)]$.

It can be easily verified that

$$\nabla_{n^{(P)},m^{(S-P)}}[\phi(n,m)] = \nabla_{n^{(1)}}\Big\{\nabla_{n^{(P-1)},m^{(S-P)}}[\phi(n,m)]\Big\} \ , \tag{12}$$

and

$$\nabla_{n^{(P)},m^{(S-P)}}[\phi(n,m)] = \nabla_{m^{(1)}}\Big\{\nabla_{n^{(P)},m^{(S-1-P)}}[\phi(n,m)]\Big\} \ . \tag{13}$$

Lemma 1: Assume we have applied, in some arbitrary sequence, P times the operator ∇_n, and $K-P$ times the operator ∇_m, to $\phi(n,m)$. Then, the resulting signal is given by

$$\nabla_{n^{(P)},m^{(K-P)}}[\phi(n,m)] = \sum_{q=0}^{K-P}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\binom{K-P}{q}\phi(n+p\tau_n, m+q\tau_m) \ . \tag{14}$$

Proof: By induction. Let K=1. Hence, two cases are possible: $P=1$, and $P=0$. For the $P=1$ case we have

$$\begin{aligned}\nabla_{n^{(1)},m^{(0)}}[\phi(n,m)] &= \sum_{p=0}^{1}(-1)^p\binom{1}{p}\phi(n+p\tau_n, m) \\ &= \phi(n,m) - \phi(n+\tau_n, m) \ ,\end{aligned} \tag{15}$$

as the difference operator in the n direction, ∇_n, is defined. Similar derivation for the $P=0$ case, results in the definition of the difference operator in the m direction.

Assume now that the lemma holds for some arbitrary K. We shall prove that it holds for $K+1$ as well. Assume that the $K+1$-th application of the phase difference operator is in the m direction. Similar derivation holds in the alternative case, *i.e.*, the case in which the $K+1$ application of the phase difference operator is in the n direction.

$$
\begin{aligned}
&\nabla_{n^{(P)},m^{(K+1-P)}}\Big[\phi(n,m)\Big] \\
&= \nabla_m\Big[\nabla_{n^{(P)},m^{(K-P)}}[\phi(n,m)]\Big] \\
&= \nabla_m\Big[\sum_{q=0}^{K-P}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\binom{K-P}{q}\phi(n+p\tau_n,m+q\tau_m)\Big] \\
&= \sum_{q=0}^{K-P}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\binom{K-P}{q}\phi(n+p\tau_n,m+q\tau_m) \\
&\quad -\sum_{q=0}^{K-P}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\binom{K-P}{q}\phi(n+p\tau_n,m+(q+1)\tau_m) \\
&= \sum_{q=0}^{K-P}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\binom{K-P}{q}\phi(n+p\tau_n,m+q\tau_m) \\
&\quad +\sum_{\ell=1}^{K-P+1}\sum_{p=0}^{P}(-1)^{p+\ell}\binom{P}{p}\binom{K-P}{\ell-1}\phi(n+p\tau_n,m+\ell\tau_m) \\
&= \sum_{p=0}^{P}(-1)^{p}\binom{P}{p}\phi(n+p\tau_n,m) \\
&\quad +\sum_{q=1}^{K-P}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\Big[\binom{K-P}{q}+\binom{K-P}{q-1}\Big]\phi(n+p\tau_n,m+q\tau_m) \\
&\quad +\sum_{p=0}^{P}(-1)^{p+K-P+1}\binom{P}{p}\phi(n+p\tau_n,m+(K-P+1)\tau_m) \\
&= \sum_{q=0}^{K-P+1}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\binom{K-P+1}{q}\phi(n+p\tau_n,m+q\tau_m)\ .
\end{aligned}
\tag{16}
$$

The second equality in (16) results from the induction assumption, and the third equality is due to the definition of the operator ∇_m. The last equality results from the identity

$$\binom{Q-1}{q} + \binom{Q-1}{q-1} = \binom{Q}{q} . \tag{17}$$

∎

2.3 Alternative Representations of the PD_n and PD_m Operators

Let $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]$ denote the signal obtained by sequentially applying the phase difference operator $\mathrm{PD}_{n^{(1)}}$ P times, and the phase difference operator $\mathrm{PD}_{m^{(1)}}$ $S-P$ times, to some complex-valued 2-D signal $v(n,m)$.

Lemma 2:

$$\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \prod_{q=0}^{S-P} \left\{ \prod_{p=0}^{P} \left\{ \left[v^{\left(*^{(p+q)}\right)}(n+p\tau_n, m+q\tau_m) \right] \right\}^{\binom{P}{p}} \right\}^{\binom{S-P}{q}} \tag{18}$$

where we define

$$v^{\left(*^{(p+q)}\right)}(n+p\tau_n, m+q\tau_m) = \begin{cases} v(n+p\tau_n, m+q\tau_m), & p+q \text{ even} \\ v^*(n+p\tau_n, m+q\tau_m), & p+q \text{ odd} \end{cases} . \tag{19}$$

Proof: By induction. Let S=1. Hence, two cases are possible: $P=1$, and $P=0$. For the $P=1$ case we have

$$\begin{aligned} \mathrm{PD}_{n^{(1)},m^{(0)}}[v(n,m)] &= \prod_{p=0}^{1} \left\{ \left[v^{\left(*^{(p)}\right)}(n+p\tau_n, m) \right] \right\}^{\binom{1}{p}} \\ &= v(n,m)v^*(n+\tau_n, m) \ , \end{aligned} \tag{20}$$

as $\mathrm{PD}_{n^{(1)}}$ is defined. Similarly, for the $P=0$ case we have

$$\begin{aligned} \mathrm{PD}_{n^{(0)},m^{(1)}}[v(n,m)] &= \prod_{q=0}^{1} \left\{ \left[v^{\left(*^{(q)}\right)}(n, m+q\tau_m) \right] \right\}^{\binom{1}{q}} \\ &= v(n,m)v^*(n, m+\tau_m) \ , \end{aligned} \tag{21}$$

as $\mathrm{PD}_{m^{(1)}}$ is defined.

Assume now that the lemma holds for some arbitrary $S-1$. We shall prove that it holds for S as well. Assume that the Sth application of the phase difference operator is in the m direction. Similar derivation holds for the case in which the Sth application of the phase difference operator is in the n direction. Hence, using Definition 1, equation (5) (or alternatively (7)),

$$
\begin{aligned}
&\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] \\
&= \mathrm{PD}_{n^{(P)},m^{(S-1-P)}}[v(n,m)]\left(\mathrm{PD}_{n^{(P)},m^{(S-1-P)}}[v(n,m+\tau_m)]\right)^* \\
&= \prod_{q=0}^{S-1-P}\left\{\prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p+q)}\right)}(n+p\tau_n,m+q\tau_m)\right]\right\}^{\binom{P}{p}}\right\}^{\binom{S-1-P}{q}} \\
&\quad\cdot\prod_{q=0}^{S-1-P}\left\{\prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p+q+1)}\right)}(n+p\tau_n,m+(q+1)\tau_m)\right]\right\}^{\binom{P}{p}}\right\}^{\binom{S-1-P}{q}} \\
&= \prod_{q=0}^{S-1-P}\left\{\prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p+q)}\right)}(n+p\tau_n,m+q\tau_m)\right]\right\}^{\binom{P}{p}}\right\}^{\binom{S-1-P}{q}} \\
&\quad\cdot\prod_{\ell=1}^{S-P}\left\{\prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p+\ell)}\right)}(n+p\tau_n,m+\ell\tau_m)\right]\right\}^{\binom{P}{p}}\right\}^{\binom{S-1-P}{\ell-1}} \\
&= \prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p)}\right)}(n+p\tau_n,m)\right]\right\}^{\binom{P}{p}} \\
&\quad\cdot\prod_{q=1}^{S-1-P}\left\{\prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p+q)}\right)}(n+p\tau_n,m+q\tau_m)\right]\right\}^{\binom{P}{p}}\right\}^{\left[\binom{S-1-P}{q}+\binom{S-1-P}{q-1}\right]} \\
&\quad\cdot\prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p+S-P)}\right)}(n+p\tau_n,m+(S-P)\tau_m)\right]\right\}^{\binom{P}{p}} \\
&= \prod_{q=0}^{S-P}\left\{\prod_{p=0}^{P}\left\{\left[v^{\left(*^{(p+q)}\right)}(n+p\tau_n,m+q\tau_m)\right]\right\}^{\binom{P}{p}}\right\}^{\binom{S-P}{q}} .
\end{aligned}
\tag{22}
$$

∎

The next corollary gives an alternative recursive definition of the 2-D polynomial phase differencing operator.

Corollary: Assume we have sequentially applied, in some arbitrary sequence, P times the phase difference operator $\mathrm{PD}_{n^{(1)}}$, and $S-P$ times the phase difference operator $\mathrm{PD}_{m^{(1)}}$, to the signal $v(n,m)$. Then, the resulting signal is given by

$$\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \mathrm{PD}_{n^{(1)}}\left[\mathrm{PD}_{n^{(P-1)},m^{(S-P)}}[v(n,m)]\right] , \tag{23}$$

and similarly by

$$\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] = \mathrm{PD}_{m^{(1)}}\left[\mathrm{PD}_{n^{(P)},m^{(S-1-P)}}[v(n,m)]\right] . \tag{24}$$

Proof: The proof follows immediately from the definition of $\mathrm{PD}_{m^{(1)}}$, (or $\mathrm{PD}_{n^{(1)}}$), and (22).

In sections 3 through 4, in order to simplify the presentation, we discuss the case in which there is no observation noise and $A \equiv 1$. Hence, in (1), $y(n,m) = v(n,m) = \exp\{j\phi(n,m)\}$. We specialize the results of this section by assuming two specific polynomial models for the phase function. These are the triangular support polynomial phase model and the rectangular support polynomial phase model.

3 The Rectangular Support Polynomial Phase Model

3.1 Model Definition and Properties

Assume $\phi(n,m)$ in (2) obeys the following polynomial model

$$\phi(n,m) = \sum_{k=0}^{P}\sum_{\ell=0}^{Q} c(k,\ell) n^k m^\ell . \tag{25}$$

where P and Q are two non-negative integers denoting the order of the polynomial in n and m respectively. In the following we denote this rectangular support polynomial phase function by $\phi_{P,Q}(n,m)$. As an example, Fig. 1 illustrates the support for $P = Q = 3$.

Let us first give a brief heuristic explanation of the idea behind the operation of the PD operator, when applied to a rectangular support polynomial phase signal. Consider the polynomial phase function in (25), and assume for the moment that m and n are continuous variables. By differentiating the phase of the observed signal Q times along the m axis, we eliminate the dependence on m, leaving a polynomial in n. Since the phase function is independent of m we can average the 2-D signal along the m-axis, to reduce it to a 1-D signal. The 1-D signal has a constant amplitude and a polynomial phase of order P.

Differentiating the phase of this 1-D signal $P-1$ times with respect to n, we get a complex exponential whose phase is a first order polynomial in n. It can be shown that the

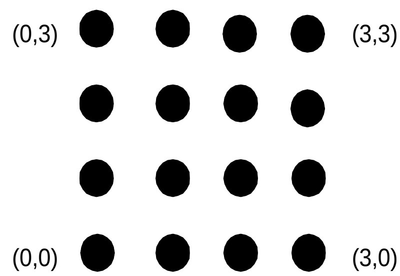

Figure 1. Rectangular support with $P = Q = 3$.

frequency of this complex exponential is a function of the highest order coefficient of the 1-D phase polynomial, and other known quantities. Therefore, by estimating the frequency of the complex exponential we obtain an estimate of the highest order polynomial coefficient. By removing the phase contribution related to this coefficient, and repeating the procedure described above, the other phase coefficients can be estimated as well. Having completed the estimation of the 1-D signal, it is now possible to reduce the order of the 2-D signal and repeat the entire process, until all the phase parameters have been estimated. The details of how this is done in practice follow.

Lemma 3:

$$\nabla_{m^{(Q)}}\{\phi_{P,Q}(n,m)\} = \sum_{k=0}^{P} c_Q(k) n^k \tag{26}$$

where

$$c_Q(k) = c(k,Q)Q!(-\tau_m)^Q \ , k = 0, \ldots, P \ . \tag{27}$$

Hence, $\nabla_{m^{(Q)}}\{\phi_{P,Q}(n,m)\}$ is a 2-D polynomial function of n only.

Proof: By induction. Let Q=1. Hence,

$$\begin{aligned}
\nabla_m\{\phi_{P,1}(n,m)\} &= \phi_{P,1}(n,m) - \phi_{P,1}(n,m+\tau_m) \\
&= \sum_{k=0}^{P}\sum_{\ell=0}^{1} c(k,\ell)n^k[m^\ell - (m+\tau_m)^\ell] \\
&= \sum_{k=0}^{P} -(\tau_m)c(k,1)n^k
\end{aligned}$$

$$= \sum_{k=0}^{P} c_1(k) n^k \ , \qquad (28)$$

which is a function of n only.

Assume now that the lemma holds for some arbitrary Q. We shall prove it holds for $Q+1$ as well.

$$\begin{aligned} \nabla_{m^{(Q+1)}}\Big\{\phi_{P,Q+1}(n,m)\Big\} &= \nabla_{m^{(Q+1)}}\Big\{\sum_{k=0}^{P} c(k,Q+1)n^k m^{Q+1} + \sum_{k=0}^{P}\sum_{\ell=0}^{Q} c(k,\ell)n^k m^\ell\Big\} \\ &= \sum_{k=0}^{P} c(k,Q+1)n^k \nabla_{m^{(Q+1)}}\{m^{Q+1}\} + \nabla_m\Big\{\nabla_{m^{(Q)}}\{\phi_{P,Q}(n,m)\}\Big\} \\ &= \sum_{k=0}^{P} c(k,Q+1)n^k \nabla_{m^{(Q)}}\Big\{\nabla_m\{m^{Q+1}\}\Big\} \\ &= \sum_{k=0}^{P} c(k,Q+1)n^k \nabla_{m^{(Q)}}\Big\{m^{Q+1} - (m+\tau_m)^{Q+1}\Big\} \\ &= \sum_{k=0}^{P} c(k,Q+1)n^k \nabla_{m^{(Q)}}\Big\{-\sum_{i=0}^{Q}\binom{Q+1}{i}\tau_m^{Q+1-i} m^i\Big\} \ , \qquad (29) \end{aligned}$$

where the second equality results from the linearity of the difference operator and the third is due to the induction assumption $\nabla_{m^{(Q)}}\{\phi_{P,Q}(n,m)\} = \sum_{k=0}^{P} c_Q(k)n^k$ and the fact that $\nabla_m\{\sum_{k=0}^{P} c_Q(k)n^k\} \equiv 0$.

Note that $\sum_{i=0}^{Q}\binom{Q+1}{i}\tau_m^{Q+1-i}m^i$ is a 1-D polynomial in m, of order Q. Hence, [10],

$$\nabla_{m^{(Q-1)}}\Big\{-\sum_{i=0}^{Q}\binom{Q+1}{i}\tau_m^{Q+1-i}m^i\Big\} = -(\binom{Q+1}{Q}\tau_m)Q!(-\tau_m)^{Q-1}m + \gamma_Q(\tau_m) \qquad (30)$$

where $\gamma_Q(\tau_m)$ in not a function of m. Substituting (30) into (29) we have

$$\begin{aligned} \nabla_{m^{(Q+1)}}\Big\{\phi_{P,Q+1}(n,m)\Big\} &= \sum_{k=0}^{P} c(k,Q+1)n^k \nabla_m\Big\{-(Q+1)!(-\tau_m)^Q m + \gamma_Q(\tau_m)\Big\} \\ &= -\sum_{k=0}^{P} c(k,Q+1)n^k\Big\{(Q+1)!(-\tau_m)^Q[m-(m+\tau_m)]\Big\} \\ &= \sum_{k=0}^{P} c(k,Q+1)n^k (Q+1)!(-\tau_m)^{Q+1} \ . \ \blacksquare \qquad (31) \end{aligned}$$

We can now state the main result of this section. This result establishes the functional relation between the phase coefficients of the 2-D signal $\mathrm{PD}_{m^{(Q)}}[v(n,m)]$, obtained by successively applying the operator $\mathrm{PD}_m[\cdot]$ to the polynomial phase signal $v(n,m)$ whose phase function is given in (25), and the phase coefficients of $v(n,m)$.

3.2 The Representation of $\mathrm{PD}_{m^{(Q)}}[v(n,m)]$

Theorem 1: Let $v(n,m)$ be given by (2), (25). Then, the signal $\mathrm{PD}_{m^{(Q)}}[v(n,m)]$ is polynomial phase in n only, and is given by

$$\mathrm{PD}_{m^{(Q)}}[v(n,m)] = \exp\left\{ j \sum_{k=0}^{P} c_Q(k) n^k \right\} \ , n = 0,1,\ldots,N-1 \ , m = 0,1,\ldots,M-1-Q\tau_m \ , \tag{32}$$

where its polynomial phase coefficients are given by

$$c_Q(k) = c(k,Q)Q!(-\tau_m)^Q \ , k = 0,\ldots,P \ . \tag{33}$$

Proof: Consider the 2-D signal

$$\begin{aligned}
\exp\left\{ j \sum_{k=0}^{P} c_Q(k) n^k \right\} &= \exp\left\{ j \nabla_{m^{(Q)}} \{ \phi_{P,Q}(n,m) \} \right\} \\
&= \exp\left\{ j \sum_{q=0}^{Q} (-1)^q \binom{Q}{q} \phi_{P,Q}(n, m+q\tau_m) \right\} \\
&= \prod_{q=0}^{Q} \exp\left\{ j (-1)^q \binom{Q}{q} \phi_{P,Q}(n, m+q\tau_m) \right\} \\
&= \prod_{q=0}^{Q} \left\{ \left[\exp\{ j\phi_{P,Q}(n, m+q\tau_m) \} \right]^{(-1)^q} \right\}^{\binom{Q}{q}} \\
&= \prod_{q=0}^{Q} \left\{ v^{(*^{(q)})}(n, m+q\tau_m) \right\}^{\binom{Q}{q}} \\
&= \mathrm{PD}_{m^{(Q)}}[v(n,m)] \ ,
\end{aligned} \tag{34}$$

where the first equality is due to Lemma 3, the second equality is due to Lemma 1, with $P = 0$ and $K = Q$, and the last equality is due to Lemma 2, with $P = 0$ and $S = Q$. ∎

Note from the PD operator definition (5) and Theorem 1 that for a 2-D polynomial phase signal $v(n,m)$ of order (P,Q),

$$\begin{aligned}\mathrm{PD}_{m^{(Q+1)}}[v(n,m)] &= \mathrm{PD}_{m^{(Q)}}[v(n,m)]\Big(\mathrm{PD}_{m^{(Q)}}[v(n,m+\tau_m)]\Big)^* \\ &= \exp\left\{j\sum_{k=0}^{P} c_Q(k)n^k\right\}\left(\exp\left\{j\sum_{k=0}^{P} c_Q(k)n^k\right\}\right)^* \\ &= 1 \ . \end{aligned} \qquad (35)$$

Hence, for all $L > Q$ applying the operator $\mathrm{PD}_{m^{(L)}}[\cdot]$ to a 2-D polynomial phase signal $v(n,m)$ of order (P,Q) yields

$$\mathrm{PD}_{m^{(L)}}[v(n,m)] \equiv 1 \ . \qquad (36)$$

3.3 The Parameter Estimation Algorithm

Consider the observed signal given by (2), (25), where P and Q are two non-negative integers, which initially, we assume to be known. We now present an algorithm for sequentially estimating the parameters $c(p,q)$, $p = 0,\ldots,P$, $q = 0,\ldots,Q$ of the 2-D polynomial phase signal.

Theorem 1 implies that applying the operator $\mathrm{PD}_{m^{(Q)}}[\cdot]$ to the observed signal $v(n,m)$ eliminates the dependence on m of the resulting 2-D signal, thus producing a 2-D polynomial phase signal whose phase is polynomial in n only. Since the phase function is independent of m we can average the 2-D signal along the m-axis, to obtain a 1-D signal whose parametric representation is identical to that of the 2-D signal. This 1-D signal is a *constant amplitude polynomial phase* signal of order P. It is given by

$$\begin{aligned} x^{(P)}(n) &= \frac{1}{M - Q\tau_m}\sum_{m=0}^{M-1-Q\tau_m} \mathrm{PD}_{m^{(Q)}}[v(n,m)] \\ &= \exp\left\{j\sum_{k=0}^{P} c_Q(k)n^k\right\} , \qquad n = 0,1,\ldots,N-1 \ . \end{aligned} \qquad (37)$$

Following [10] we define the 1-D polynomial phase difference operator.

Definition 3: Let τ_n be some positive constant. Define

$$\mathrm{PD}^0[x(n)] \stackrel{\Delta}{=} x(n) \ , n = 0,1,\ldots,N-1 \ , \qquad (38)$$

and in general

$$\mathrm{PD}^P[x(n)] \triangleq \mathrm{PD}^{P-1}[x(n)]\left(\mathrm{PD}^{P-1}[x(n+\tau_n)]\right)^* ,$$
$$n = 0, 1, \ldots, N-1-P\tau_n \ . \tag{39}$$

As before, it can be shown that the PD operator can also be defined recursively, with the PD operator of order P being obtained by P repeated applications of the first order operator. The Pth order ambiguity function of some 1-D signal $x(n)$ is defined as the Fourier transform of $\mathrm{PD}^{P-1}[x(n)]$, [20].

Given observations of a constant amplitude polynomial phase 1-D signal, (37), it is straightforward to show that

$$\mathrm{PD}^{P-1}[x^{(P)}(n)] = \exp\left\{j\omega_P n + \phi_P\right\} , \tag{40}$$

where

$$\omega_P = (-1)^{P-1} P! \tau_n^{P-1} c_Q(P), \tag{41}$$

and ϕ_p is a function of $P, \tau_n, c_Q(P)$, and $c_Q(P-1)$, but not of n.

Note that $\mathrm{PD}^{P-1}[x^{(P)}(n)]$ is a 1-D complex exponential, whose frequency ω_P is a function of the highest order coefficient, $c_Q(P)$, of the phase polynomial.

We have thus shown that given observations on some 2-D constant amplitude, polynomial phase signal $v(n,m)$, this non-homogeneous 2-D signal can be reduced to a 1-D stationary exponential signal. This is done by computing $\mathrm{PD}_{m(Q)}[v(n,m)]$, as defined in (5), followed by computing $\mathrm{PD}^{P-1}[x^{(P)}(n)]$ in (40), where $x^{(P)}(n)$ is the average of $\mathrm{PD}_{m(Q)}[v(n,m)]$ along the m-axis, as given by (37). Moreover, using (41), and (33), we have that the highest order coefficient of the 2-D phase polynomial $c(P,Q)$ can be expressed as the following function of ω_P

$$c(P,Q) = \frac{\omega_P}{(-1)^{P+Q-1} P! \tau_n^{P-1} Q! \tau_m^Q} \ . \tag{42}$$

Hence, estimating ω_P using any standard frequency estimation technique, results in an estimate of $c(P,Q)$. In this work we estimate the frequency of the exponential using a search for the maximum of the absolute value of the signal Discrete Fourier Transform (DFT).

Multiplying the 1-D signal $x^{(P)}(n)$, by $\exp\{-j\hat{c}_Q(P)n^P\}$ results in a new 1-D signal, whose phase is of order $P-1$. We can therefore repeat the procedure used to estimate $c_Q(P)$, to obtain an estimate of $c_Q(P-1)$, and thus an estimate of $c(P-1,Q)$ as well. By repeating for all $p = P, \ldots, 1$ the two basic steps of estimating $c_Q(p)$ (finding the maxima of $|\mathrm{DFT}(\mathrm{PD}^{p-1}[x^{(p)}(n)])|$, followed by multiplying the already reduced order 1-D polynomial phase signal, $x^{(p)}(n)$, by $\exp\{-j\hat{c}_Q(p)n^p\}$), we obtain estimates for all $c(p,Q)\ , p = 1, \ldots, P$. The resulting signal $x^{(0)}(n)$, is a constant phase 1-D signal. Taking the average of the imaginary part of the logarithm of this signal, and scaling it using (33), we obtain an estimate for $c(0,Q)$. We have thus completed the estimation of all the 2-D phase polynomial coefficients, $c(p,Q)\ , p = 0, \ldots, P$, which correspond to Q, the highest power in m.

In the next step of the algorithm we multiply $v(n,m)$ by $\exp\{-j\sum_{p=0}^{P}\hat{c}(p,Q)n^p m^Q\}$ to obtain a 2-D polynomial phase signal of order $(P, Q-1)$. Having reduced the order from Q to $Q-1$, we can now repeat the procedure described above, to estimate $c(p,Q-1)\ , p = 0, \ldots, P$, and again to estimate $c(p, Q-2)\ , p = 0, \ldots, P$, and so on, until all the phase parameters have been estimated.

The amplitude of the polynomial phase signal can be estimated by multiplying the observed signal by $\exp\{-j\sum_{p=0}^{P}\sum_{q=0}^{Q}\hat{c}(p,q)n^p m^q\}$, and averaging the result (Ideally, the resulting 2-D signal is a constant with amplitude A). The algorithm is summarized in Table 1.

It was shown in (36) that overestimating the order of the phase polynomial yields zero values for the non-existing coefficients. In other words, for $L > Q$ applying the operator $\mathrm{PD}_{m^{(L)}}[\cdot]$ to a 2-D polynomial phase signal $v(n,m)$ of order (P,Q), yields $\mathrm{PD}_{m^{(L)}}[v(n,m)] \equiv 1$. Since a similar property exists for the 1-D phase difference operator, the algorithm results in $c(p,q) \equiv 0$ for $p > P$, or $q > Q$, and hence allows for relatively simple order estimation in cases where the polynomial order (P,Q) is unknown.

Assuming some arbitrarily high P and Q, the decision that $\hat{c}(k,\ell) = 0$ can be based on comparison with the Cramer-Rao bound, which is developed in Section 5. In the presence of observation noise, we decide that $c(k,\ell) = 0$ whenever $|\hat{c}(k,\ell)|$ is not considerably higher than $\{\mathrm{CRB}[c(k,\ell)]\}^{\frac{1}{2}}$.

4 The Triangular Support Polynomial Phase Model

Assume $\phi(n,m)$ in (2) obeys the following polynomial model

Let $v^{(P,Q)}(n,m) = v(n,m),\ n = 0,\ldots,N-1,\ m = 0,\ldots,M-1.$

For q=Q, ..., 0

$x^{(P)}(n) = \frac{1}{M-q\tau_m}\sum_{m=0}^{M-1-q\tau_m} \mathrm{PD}_{m^{(q)}}[v^{(P,q)}(n,m)], \quad n = 0,\ldots,N-1$

For p=P, ..., 1

$\hat{\omega}_p = \underset{\omega}{\operatorname{argmax}} \left| \mathrm{DFT}(\mathrm{PD}^{p-1}[x^{(p)}(n)]) \right|$

$\hat{c}_q(p) = \frac{\hat{\omega}_p}{(-1)^{p-1}p!\tau_n^{p-1}}$

$\hat{c}(p,q) = \frac{\hat{c}_q(p)}{q!(-\tau_m)^q}$

$x^{(p-1)}(n) = x^{(p)}(n)\exp\{-j\hat{c}_q(p)n^p\}$

end

$c(0,q) = \frac{1}{q!(-\tau_m)^q N}\sum_{n=0}^{N-1} \mathrm{Im}\{\log(x^{(0)}(n))\}$

$v^{(P,q-1)}(n,m) = v^{(P,q)}(n,m)\exp\{-j\sum_{p=0}^{P}\hat{c}(p,q)n^p m^q\}$

end

Table 1. The Rectangular Support Polynomial Phase Signal Estimation Algorithm.

$$\phi(n,m) = \sum_{(k,\ell)\in I} c(k,\ell)n^k m^\ell \ , \qquad (43)$$

where $I = \{0 \leq k,\ell \text{ and } 0 \leq k+\ell \leq S+1\}$. In the following we denote this phase function by $\phi_{S+1}(n,m)$ and call it a triangular support 2-D polynomial of *total-degree* $S+1$. Thus, one might think of the phase polynomial $\phi_S(n,m)$, as if it has S 'layers' since increasing S by one adds a 'layer' of additional $S+2$ parameters to the phase model. To further illustrate the definition we depict in Fig. 2 a triangular support of total-degree 4.

Before addressing in detail the properties of the signal produced by applying the PD operator to a signal with triangular support polynomial phase, we would like to refer the interested reader to [2] where a brief heuristic explanation of the idea behind the proposed

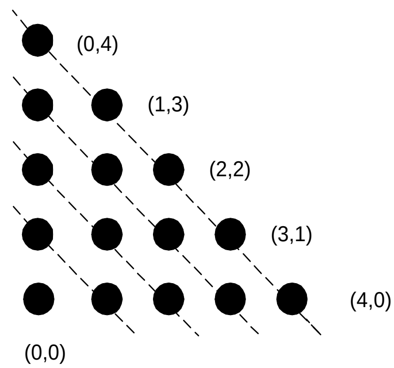

Figure 2. Triangular support of total-degree 4. Diagonal lines indicate layers 1 through 4.

algorithm for estimating the parameters of constant amplitude triangular support polynomial phase signals, can be found.

Note that applying any of the operators $\mathrm{PD}_{m^{(1)}}[\cdot]$, or $\mathrm{PD}_{n^{(1)}}[\cdot]$ to a 2-D polynomial phase signal of total-degree $S+1$, results in a 2-D polynomial phase signal of total-degree S.

Lemma 4:

$$\nabla_{n^{(P)},m^{(S-P)}}\Big[\sum_{k=0}^{S+1} c(k,S+1-k)n^k m^{S+1-k}\Big] = \omega_S n + \nu_S m + \tilde{\gamma}_S(\tau_n,\tau_m) \ , \qquad (44)$$

where,

$$\omega_S = (-1)^S c(P+1,S-P)(P+1)!(S-P)!\tau_n^P \tau_m^{S-P} \ , \qquad (45)$$

$$\nu_S = (-1)^S c(P,S+1-P)P!(S+1-P)!\tau_n^P \tau_m^{S-P} \ , \qquad (46)$$

and $\tilde{\gamma}_S(\tau_n,\tau_m)$ is not a function of m nor n.

Proof: From the properties of the 1-D PD operator, [10], we have that

$$\nabla_{m^{(M-1)}}[m^M] = M!(-\tau_m)^{M-1}m + \gamma_M(\tau_m) \ , \qquad (47)$$

where $\gamma_M(\tau_m)$ is not a function of m,

$$\nabla_{m^{(M)}}[m^M] = M!(-\tau_m)^M \ , \qquad (48)$$

and

$$\nabla_{m^{(M+1)}}[m^M] = 0 \ . \tag{49}$$

Hence,

$$\begin{aligned}
&\nabla_{m^{(S-P)}}\Big[\nabla_{n^{(P)}}\Big[\sum_{k=0}^{S+1}[c(k,S+1-k)n^k m^{S+1-k}]\Big]\Big] \\
&= \nabla_{m^{(S-P)}}\Big[\nabla_{n^{(P)}}\Big[\sum_{k=0}^{P-1}[c(k,S+1-k)m^{S+1-k}]n^k + \sum_{k=P}^{S+1}[c(k,S+1-k)m^{S+1-k}]n^k\Big]\Big] \\
&= \sum_{k=P}^{S+1}\nabla_{n^{(P)}}\Big\{n^k\nabla_{m^{(S-P)}}[c(k,S+1-k)m^{S+1-k}]\Big\} \\
&= \nabla_{n^{(P)}}\Big\{n^P\nabla_{m^{(S-P)}}\Big[c(P,S+1-P)m^{S+1-P}\Big]\Big\} \\
&\quad + \nabla_{n^{(P)}}\Big\{n^{P+1}\nabla_{m^{(S-P)}}\Big[c(P+1,S+1-P-1)m^{S+1-P-1}\Big]\Big\} \\
&= c(P,S+1-P)\nabla_{n^{(P)}}\Big\{n^P\Big\}\nabla_{m^{(S-P)}}\Big\{m^{S+1-P}\Big\} \\
&\quad + c(P+1,S-P)\nabla_{n^{(P)}}\Big\{n^{P+1}\Big\}\nabla_{m^{(S-P)}}\Big\{m^{S-P}\Big\} \\
&= c(P,S+1-P)P!(-\tau_n)^P\Big[(S+1-P)!(-\tau_m)^{S-P}m + \gamma_{S+1-P}(\tau_m)\Big] \\
&\quad + c(P+1,S-P)\Big[(P+1)!(-\tau_n)^P n + \gamma_{P+1}(\tau_n)\Big](S-P)!(-\tau_m)^{S-P} \\
&= \nu_S m + \omega_S n + \tilde{\gamma}_S(\tau_n,\tau_m) \ ,
\end{aligned} \tag{50}$$

where

$$\tilde{\gamma}_S(\tau_n,\tau_m) = c(P,S+1-P)P!(-\tau_n)^P\gamma_{S+1-P}(\tau_m)+c(P+1,S-P)\gamma_{P+1}(\tau_n)(S-P)!(-\tau_m)^{S-P} \ . \tag{51}$$

The second and third equalities in (50) are due to (49) and the linearity of the difference operators. Note that since $\gamma_{S+1-P}(\tau_m)$, $\gamma_{P+1}(\tau_n)$ are not functions of m nor n, $\tilde{\gamma}_S(\tau_n,\tau_m)$ is also not a function of m nor n. ∎

Lemma 5:

$$\nabla_{n^{(P)},m^{(S-P)}}\Big[\phi_{S+1}(n,m)\Big] = \omega_S n + \nu_S m + \gamma_S(\tau_n,\tau_m) \ , \tag{52}$$

where ω_S, ν_S, are given by (45) and (46), respectively, and $\gamma_S(\tau_n,\tau_m)$ is not a function of m nor n.

Proof: By induction. Let S=1. Hence, two cases are possible: $P = 1$, and $P = 0$. For the $P = 1$ case we have

$$\begin{aligned}\nabla_{n^{(1)},m^{(0)}}[\phi_2(n,m)] &= \phi_2(n,m) - \phi_2(n+\tau_n, m) \\ &= (-1)[2c(2,0)\tau_n n + c(1,1)\tau_n m] + [-c(1,0)\tau_n - c(2,0)\tau_n^2] \ , \end{aligned} \quad (53)$$

which satisfies (52) for $S = 1$ and $P = 1$. Similarly, for the $P = 0$ case we have

$$\begin{aligned}\nabla_{n^{(0)},m^{(1)}}[\phi_2(n,m)] &= \phi_2(n,m) - \phi_2(n, m+\tau_m) \\ &= (-1)[2c(0,2)\tau_m m + c(1,1)\tau_m n] + [-c(0,1)\tau_m - c(0,1)\tau_m^2] \ , \end{aligned} \quad (54)$$

which satisfies (52) for $S = 1$ and $P = 0$.

Assume now that the lemma holds for some arbitrary $S-1$. We shall prove it holds for S as well. Assume that the Sth application of the ∇ difference operator is in the m direction. Similar derivation holds in the alternative case, *i.e.*, the case in which the Sth application of the ∇ difference operator is in the n direction. Using the definition of $\phi_{S+1}(n,m)$ in (43) we have

$$\begin{aligned}
&\nabla_{n^{(P)},m^{(S-P)}}\Big[\phi_{S+1}(n,m)\Big] \\
&= \nabla_{n^{(P)},m^{(S-P)}}\Big[\sum_{k=0}^{S+1} c(k, S+1-k) m^{S+1-k} n^k + \phi_S(n,m)\Big] \\
&= \nabla_{n^{(P)},m^{(S-P)}}\Big[\sum_{k=0}^{S+1} c(k, S+1-k) m^{S+1-k} n^k\Big] + \nabla_m\Big\{\nabla_{n^{(P)},m^{(S-1-P)}}\Big[\phi_S(n,m)\Big]\Big\} \\
&= \nabla_{n^{(P)},m^{(S-P)}}\Big[\sum_{k=0}^{S+1} c(k, S+1-k) m^{S+1-k} n^k\Big] + \nabla_m\Big[\omega_{S-1} n + \nu_{S-1} m + \gamma_{S-1}(\tau_n,\tau_m)\Big] \\
&= \omega_S n + \nu_S m + \Big\{\tilde{\gamma}_S(\tau_n,\tau_m) + \nabla_m\Big[\omega_{S-1} n + \nu_{S-1} m + \gamma_{S-1}(\tau_n,\tau_m)\Big]\Big\} \\
&= \omega_S n + \nu_S m + \gamma_S(\tau_n,\tau_m) \ ,
\end{aligned} \quad (55)$$

since $\nabla_m\Big[\omega_{S-1} n + \nu_{S-1} m + \gamma_{S-1}(\tau_n,\tau_m)\Big]$ is not a function of n nor m. The third equality in (55) results from the induction assumption, and the fourth equality is due to Lemma 4. ∎

Theorem 2: Let $v(n,m)$ be given by (2) and (43). Then, the signal $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]$ is a 2-D exponential given by

$$\begin{aligned}\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] &= \exp\Big\{j[\omega_S n + \nu_S m + \gamma_S(\tau_n,\tau_m)]\Big\} \ , \\ & n = 0,1,\ldots,N-1-P\tau_n \ , m = 0,1,\ldots,M-1-(S-P)\tau_m \ , \end{aligned} \quad (56)$$

where

$$\omega_S = (-1)^S c(P+1, S-P)(P+1)!(S-P)!\tau_n^P \tau_m^{S-P} \ , \tag{57}$$

$$\nu_S = (-1)^S c(P, S+1-P)P!(S+1-P)!\tau_n^P \tau_m^{S-P} \ , \tag{58}$$

and $\gamma_S(\tau_n, \tau_m)$ is not a function of m nor n.

Proof: Consider the 2-D signal

$$\begin{aligned}
&\exp\left\{j[\omega_S n + \nu_S m + \gamma_S(\tau_n, \tau_m)]\right\} \\
&\quad = \exp\left\{j\nabla_{n^{(P)},m^{(S-P)}}[\phi_{S+1}(n,m)]\right\} \\
&\quad = \exp\left\{j\sum_{q=0}^{S-P}\sum_{p=0}^{P}(-1)^{p+q}\binom{P}{p}\binom{S-P}{q}\phi_{S+1}(n+p\tau_n, m+q\tau_m)\right\} \\
&\quad = \prod_{q=0}^{S-P}\prod_{p=0}^{P}\exp\left\{j(-1)^{p+q}\binom{P}{p}\binom{S-P}{q}\phi_{S+1}(n+p\tau_n, m+q\tau_m)\right\} \\
&\quad = \prod_{q=0}^{S-P}\left\{\prod_{p=0}^{P}\left\{\left[\exp\{j\phi_{S+1}(n+p\tau_n, m+q\tau_m)\}\right]^{(-1)^{p+q}}\right\}^{\binom{P}{p}}\right\}^{\binom{S-P}{q}} \\
&\quad = \prod_{q=0}^{S-P}\left\{\prod_{p=0}^{P}\left\{\left[v^{(*^{(p+q)})}(n+p\tau_n, m+q\tau_m)\right]\right\}^{\binom{P}{p}}\right\}^{\binom{S-P}{q}} \\
&\quad = \mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)] \ ,
\end{aligned} \tag{59}$$

where the first equality is due to Lemma 5, the second equality is due to Lemma 1, and the last equality is due to Lemma 2. ∎

Note from the definition of the phase difference operators in Definition 1, and Theorem 2 that for a 2-D polynomial phase signal $v(n,m)$ of total-degree S, $\mathrm{PD}_{n^{(P)},m^{(S-P)}}[v(n,m)]$ is not a function of n nor m. Hence, for all $L \geq S$ applying in some arbitrary sequence, P times the operator $\mathrm{PD}_{n^{(1)}}$, and $L-P$ times the operator $\mathrm{PD}_{m^{(1)}}$, to a 2-D polynomial phase signal $v(n,m)$ of total-degree S yields a constant amplitude signal.

An algorithm for estimating the parameters of 2-D constant amplitude triangular support polynomial phase signals, based on the results of this section, is given in [2]. In sections 6 and 7 we investigate its performance in the presence of observation noise. A detailed analytic evaluation of the performance of this algorithm in the presence of observation noise is presented in [9].

5 The CRB of a 2-D Polynomial Phase Signal in Noise

In this section we derive the Cramer-Rao Bound (CRB) on the error variance in estimating the phase parameters when the signal is observed in white additive Gaussian noise, *i.e.*, the observed field is $\{y(n,m)\}$ given by (1), (2) and (43), (or (25)).

Define,

$$\mathbf{y} = [y(0,0),\ldots,y(0,M-1),y(1,0),\ldots,y(1,M-1),\ldots,\ldots,y(N-1,0),\ldots,y(N-1,M-1)]^T , \tag{60}$$

and similarly let all the phase parameters, of the triangular support phase model be assembled, 'layer' after 'layer' into a vector $\mathbf{c}$

$$\mathbf{c} = [c(0,0);c(0,1),c(1,0);c(0,2),c(1,1),c(2,0);\ldots,\ldots;c(0,S),\ldots,c(S,0)]^T , \tag{61}$$

where we use ';' to distinguish 'layer' from 'layer'. Hence $\mathbf{c}$ is an $\frac{(S+2)(S+1)}{2}$ dimensional vector. For the case of a rectangular support phase model we define $\mathbf{c}$ to be the $(P+1)(Q+1)$ dimensional vector

$$\mathbf{c} = [c(0,0),\ldots,c(0,Q),c(1,0),\ldots,c(1,Q),\ldots,\ldots,c(P,0),\ldots,c(P,Q)]^T . \tag{62}$$

Also let $\mathbf{t}$ be a $NM\times 2$ matrix such that each row of $\mathbf{t}$ contains the pair of indices (n,m) where $n=0,\ldots,N-1;\ m=0,\ldots,M-1$. In the following we will use the following shorthand vector notation for functions of space. Given a scalar function $f(n,m)$, we will denote the column vector consisting of the values of $f(n,m),\ n=0,\ldots,N-1;\ m=0,\ldots,M-1$ by $f(\mathbf{t})$. Using this notation, we denote the vector of phase values of the signal by $\phi(\mathbf{t})$. Hence we can write

$$\mathbf{\Phi} = e^{j\phi(\mathbf{t})} , \tag{63}$$

where $e^{j\phi(\mathbf{t})}$ is an MN dimensional column vector.

Using the above definitions we can rewrite (1) in the following matrix representation:

$$\mathbf{y} = A\mathbf{\Phi} + \mathbf{n} \tag{64}$$

Note that $\mathbf{y}$ is a linear function of the amplitude parameter A, while the phase parameters enter non-linearly through $\mathbf{\Phi}$.

Since the observation noise is assumed to be a complex valued Gaussian field such that its real and imaginary components are independent real Gaussian white noise fields each with zero mean and variance $\frac{\sigma^2}{2}$, the probability density function of the observations is given by

$$p(\mathbf{y};\boldsymbol{\theta}) = \frac{1}{(\pi\sigma^2)^{MN}}\exp\Big\{-\frac{1}{\sigma^2}||\mathbf{y} - A\mathbf{\Phi}||^2\Big\} \tag{65}$$

where $\boldsymbol{\theta} = [\mathbf{c}^T A]^T$ denotes the vector of unknown parameters. The log-likelihood function Λ is then given by

$$\Lambda = -MN \ln \pi - MN \ln \sigma^2 - \frac{1}{\sigma^2}||\mathbf{y} - A\mathbf{\Phi}||^2 \ . \tag{66}$$

To derive the Cramer-Rao bound we use the well-known formula which states that the elements of the Fisher Information Matrix (FIM) are given by

$$\mathbf{F}_{ij} = -E\Big\{\frac{\partial^2\Lambda}{\partial\theta_i\partial\theta_j}\Big\}, \tag{67}$$

and the CRB is simply the inverse of the FIM [12]. Thus, to evaluate the FIM we need to compute the derivatives of the log-likelihood function with respect to the various parameters of interest, and take their expected value.

5.1 Computation of the Derivatives

Taking the partial derivatives of Λ, we have

$$\begin{aligned}\frac{\partial\Lambda}{\partial c(k,\ell)} &= \frac{1}{\sigma^2}\Big\{A\frac{\partial\mathbf{\Phi}^H}{\partial c(k,\ell)}(\mathbf{y} - A\mathbf{\Phi}) + A(\mathbf{y} - A\mathbf{\Phi})^H\frac{\partial\mathbf{\Phi}}{\partial c(k,\ell)}\Big\} \\ &= \frac{2}{\sigma^2}\mathrm{Re}\Big\{A\frac{\partial\mathbf{\Phi}^H}{\partial c(k,\ell)}(\mathbf{y} - A\mathbf{\Phi})\Big\} \quad .\end{aligned} \tag{68}$$

Hence for all (k,ℓ) and (p,q),

$$-E\Big\{\frac{\partial^2\Lambda}{\partial c(k,\ell)\partial c(p,q)}\Big\} = \frac{2}{\sigma^2}\mathrm{Re}\Big\{A\frac{\partial\mathbf{\Phi}^H}{\partial c(k,\ell)}A\frac{\partial\mathbf{\Phi}}{\partial c(p,q)}\Big\} \quad . \tag{69}$$

Let

$$\mathbf{e}_1 = [0,1,\ldots,(N-1)]^T \otimes \mathbf{1_M} \ , \tag{70}$$

$$\mathbf{e}_2 = \mathbf{1}_N \otimes [0,1,\ldots,(M-1)]^T \ , \tag{71}$$

where $\mathbf{1_M}$ and $\mathbf{1_N}$ are M-dimensional and N-dimensional column vectors of ones, respectively, and $\otimes$ is the Kronecker product. In other words, $\mathbf{e}_1$ is the first column of $\mathbf{t}$, and $\mathbf{e}_2$

is its second column. Evaluating now the partial derivatives of $\mathbf{\Phi}$ with respect to the phase parameters yields

$$
\begin{aligned}
A\frac{\partial \mathbf{\Phi}}{\partial c(k,\ell)} &= jA\mathbf{e}_1^k \cdot \mathbf{e}_2^\ell \cdot \mathbf{\Phi} \\
&= j\mathbf{e}_1^k \cdot \mathbf{e}_2^\ell \cdot v(\mathbf{t}) \quad , \qquad (72)
\end{aligned}
$$

where $\cdot$ denotes element by element multiplication of the entries of the vectors, and $v(\mathbf{t})$ is the vector of the signal values at each lattice point (n,m), $n=0,\ldots,N-1;\ m=0,\ldots,M-1$.

Substituting (72) into (69), we have

$$
\begin{aligned}
-E\Big\{\frac{\partial^2\Lambda}{\partial c(k,\ell)\partial c(p,q)}\Big\} &= \frac{2}{\sigma^2}\mathrm{Re}\Big\{(jA\mathbf{e}_1^k \cdot \mathbf{e}_2^\ell \cdot \mathbf{\Phi})^H(jA\mathbf{e}_1^p \cdot \mathbf{e}_2^q \cdot \mathbf{\Phi})\Big\} \\
&= \frac{2A^2}{\sigma^2}\left(\mathbf{e}_1^k \cdot \mathbf{e}_2^\ell\right)^T\left(\mathbf{e}_1^p \cdot \mathbf{e}_2^q\right) \\
&= \frac{2A^2}{\sigma^2}\left(\mathbf{e}_1^{k+p}\right)^T\mathbf{e}_2^{\ell+q} \\
&= \frac{2A^2}{\sigma^2}\sum_{i=1}^{NM}\mathbf{e}_1^{k+p}(i)\mathbf{e}_2^{\ell+q}(i) \\
&= \frac{2A^2}{\sigma^2}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}n^{k+p}m^{\ell+q} \\
&= \frac{2A^2}{\sigma^2}\sum_{n=0}^{N-1}n^{k+p}\sum_{m=0}^{M-1}m^{\ell+q} \ . \qquad (73)
\end{aligned}
$$

Rewriting (73) in a matrix form we obtain the FIM block ,$-E\Big\{\frac{\partial^2\Lambda}{\partial \mathbf{c}^2}\Big\}$, which corresponds to the phase parameters. Note that (73) implies that the FIM block which corresponds to the phase parameters has a separable representation, although the phase is modeled by a non-separable two-dimensional polynomial and the fundamental algebra theorem is not valid for two-dimensional polynomials.

Similarly, using (68) and (72) we find that

$$
\begin{aligned}
-E\Big\{\frac{\partial^2\Lambda}{\partial c(k,\ell)\partial A}\Big\} &= \frac{2}{\sigma^2}\mathrm{Re}\Big\{A\frac{\partial \mathbf{\Phi}^H}{\partial c(k,\ell)}\mathbf{\Phi}\Big\} \\
&= \frac{2}{\sigma^2}\mathrm{Re}\Big\{(jA\mathbf{e}_1^k \cdot \mathbf{e}_2^\ell \cdot \mathbf{\Phi})^H\mathbf{\Phi})\Big\} \\
&= 0 \ , \qquad (74)
\end{aligned}
$$

since $(A\mathbf{e}_1^k \cdot \mathbf{e}_2^\ell \cdot \mathbf{\Phi}^H \mathbf{\Phi})$ is purely real.

Taking the partial derivative of Λ with respect to the amplitude parameter we have

$$\frac{\partial \Lambda}{\partial A} = \frac{2}{\sigma^2} \mathrm{Re}\Big\{ \mathbf{\Phi}^H (\mathbf{y} - A\mathbf{\Phi}) \Big\} \quad . \tag{75}$$

Hence,

$$\begin{aligned} -E\Big\{ \frac{\partial^2 \Lambda}{\partial^2 A} \Big\} &= \frac{2}{\sigma^2} \mathrm{Re}\Big\{ \mathbf{\Phi}^H \mathbf{\Phi} \Big\} \\ &= \frac{2NM}{\sigma^2} \ . \end{aligned} \tag{76}$$

In the case where the noise variance is unknown the FIM needs to be augmented by a row and a column corresponding to derivatives with respect to σ^2. Using (68) we obtain

$$-E\Big\{ \frac{\partial^2 \Lambda}{\partial c(k,\ell) \partial \sigma^2} \Big\} = 0 \ . \tag{77}$$

Similarly, using (75)

$$-E\Big\{ \frac{\partial^2 \Lambda}{\partial A \partial \sigma^2} \Big\} = 0 \ . \tag{78}$$

Finally,

$$-E\Big\{ \frac{\partial^2 \Lambda}{\partial^2 \sigma^2} \Big\} = \frac{NM}{\sigma^4} \ . \tag{79}$$

We therefore conclude that the FIM in (67) is block diagonal, for *any* polynomial phase model. Hence the CRB's on the estimation of the phase parameters, the amplitude parameter, and the observation noise variance are decoupled, where

$$\mathrm{CRB}(A) = \frac{\sigma^2}{2NM} \quad , \tag{80}$$

and

$$\mathrm{CRB}(\sigma^2) = \frac{\sigma^4}{MN} \, . \tag{81}$$

From the decoupling of the bounds and (73) we conclude that the bound on the phase parameters is a function only of the total-degree of the triangular support 2-D polynomial phase function, (or the order (P,Q) of the rectangular support 2-D polynomial phase function), and is independent of the phase parameters. In other words, all the triangular support 2-D polynomial phase signals of total-degree $S+1$ and amplitude A will have the same values of $\mathrm{CRB}(\mathbf{c})$. Similarly, all the rectangular support 2-D polynomial phase signals of order (P,Q)

and amplitude A will have the same values of CRB($\mathbf{c}$). Let SNR $= \frac{A^2}{\sigma^2}$ denote the signal-to-noise ratio. Using (73) we conclude that the CRB on the error variance in estimating the phase parameters $c(k,\ell)$ is inversely proportional to the SNR. Similarly, (80) implies that the CRB on the amplitude parameter is independent of the signal phase and amplitude. Hence all constant amplitude polynomial phase signals, will have the same value of CRB(A).

Finally, we note that since a 2-D exponential is a 2-D polynomial phase signal of total-degree 1, we obtain as a special case of our derivation the bound on the error variance in estimating the frequency, phase, and amplitude parameters of a 2-D exponential, observed in additive complex valued, white Gaussian noise. The latter is a common problem in spectral analysis of images.

For the case of rectangular support polynomial phase signals a more compact representation of the CRB can be obtained. Its derivation is presented in section 5.2.

5.2 The CRB for a Rectangular Support Polynomial Phase Signal

Rewriting (73) using matrix notation we have

$$-E\left\{\frac{\partial^2 \Lambda}{\partial \mathbf{c}^2}\right\} = \frac{2A^2}{\sigma^2}\mathbf{H}_1 \otimes \mathbf{H}_2 \ , \tag{82}$$

where the matrix $-E\left\{\frac{\partial^2 \Lambda}{\partial \mathbf{c}^2}\right\}$ is $(P+1)(Q+1) \times (P+1)(Q+1)$ dimensional. Its $((k+1)Q+(\ell+1),(p+1)Q+(q+1))$ element, is the FIM element which corresponds to the parameters $c(k,\ell), c(p,q)$. $\mathbf{H}_1$ is the $(P+1)\times(P+1)$ matrix and $\mathbf{H}_2$ is the $(Q+1)\times(Q+1)$ matrix which are given by

$$(\mathbf{H}_1)_{(k+1,p+1)} = \sum_{n=0}^{N-1} n^{k+p} \ , \tag{83}$$

and

$$(\mathbf{H}_2)_{(\ell+1,q+1)} = \sum_{m=0}^{M-1} m^{\ell+q} \ . \tag{84}$$

Using (82) we obtain the following compact representation for the CRB on the phase parameters

$$\text{CRB}(\mathbf{c}) = \frac{\sigma^2}{2A^2}\mathbf{H}_1^{-1} \otimes \mathbf{H}_2^{-1} \ . \tag{85}$$

6 The Asymptotic CRB

The derivation of the exact CR bound presented in the previous section requires inversion of the matrix $-E\left\{\frac{\partial^2\Lambda}{\partial \mathbf{c}^2}\right\}$ in (73) whose elements are expressed in terms of products of sums. In order to get further insight into the behavior of the bound on estimating the 2-D phase parameters we derive in this section an approximation for the CRB assuming that N and M are large compared to the polynomial total-degree $S+1$ for the triangular support model, or alternatively that N and M are large compared to P and Q for the rectangular support polynomial phase model.

Evaluating the sum expressions (73) we have, [14]

$$\sum_{n=0}^{N-1} n^{k+p} = \begin{cases} \frac{N^{k+p+1}}{k+p+1} + \frac{1}{2}N^{k+p} + \frac{(k+p)B_1 N^{k+p-1}}{2!} + \ldots, & k+p > 0 \\ N, & k+p = 0 \end{cases} ; \quad (86)$$

where $\{B_k\}$ are Bernoulli numbers. Similar expression is obtained for the summation over m. Hence an approximate expression for (73) is given by

$$\begin{aligned} -E\left\{\frac{\partial^2\Lambda}{\partial c(k,\ell)\partial c(p,q)}\right\} & \\ &= \frac{2A^2}{\sigma^2} N^{k+p+1}\left(\frac{1}{k+p+1} + O(N^{-1})\right) M^{\ell+q+1}\left(\frac{1}{\ell+q+1} + O(M^{-1})\right) \\ &= \frac{2A^2}{\sigma^2} N^{k+p+1} M^{\ell+q+1}\left(\frac{1}{(k+p+1)(\ell+q+1)} + O(N^{-1}) + O(M^{-1})\right) . \end{aligned} \quad (87)$$

If we ignore the $O(N^{-1}$ and $M^{-1})$ terms, we get a simple closed form expression for the elements of the FIM. Inverting this matrix results in the asymptotic CRB for the phase parameters. In section 6.1 we show that for the case of rectangular support polynomial phase signals a closed form expression of the asymptotic CRB can be obtained.

6.1 The Asymptotic CRB for Rectangular Support Phase Model

From (73) we have

$$-E\left\{\frac{\partial^2\Lambda}{\partial \mathbf{c}^2}\right\}_{(k+1)Q+(\ell+1),(p+1)Q+(q+1)} = \frac{2A^2}{\sigma^2}\sum_{n=0}^{N-1} n^{k+p} \sum_{m=0}^{M-1} m^{\ell+q} . \quad (88)$$

Using (82) and (87) we get

$$-E\left\{\frac{\partial^2\Lambda}{\partial\mathbf{c}^2}\right\} = \frac{2A^2}{\sigma^2}N\left\{\mathbf{G}_1\left(\tilde{\mathbf{H}}_1 + O(N^{-1})\right)\mathbf{G}_1\right\} \otimes M\left\{\mathbf{G}_2\left(\tilde{\mathbf{H}}_2 + O(M^{-1})\right)\mathbf{G}_2\right\}$$
$$= \frac{2A^2}{\sigma^2}NM(\mathbf{G}_1 \otimes \mathbf{G}_2)\left\{\left(\tilde{\mathbf{H}}_1 + O(N^{-1})\right) \otimes \left(\tilde{\mathbf{H}}_2 + O(M^{-1})\right)\right\}(\mathbf{G}_1 \otimes \mathbf{G}_2)$$
$$= \frac{2A^2}{\sigma^2}NM(\mathbf{G}_1 \otimes \mathbf{G}_2)\left(\tilde{\mathbf{H}}_1 \otimes \tilde{\mathbf{H}}_2 + O(N^{-1}) + O(M^{-1})\right)(\mathbf{G}_1 \otimes \mathbf{G}_2)\ , \qquad (89)$$

where

$$\mathbf{G}_1 = \mathrm{diag}\{1, \mathrm{N}, \ldots, \mathrm{N}^{\mathrm{P}}\}\ , \qquad (90)$$

$$\mathbf{G}_2 = \mathrm{diag}\{1, \mathrm{M}, \ldots, \mathrm{M}^{\mathrm{Q}}\}\ , \qquad (91)$$

$$(\tilde{\mathbf{H}}_1)_{k+1,p+1} = \frac{1}{k+p+1}\ , \qquad (92)$$

and

$$(\tilde{\mathbf{H}}_2)_{\ell+1,q+1} = \frac{1}{\ell+q+1}\ . \qquad (93)$$

Note that $\tilde{\mathbf{H}}_1$, $\tilde{\mathbf{H}}_1$ are Hilbert matrices. Inverting (89) we have

$$\mathrm{CRB}(\mathbf{c}) = \frac{\sigma^2}{2A^2}\frac{1}{NM}(\mathbf{G}_1^{-1} \otimes \mathbf{G}_2^{-1})\left(\tilde{\mathbf{H}}_1^{-1} \otimes \tilde{\mathbf{H}}_2^{-1} + O(N^{-1}) + O(M^{-1})\right)(\mathbf{G}_1^{-1} \otimes \mathbf{G}_2^{-1})\ , \quad (94)$$

where the elements of the inverse of the Hilbert matrix $\tilde{\mathbf{H}}_1$ are given by (*e.g.*, [11]),

$$(\tilde{\mathbf{H}}_1^{-1})_{k+1,p+1} = \frac{(-1)^{k+p}(P+k+1)(P+p+1)}{k+p+1}\binom{P+k}{k}\binom{P}{k}\binom{P+p}{p}\binom{P}{p}\ , \qquad (95)$$

and similarly for the elements of $\tilde{\mathbf{H}}_2^{-1}$. In particular, note that the diagonal elements of $\tilde{\mathbf{H}}_1^{-1} \otimes \tilde{\mathbf{H}}_2^{-1}$, (which are obtained for $k = p$, $\ell = q$), are given by

$$(\tilde{\mathbf{H}}_1^{-1} \otimes \tilde{\mathbf{H}}_2^{-1})_{(k+1)Q+(\ell+1),(k+1)Q+(\ell+1)}$$
$$= \frac{(P+k+1)^2}{2k+1}\left[\binom{P+k}{k}\binom{P}{k}\right]^2 \frac{(Q+\ell+1)^2}{2\ell+1}\left[\binom{Q+\ell}{\ell}\binom{Q}{\ell}\right]^2 . \quad (96)$$

Substituting (96) into (94) yields the following approximate bound:

$$\mathrm{var}\{\hat{c}(k,\ell)\} \geq \frac{1}{2\ N^{2k+1}\ M^{2\ell+1}\ \mathrm{SNR}}$$
$$\cdot\left[\frac{1}{2k+1} + O(N^{-1})\right]\left[(P+k+1)\binom{P+k}{k}\binom{P}{k}\right]^2$$
$$\cdot\left[\frac{1}{2\ell+1} + O(M^{-1})\right]\left[(Q+\ell+1)\binom{Q+\ell}{\ell}\binom{Q}{\ell}\right]^2 , \qquad (97)$$

where $\mathrm{SNR} = \frac{A^2}{\sigma^2}$ is the signal-to-noise ratio.

We therefore conclude that the CRB on the error variance in estimating the phase parameters $c(k,\ell), k = 0,\ldots,P;\ \ell = 0,\ldots,Q$ is approximately inversely proportional to the SNR, and to $N^{2k+1}M^{2\ell+1}$ where N, M are the numbers of data measurements in each axis.

7 Numerical Examples

In this section we present some numerical examples to illustrate the operation of the 2-D phase difference operator, as well as the parameter estimation algorithms which are based on the operator. We begin with an example that illustrates the operation of the rectangular support phase model estimation algorithm in the presence of noise. An illustrative example of the operation of the triangular support phase model estimation algorithm can be found in [2]. We also investigate the performance of the estimation algorithms for triangular and rectangular support polynomial phase signals by Monte-Carlo simulations and by comparing the estimation error variance with the corresponding exact and asymptotic Cramer-Rao bounds.

7.1 The 2-D Difference Operator: Example

Consider a unit amplitude rectangular support polynomial phase signal with $P = 1$ and $Q = 2$. The observations are subject to an additive white Gaussian noise, such that the SNR = 0dB. In this example the observed field dimensions are $N = 100,\ M = 100$. The phase coefficients are given by

$$\mathbf{c} = \begin{bmatrix} 1 & 1.5\cdot 10^{-1} & 5\cdot 10^{-3} & -1.2\cdot 10^{-1} & 4.5\cdot 10^{-3} & 6.5\cdot 10^{-5} \end{bmatrix}\ , \tag{98}$$

where $\mathbf{c}$ is defined as in (62).

The image of the real part of the observed field $y(n,m)$ is shown in Figure 3, and a plot of the absolute value of its Fourier transform is shown on its right. It is clear from these two figures that the observed signal is nonhomogeneous and is of a broad bandwidth.

Figures 4 through 7 illustrate two steps of the parameter estimation algorithm. Applying the operator $\mathrm{PD}_n^{(1)}$ to the signal, yields a 2-D polynomial phase signal which is a function

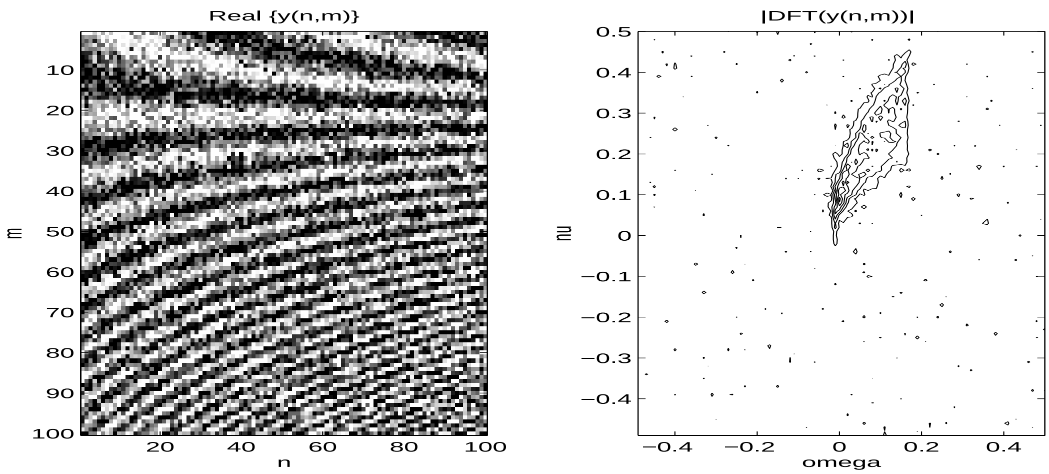

Figure 3. The real part of the observed signal (left), and the magnitude of the signal DFT (right).

of m but not of n. Its phase is a polynomial of order two in m. The real part of this signal is shown in the upper left image of Figure 4, and the absolute value of its DFT is shown in the upper right of the same figure. Note that due to the presence of the observation noise, the signal is not constant along the n-axis. Averaging the resulting signal, $\mathrm{PD}_{n^{(1)}}[y(n,m)]$, along the n axis, yields noisy measurements of a 1-D polynomial phase signal, whose phase is of order two in m. The real part of this signal is shown in the bottom left plot of Figure 4, and the absolute value of its DFT is shown in the bottom right of the figure.

Applying the 1-D phase difference operator once, reduces the phase of the 1-D signal to a first order polynomial, *i.e.*, in the absence of noise the resulting signal is a complex exponential. The absolute value of the DFT of this signal is shown in the left plot of Figure 5. Estimating the frequency of the peak results in the estimate of $c(1,2)$. We therefore see that a broad band non-homogeneous 2-D signal, has been reduced to a 1-D stationary signal, in a way that enables us to estimate one of its parameters.

Multiplying the 1-D signal by $\exp\{-j\hat{c}_1(2)m^2\}$ reduces the order of the 1-D polynomial phase from two to one. In the noiseless case this yields another 1-D, stationary, exponential signal, whose DFT magnitude is shown in the right plot of Figure 5. Estimating the frequency of this exponential results in an estimate of $c(1,1)$. Multiplying the 1-D signal by $\exp\{-j\hat{c}_1(1)m\}$ reduces the order of the 1-D polynomial phase from one to zero. $c_1(0)$ can now be computed as the arithmetic average of the imaginary part of the logarithm of the signal. Hence, $c(1,0)$ can be computed, using (33). At this point we have obtained estimates

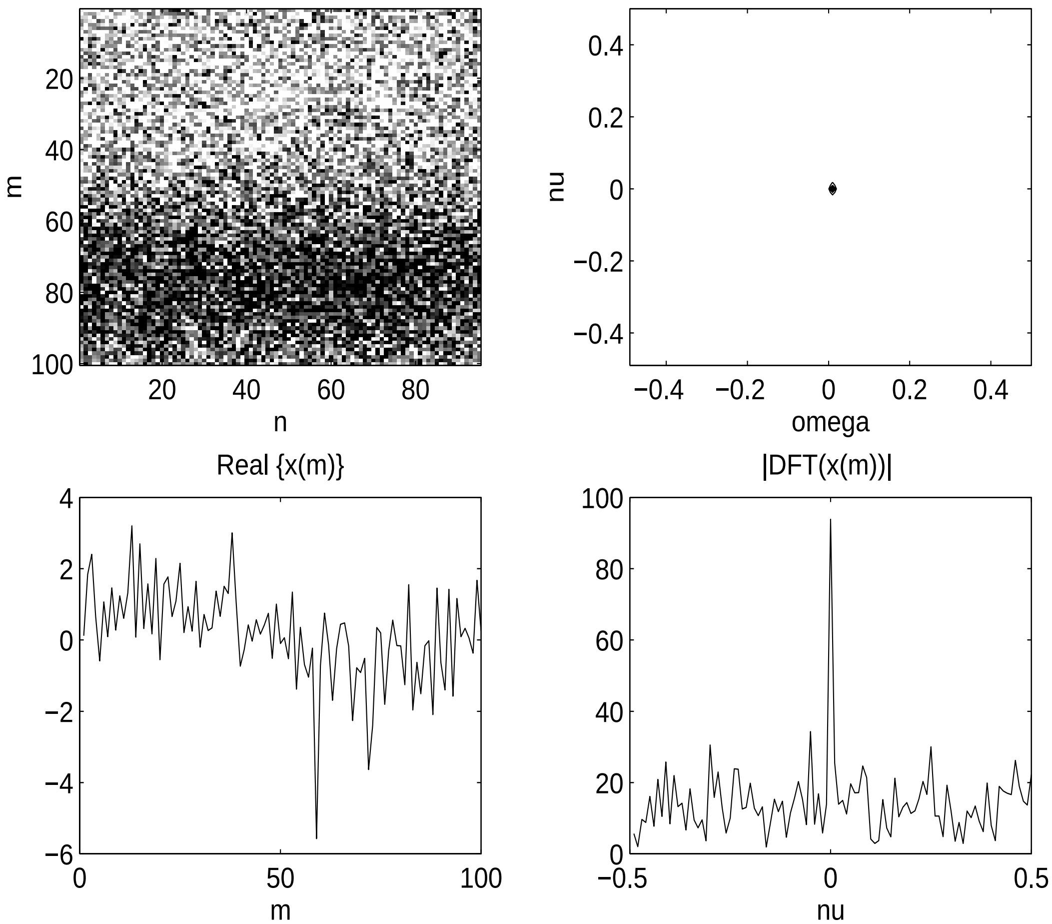

Figure 4. The 2-D polynomial phase signal after applying the operator $\mathrm{PD}_{n^{(1)}}$. Top left: Real part of the resulting 2-D signal. Top right: Absolute value of the 2-D DFT of the resulting signal. Bottom left: Real part of the 1-D signal obtained by averaging the 2-D signal along the n-axis. Bottom right: Absolute value of the 1-D signal DFT.

for $c(1,0), c(1,1)$, and $c(1,2)$.

Multiplying $y(n,m)$ by $\exp\{-j\sum_{\ell=0}^{2}\hat{c}(1,\ell)m^{\ell}n\}$ yields, in the noise free case, a new polynomial phase signal whose degree is zero in the n dimension and two in the m dimension. The real part of this signal is shown in the upper left image of Figure 6, and the absolute value of its DFT is shown in the upper right part of the same figure. Averaging this 2-D signal along the n axis, we obtain noisy observations on a 1-D polynomial phase signal, whose phase is of order two in m. The real part of the 1-D signal is shown in the bottom left plot of Figure 6, and the absolute value of its DFT is shown in the bottom right of the figure.

Applying the 1-D phase difference operator once, reduces the 1-D signal to a complex

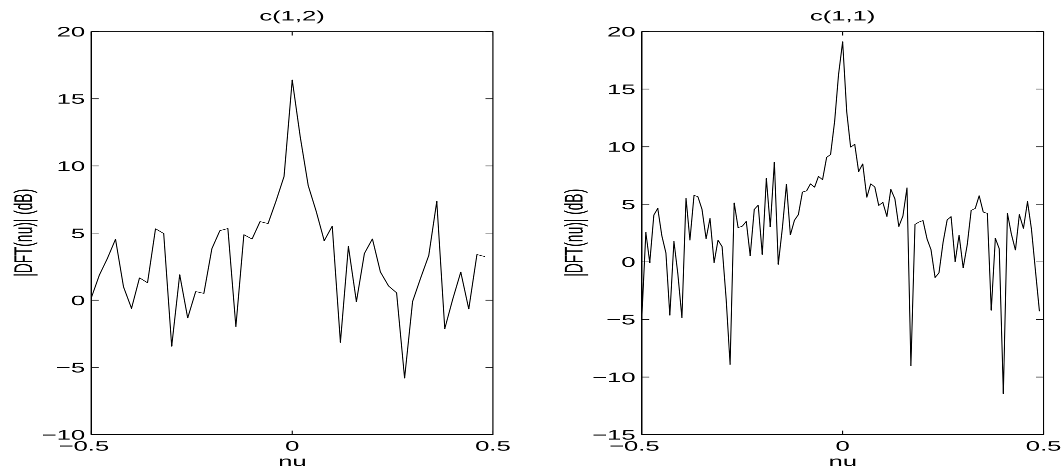

Figure 5. The 1-D exponentials obtained by sequentially applying the 1-D polynomial phase difference operator to $x(m)$. Estimating the frequency of the exponential in the left plot produces an estimate of $c(1,2)$. Estimating the frequency of the exponential in the right plot produces an estimate of $c(1,1)$.

exponential in noise. The absolute value of this signal DFT is shown in the left plot of Figure 7. Estimating the frequency of the peak results in the estimate of $c(0,2)$. Multiplying the 1-D signal by $\exp\{-j\hat{c}_0(2)m^2\}$ reduces the degree of the 1-D polynomial phase signal to one, yielding an exponential signal in noise. Its DFT magnitude is shown in the right hand-side plot of Figure 7. Estimating the frequency of this exponential results in an estimate for $c(0,1)$. Multiplying the 1-D signal by $\exp\{-j\hat{c}_0(1)m\}$ reduces the order of the 1-D polynomial phase from one to zero. $c_0(0)$ can now be computed as the arithmetic average of the imaginary part of the logarithm of the residual signal. Hence, $c(0,0)$ can be computed. At this point we have completed the estimation of all the phase parameters of the 2-D signal. The estimated parameter vector is given by

$$\hat{\mathbf{c}} = \left[\, 1.003,\ 1.603 \cdot 10^{-1},\ 4.879 \cdot 10^{-3}, -1.197 \cdot 10^{-1},\ 4.283 \cdot 10^{-3},\ 6.748 \cdot 10^{-5} \,\right] . \quad (99)$$

7.2 Polynomial Phase Order Estimation

Consider a unit amplitude triangular support polynomial phase signal of total-degree 2. The observations are subject to an additive white Gaussian noise, such that the SNR = 10dB. In this example the observed field dimensions are $N = 100,\ M = 100$. The phase coefficients are given by

$$\mathbf{c} = \left[\, 1; 4.5 \cdot 10^{-1}, 8.2 \cdot 10^{-1}; -1.5 \cdot 10^{-3}, 16 \cdot 10^{-3}, -2.2 \cdot 10^{-3} \,\right] , \quad (100)$$

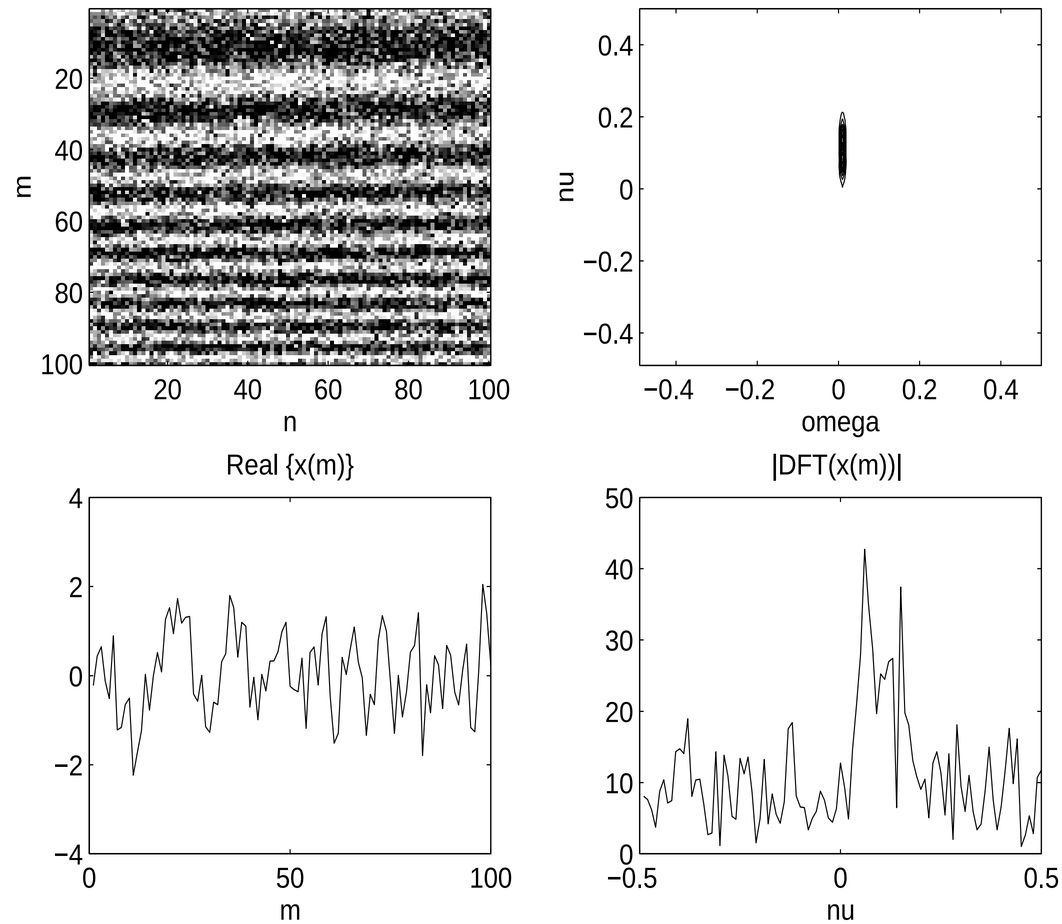

Figure 6. The 2-D polynomial phase signal after applying the operator $\mathrm{PD}_{n^{(0)}}$ to the signal $y(n,m)\cdot\exp\{-j\sum_{\ell=0}^{2}\hat{c}(1,\ell)m^{\ell}n\}$. Top left: Real part of the resulting 2-D signal. Top right: Absolute value of the 2-D DFT of the resulting signal. Bottom left: Real part of the 1-D signal obtained by averaging the 2-D signal along the n-axis. Bottom right: Absolute value of the 1-D signal DFT.

where $\mathbf{c}$ is defined as in (61).

It was shown in section 4 that in the absence of observation noise, overestimating the order of the phase polynomial yields zero estimated coefficients for the non-existing coefficients, *i.e.*, for all $L \geq S$ applying L times, in any order, the operators $\mathrm{PD}_{n^{(1)}}[\cdot]$ and $\mathrm{PD}_{m^{(1)}}[\cdot]$ to a 2-D polynomial phase signal $y(n,m)$ of total-degree S yields a constant amplitude signal. In the presence of observation noise, we decide that $c(k,\ell)=0$ whenever $|\hat{c}(k,\ell)|$ is not considerably higher than $\{\mathrm{CRB}[c(k,\ell)]\}^{\frac{1}{2}}$. In the following example we illustrate the operation of the proposed procedure.

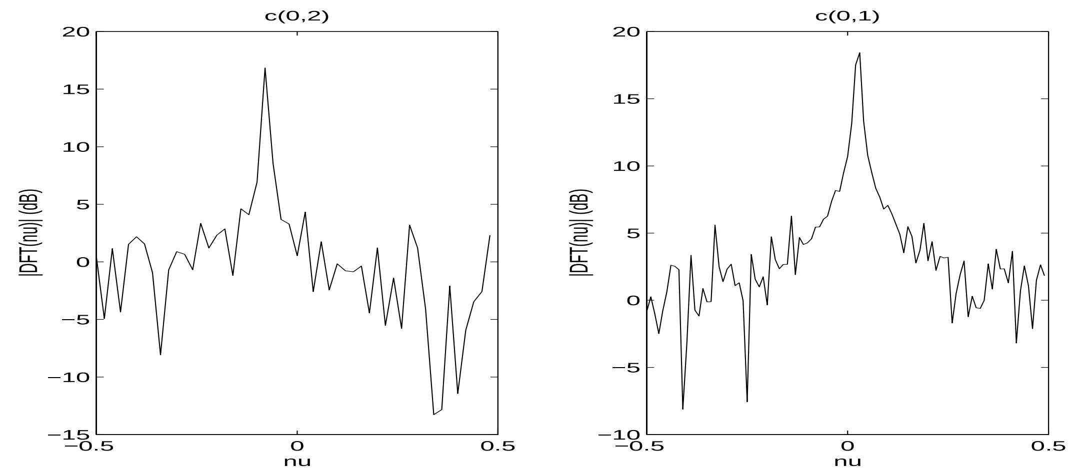

Figure 7. The 1-D exponentials obtained by sequentially applying the 1-D polynomial phase difference operator to $x(m)$. Estimating the frequency of the exponential in the left plot produces an estimate of $c(0,2)$. Estimating the frequency of the exponential in the right plot produces an estimate of $c(0,1)$.

Assume that $\hat{S} = 3$, while in fact, $S = 2$. The resulting estimated parameters and the corresponding square root values of the CRB are listed in Table 2. It is clear from the example that the proposed order estimation procedure would result in the correct result, since the estimated values of the non-existing coefficients are not considerably higher than the corresponding square root values of the CRB. Thus, we have here a relatively simple and effective order estimation procedure. We start with an assumed bound on the total-degree, and determine the true order by the procedure illustrated above. A similar procedure can be applied for rectangular support polynomial phase signals.

Parameter	True Value	Estimated Value	$\{\mathrm{CRB}[c(k,\ell)]\}^{\frac{1}{2}}$
$c(0,0)$	1	1.04	$1.8 \cdot 10^{-2}$
$c(0,1)$	$4.5 \cdot 10^{-1}$	$4.49 \cdot 10^{-1}$	$9.38 \cdot 10^{-4}$
$c(1,0)$	$8.2 \cdot 10^{-1}$	$8.18 \cdot 10^{-1}$	$9.38 \cdot 10^{-4}$
$c(0,2)$	$-1.5 \cdot 10^{-3}$	$-1.5 \cdot 10^{-3}$	$1.86 \cdot 10^{-5}$
$c(1,1)$	$16 \cdot 10^{-3}$	$16 \cdot 10^{-3}$	$1.48 \cdot 10^{-5}$
$c(2,0)$	$-2.2 \cdot 10^{-3}$	$-2.15 \cdot 10^{-3}$	$1.86 \cdot 10^{-5}$
$c(0,3)$	0	$2.3 \cdot 10^{-9}$	$1.18 \cdot 10^{-7}$
$c(1,2)$	0	$3.09 \cdot 10^{-8}$	$1.04 \cdot 10^{-7}$
$c(2,1)$	0	$-1.55 \cdot 10^{-7}$	$1.04 \cdot 10^{-7}$
$c(3,0)$	0	$-2.39 \cdot 10^{-7}$	$1.18 \cdot 10^{-7}$

Table 2. The overestimated model and the corresponding CRB values.

7.3 The Performance of the Parameter Estimation Algorithms

In this section we illustrate the performance of the proposed parameter estimation algorithms by Monte Carlo simulations. We compare the variance of the estimation errors of the suggested algorithms with the corresponding CRB's. In these examples the observation noise is a complex valued, zero mean, white Gaussian noise, and we investigate the performance of the algorithms as a function of the signal to noise ratio (SNR). We also compare here the exact Cramer-Rao bound to its approximation which was developed assuming that N and M are large compared to the polynomial phase degree.

The triangular support polynomial phase signal being considered here is the same signal which was considered in Section 7.2, *i.e.*, the phase polynomial total-degree is 2. The rectangular support polynomial phase signal being considered in this example is the same signal which was considered in Section 7.1, *i.e.*, the phase polynomial is of order $(P, Q) = (1, 2)$.

In figures 8 and 9 the experimental variance plots are based on 200 independent realizations of the signal for each SNR value. The results depicted in these figures illustrate the bounds and the estimation performance for all the model parameters, for different SNR values.

From these results we see that for triangular support polynomial phase signals the estimates have a variance which is very close to the CRB, for SNR values above 5 dB. The estimation algorithm of rectangular support polynomial phase signals produces estimation error variances that are not as close to its CRB. However, computationally the estimation algorithm of rectangular support polynomial phase signals is considerably simpler than the estimation algorithm for triangular support polynomial phase signals, since it employs 1-D FFT's instead of 2-D FFT's. It should be noted that there is no way to compare the performance of the two algorithms since they are designed to estimate the parameters of different models. It is also clear that the asymptotic approximation to the exact CRB is very good, as the graphs of the exact bound and the asymptotic bound essentially overlap.

8 Conclusions

In this paper we studied the problem of parametric modeling and estimation of nonhomogeneous 2-D signals, and in particular, the class of constant modulus polynomial phase

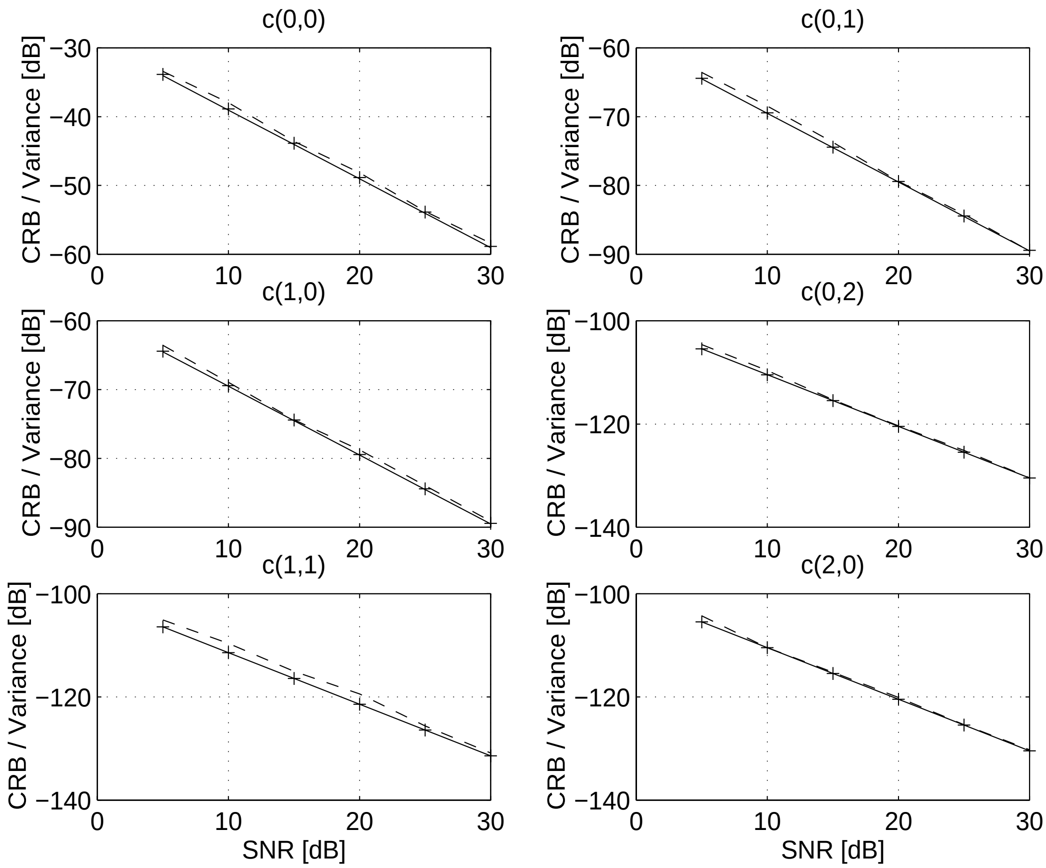

Figure 8. Performance of the triangular support polynomial phase signal estimation algorithm. Solid lines denote the exact CRB, '+' denotes the asymptotic CRB, and dashed lines denote the experimental variance of the estimates.

signals. Computationally efficient estimation algorithms for two different polynomial phase models were derived. The operation of the algorithms was discussed in some detail and illustrated by numerical examples. The performance of the suggested algorithms was evaluated by Monte-Carlo simulations and by comparing the experimental estimation error variance with the corresponding Cramer Rao lower bounds. The algorithms are shown to be robust in the presence of noise, and their performance close to the CRB, even at moderate signal to noise ratios. The problem of choosing the optimal set of the algorithm free parameters τ_n and τ_m is discussed in [9] for the triangular support polynomial phase model.

The proposed algorithms have been shown to be applicable in a wide variety of applica-

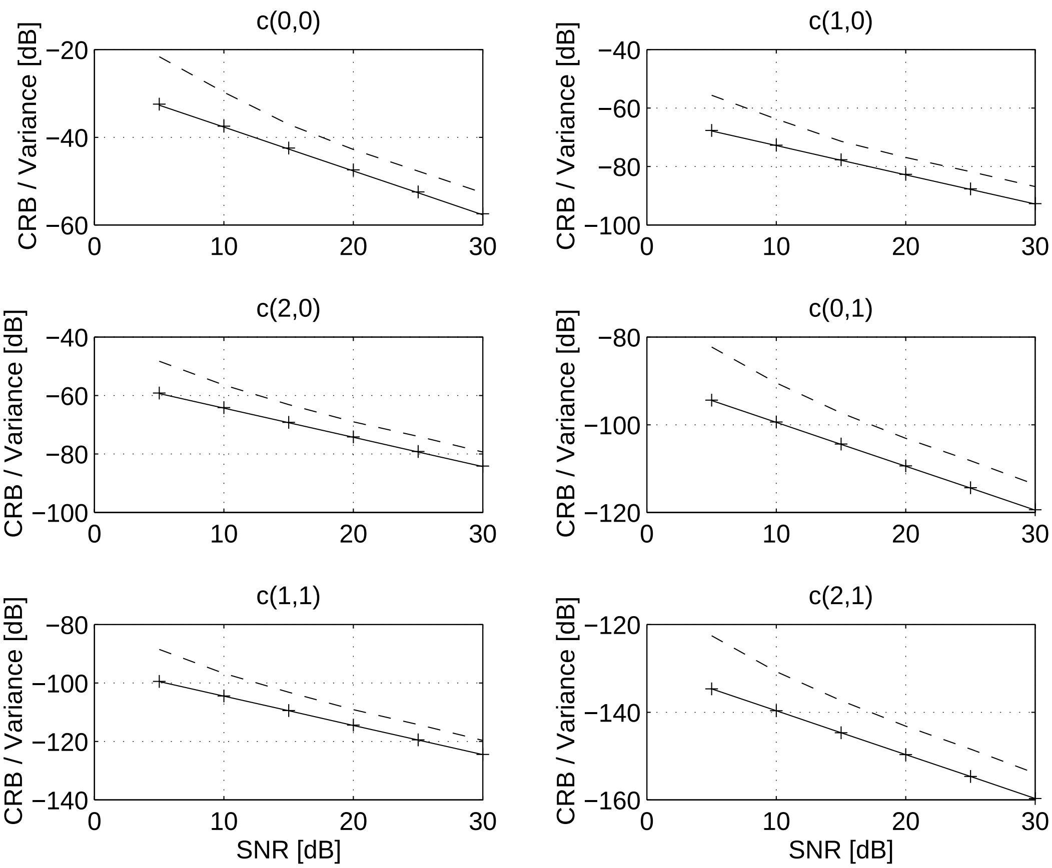

Figure 9. Performance of the rectangular support polynomial phase signal estimation algorithm. Solid lines denote the exact CRB, '+' denotes the asymptotic CRB, and dashed lines denote the experimental variance of the estimates.

tions. In [8], the algorithm is applied as the basic building block of a 2-D phase unwrapping algorithm for complex valued data. The phase unwrapping algorithm is shown to be effective in the presence of both phase aliasing and low signal to noise ratios. In [19] the same algorithm serves as the basic building block of an algorithm for estimating the orientation in space of a planar textured surface, *i.e.*, its tilt and slant, using a single image of that surface.

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