

On the Ozarow-Leung Scheme for the Gaussian Broadcast Channel with Feedback

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Abstract—In this work, we consider linear-feedback schemes for the two-user Gaussian broadcast channel with noiseless feedback. We extend the transmission scheme of [Ozarow and Leung, 1984] by applying estimators with memory instead of the memoryless estimators used by Ozarow and Leung (OL) in their original work. A recursive formulation of the mean square errors achieved by the proposed estimators is provided, along with a proof for the existence of a fixed point. This enables characterizing the achievable rates of the extended scheme. Finally, via numerical simulations it is shown that the extended scheme can improve upon the original OL scheme in terms of achievable rates, as well as achieve a low probability of error after a finite number of channel uses.

I. INTRODUCTION

WE STUDY the transmission of two independent messages over a two-user Gaussian broadcast channel (GBC) with noiseless causal feedback, referred to in the following as the GBCF, focusing on linear-feedback schemes. In [1], Ozarow and Leung presented inner and outer bounds on the capacity region of the two-user GBCF, and showed that in some scenarios it is larger than the capacity region of the GBC. In the following, we refer to the achievability scheme presented in [1] as the *OL scheme*. The OL scheme is a linear-feedback scheme that builds upon the scheme of Schalkwijk and Kailath (SK) [2], which achieves the capacity of point-to-point (PtP) Gaussian channels with noiseless causal feedback (NCF). Motivated by the optimality of the SK scheme for PtP channels, the works [3] and [1] extended this approach to the two-user Gaussian multiple-access channel with NCF (GMACF) and to the two-user GBCF, respectively. For the GMACF this extension achieves the capacity region, however, for the GBCF this extension is generally suboptimal. The OL scheme of [1] and the scheme of [3] were later extended to GBCFs and to GMACFs with more than two users as well as to Gaussian interference channels with NCF (GICFs) in [4]. These schemes were also used in [5] to stabilize (in the mean square sense) two linear, discrete-time, scalar and time-invariant systems in closed-loop, via control over GMACFs and GBCFs, respectively.

Transmission over the GBCF was also studied using tools from control theory. The work [6] derived a linear code for the two-user GBCF in which the noises at the receivers are independent. This code obtained higher achievable rates compared to

the OL scheme. Later, [7] used linear quadratic Gaussian (LQG) control to remove the restriction of independent noises in [6], and obtained a linear-feedback communications scheme for the K -user GBCF. Recently, [8] showed that for the two-user GBCF with independent noises having the same variance, the scheme of [7] is the optimal scheme subject to using a linear feedback, in the sense of maximal sum-rate.

The work [9] studied the GBCF and the GICF and derived a scheme whose achievable sum-rate approaches the full-cooperation bound as the signal-to-noise ratio (SNR) increases to infinity. Finally, in [10] it was shown that the capacity region of the GBCF with independent noises and with only a common message cannot be achieved using a coding scheme which employs linear feedback.

Note that all the works on GBCFs reviewed above focused on the achievable rates, namely, the rates are obtained as the block-length increases to infinity. From this perspective, it was shown in [7] that when the noises are independent, the LQG scheme of [7] achieves a larger rate region than the OL scheme. However, in [11] we showed that when constraining the blocklength to be finite, the OL scheme can achieve lower mean squared errors (MSEs), and therefore a lower probability of error compared to the LQG scheme (we note that although the focus of [11] is on transmission of correlated sources, this observation holds also for independent messages). In this work we propose an extension of the OL scheme which improves upon the achievable region obtained in [1], and benefits from the good performance of the OL scheme when the blocklength is finite. Next, we detail our main contributions:

Main Contributions: We focus on linear-feedback schemes as such schemes are simple to implement. In the OL scheme of [1] the receivers' errors are estimated based *only* on the last channel output. However, as the transmitted signal in the OL scheme is statistically correlated with *all* previous channel outputs, this approach is generally suboptimal. In this work we provide an *explicit* recursive formulation of the minimum MSE (MMSE) estimators which use the last *two* channel outputs, along with an explicit recursive characterization of the resulting achievable MSEs. We show that the proposed scheme has a fixed point, which enables characterizing its achievable rates as well as its MSE performance at any finite number of channel uses. We note that this is the first *explicit characterization* of an OL-type scheme which uses estimators with memory, and the first time that a fixed point property is proved for such a scheme. Furthermore, via numerical simulations we show that the extended scheme can both improve upon the original OL scheme in terms of achievable rates, and outperform the scheme of [7] in terms of probability of error after a *finite* number of channel uses. Finally, we demonstrate that in contrast to the common intuition, applying MMSE estimation based on several recent channel outputs may sometimes result in lower achievable rates than MMSE estimation based only on the most recent channel output.

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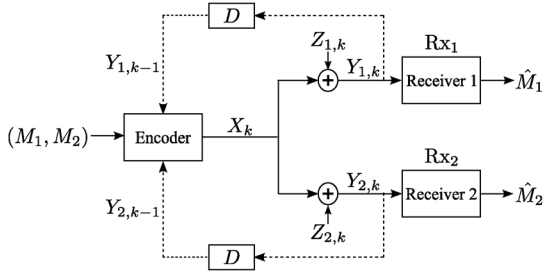


Fig. 1. The Gaussian broadcast channel with feedback links. The blocks denoted by D represent a unit delay.

The rest of this paper is organized as follows: The problem definition and the OL scheme are presented in Section II, the extended OL scheme is derived in Section III, and discussion along with numerical examples are given in Section IV.

Notations: We use upper-case letters to denote random variables, e.g., X , boldface letters to denote random column vectors, e.g., \mathbf{X} , and calligraphic letters to denote sets, e.g., \mathcal{M} . We use $\mathbb{E}\{\cdot\}$, $(\cdot)^T$ and \Re to denote the expectation, transpose, and the set of real numbers, respectively. Lastly, $\text{sgn}(x)$ denotes the sign of x , with $\text{sgn}(0) \triangleq 1$.

II. PROBLEM DEFINITION AND PREVIOUS RESULTS

A. Problem Definition

We consider communications over the GBCF, depicted in Fig. 1. All signals are real. The encoder obtains a pair of independent messages $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$, where each message is uniformly distributed over its message set. The encoder is required to send the message M_i , $i = 1, 2$, to the i 'th receiver, Rx_i , using n channel uses. The channel outputs at each receiver at time k , $k = 1, 2, \dots, n$, are given by:

$$Y_{i,k} = X_k + Z_{i,k}, \quad i = 1, 2, \quad (1)$$

where the noises $Z_{i,k} \sim \mathcal{N}(0, \sigma_i^2)$ are i.i.d over time k , and independent of (M_1, M_2) . Let $\mathbb{E}\{Z_1 Z_2\} = \rho_z \sigma_1 \sigma_2$.

A (R_1, R_2, n) code for the GBCF consists of

- 1) Two discrete message sets $\mathcal{M}_i \triangleq \{1, 2, \dots, 2^{nR_i}\}$, $i = 1, 2$.
- 2) An encoder which maps the observed message pair, $M_i \in \mathcal{M}_i$, and the received NCF at time k , into a channel input up to time k via $X_k = f_k(M_1, M_2, \mathbf{Y}_{1,1}^{k-1}, \mathbf{Y}_{2,1}^{k-1})$.
- 3) Two decoders $g_i : \Re^n \mapsto \mathcal{M}_i$, each uses its n channel outputs, $\mathbf{Y}_{i,1}^n$, to estimate M_i : $\hat{M}_i = g_i$.

The transmitted signal is subject to the average power constraint [1], [7]:

$$\sum_{k=1}^n \mathbb{E}\{X_k^2\} \leq nP. \quad (2)$$

The *probability of error* at Rx_i is defined as: $P_{e,i}^{(n)} \triangleq \Pr\{\hat{M}_i \neq M_i\}$. We say that (R_1, R_2) is an *achievable rate pair* subject to the power constraint (2) if there exists a sequence of (R_1, R_2, n) codes satisfying (2), such that $\lim_{n \rightarrow \infty} P_{e,i}^{(n)} = 0$. Next, we briefly review the OL scheme of [1].

B. A Short Review of the OL Scheme

In the OL scheme [1], prior to transmitting a channel symbol, the transmitter determines the estimation errors at the receivers based on the noiseless feedback, and then sends a linear combination of these errors. Thus, the channel output at each receiver

consists of its estimation error corrupted by a correlated noise term, which consists of the other receiver's error and additive Gaussian noise. Each receiver then updates its estimation according to its observed channel output, thereby, decreasing the variance of its estimation error.

Setup: Each message $m_i \in \mathcal{M}_i$ is mapped into a PAM constellation point, θ_i , uniformly distributed over the interval $[-0.5, 0.5]$. Next, define $\hat{\Theta}_{i,k}$ to be the estimate of the constellation point Θ_i at the i 'th receiver, after observing the k 'th channel output $Y_{i,k}$. Let $\epsilon_{i,k} \triangleq \hat{\Theta}_{i,k} - \Theta_i$ be the estimation error after k transmissions, and define $\hat{\epsilon}_{i,k-1} \triangleq \hat{\Theta}_{i,k-1} - \hat{\Theta}_{i,k}$. Thus, we can write $\epsilon_{i,k} = \epsilon_{i,k-1} - \hat{\epsilon}_{i,k-1}$. We also define $\alpha_{i,k} \triangleq \mathbb{E}\{\epsilon_{i,k}^2\}$ to be the MSEs after k transmissions, and $\rho_k \triangleq \frac{\mathbb{E}\{\epsilon_{1,k}\epsilon_{2,k}\}}{\sqrt{\alpha_{1,k}\alpha_{2,k}}}$ to be the correlation coefficient between the estimation errors.

Initialization: In the first two transmissions $X_k = \sqrt{12P} \cdot \Theta_k$, $k = 1, 2$, are sent. After the first transmission, Rx_1 estimates Θ_1 via $\hat{\Theta}_{1,1} = \frac{Y_{1,1}}{\sqrt{12P}}$. Rx_1 ignores the second transmission and sets $\hat{\Theta}_{1,2} = \hat{\Theta}_{1,1}$. Similarly, Rx_2 ignores the first transmission and sets $\hat{\Theta}_{2,2} = \frac{Y_{2,2}}{\sqrt{12P}}$. Therefore $\alpha_{i,2} = \frac{\sigma_i^2}{12P}$, and $\rho_2 = 0$.

Encoding: Let $g > 0$ be a constant which facilitates a tradeoff between R_1 and R_2 , and let $\Psi_k \triangleq \sqrt{\frac{P}{1+g^2+2g|\rho_k|}}$. At the k 'th transmission, $k \geq 3$, the transmitter sends [1, pg. 668]:

$$X_k = \Psi_{k-1} \left(\frac{\epsilon_{1,k-1}}{\sqrt{\alpha_{1,k-1}}} + \frac{\epsilon_{2,k-1}}{\sqrt{\alpha_{2,k-1}}} \cdot g \cdot \text{sgn}(\rho_{k-1}) \right), \quad (3)$$

and the corresponding channel outputs are given by (1).

Decoding: Rx_i estimates $\epsilon_{i,k-1}$, $i = 1, 2$, based *only* on $Y_{i,k}$: $\hat{\epsilon}_{i,k-1} = \mathbb{E}\{\epsilon_{i,k-1} | Y_{i,k}\} = \frac{\mathbb{E}\{\epsilon_{i,k-1} Y_{i,k}\}}{\mathbb{E}\{Y_{i,k}^2\}} Y_{i,k}$. Let $\pi_i \triangleq P + \sigma_i^2$, $\Sigma \triangleq P + \sigma_1^2 + \sigma_2^2 - \rho_z \sigma_1 \sigma_2$, and $\varsigma_i^2 \triangleq \sigma_i^2 - \rho_z \sigma_1 \sigma_2$. Then, $\alpha_{i,k}$ are given by the recursive expressions [1, Eqs. (5)–(6)]:

$$\alpha_{i,k} = \alpha_{i,k-1} \frac{\sigma_i^2 + \Psi_{k-1}^2 g^{4-2i} (1 - \rho_{k-1}^2)}{\pi_i}, \quad i = 1, 2, \quad (4)$$

where the recursive expression for ρ_k is given by [1, Eq. (7)]:

$$\rho_k = \frac{(\rho_z \sigma_1 \sigma_2 \Sigma + \varsigma_1^2 \varsigma_2^2) \rho_{k-1} - \Psi_{k-1}^2 \Sigma \cdot g (1 - \rho_{k-1}^2) \text{sgn}(\rho_{k-1})}{\sqrt{\pi_1 \pi_2} \sqrt{\sigma_1^2 + \Psi_{k-1}^2 g^2 (1 - \rho_{k-1}^2)} \sqrt{\sigma_2^2 + \Psi_{k-1}^2 (1 - \rho_{k-1}^2)}}. \quad (5)$$

In [1] it was shown that there exists a $\rho \in [0, 1]$ such that when $|\rho_{k-1}| = \rho$ then $\rho_k = -\rho_{k-1}$. This ρ is a root of the polynomial obtained by setting $\rho_k = \rho$ and $\rho_{k-1} = -\rho$ in (5). Let $\tilde{\rho}$ denote the largest root of this polynomial in $[0, 1]$. In [1] it is shown how to initialize the transmission to guarantee $|\rho_k| = \tilde{\rho} \equiv \rho_{\text{OL}}$, $k \geq 3$. After n channel uses Rx_i employs a maximum likelihood decoder to recover M_i . Let $\tilde{\Psi} \triangleq \frac{P}{1+g^2+2g\tilde{\rho}}$. Then, the rates achieved by the OL scheme are given by [1, Eq. (9)]:

$$R_i < \frac{1}{2} \log_2 \left(\frac{\pi_i}{\sigma_i^2 + \tilde{\Psi}^2 g^{4-2i} (1 - \tilde{\rho}^2)} \right) \triangleq R_i^{\text{OL}}. \quad (6)$$

III. A NEW EXTENDED OL SCHEME

The MMSE estimator of $\epsilon_{i,k-1}$ based on the channel outputs $\mathbf{Y}_{i,1}^k$, is given by $\mathbb{E}\{\epsilon_{i,k-1} | \mathbf{Y}_{i,1}^k\}$. Yet, as successive channel outputs are not independent, obtaining an explicit expression for this estimator is analytically intractable. In the OL scheme, the estimates of $\epsilon_{i,k-1}$ are generated based *only* on $Y_{i,k}$. These estimators are suboptimal since $\mathbf{Y}_{i,1}^{k-1}$ and $\epsilon_{i,k-1}$ are correlated. A natural way to improve upon the OL scheme is estimating $\epsilon_{i,k-1}$ based on $[Y_{i,k}, \mathbf{Y}_{i,1}^{k-1}]^T \triangleq \tilde{\mathbf{Y}}_{i,k}$. We refer to this as extended OL (EOL). The EOL encoding is done as in (3). Let $\mathbf{Q}_{\tilde{\mathbf{Y}}_{i,k}}$ denote the covariance matrix of the vector $\tilde{\mathbf{Y}}_{i,k}$. Since $\epsilon_{i,k-1}$ and $\tilde{\mathbf{Y}}_{i,k}$ are

jointly Gaussian, the MMSE estimator of $\epsilon_{i,k-1}$ based on $\tilde{\mathbf{Y}}_{i,k}$ is given by [12, Eq. (12.6)]:

$$\hat{\epsilon}_{i,k-1} = \mathbb{E} \left\{ \epsilon_{i,k-1} \cdot (\tilde{\mathbf{Y}}_{i,k})^T \right\} \cdot \mathbf{Q}_{\tilde{\mathbf{Y}}_{i,k}}^{-1} \cdot \tilde{\mathbf{Y}}_{i,k}. \quad (7)$$

The following theorem explicitly characterizes $\hat{\epsilon}_{i,k-1}$ in (7):

Theorem 1: The estimators $\hat{\epsilon}_{i,k-1}$ in (7) are given by:

$$\hat{\epsilon}_{1,k-1} = \frac{\Psi_{k-1} \sqrt{\alpha_{1,k-1}} (1 + g \cdot |\rho_{k-1}|)}{\pi_1^2 - \lambda_{1,k-1}^2} \times (\pi_1 Y_{1,k} - \lambda_{1,k-1} Y_{1,k-1}) \quad (8a)$$

$$\hat{\epsilon}_{2,k-1} = \frac{\Psi_{k-1} \sqrt{\alpha_{2,k-1}} (g + |\rho_{k-1}|) \text{sgn}(\rho_{k-1})}{\pi_2^2 - \lambda_{2,k-1}^2} \times (\pi_2 Y_{2,k} - \lambda_{2,k-1} Y_{2,k-1}), \quad (8b)$$

where the terms $\lambda_{1,k-1}$ and $\lambda_{2,k-1}$ are recursively given in (9) at the bottom of the page, and $\lambda_{i,k} = 0, k = 1, 2$. Furthermore, the MSEs after k transmissions, $\alpha_{i,k}$, are recursively given by:

$$\alpha_{i,k} = \alpha_{i,k-1} \frac{\pi_i^2 - \lambda_{i,k-1}^2 - P\pi_i + \Psi_{k-1}^2 g^{4-2i} (1 - \rho_{k-1}^2) \pi_i}{\pi_i^2 - \lambda_{i,k-1}^2}, \quad (10)$$

$i = 1, 2$. Finally, let $\varphi_k \triangleq \Psi_k^2 \cdot (g + |\rho_k|)(1 + g|\rho_k|) \text{sgn}(\rho_k)$. Then, ρ_k is recursively given by $\rho_k = \frac{T_{k-1}}{\Omega_{k-1}}$, where T_{k-1} and Ω_{k-1} are given in (11) at the bottom of the page.

Proof outline: Let $\lambda_{i,k-1}$ denote the off-diagonal elements of $\mathbf{Q}_{\tilde{\mathbf{Y}}_{i,k}}$ (the two off-diagonal elements of $\mathbf{Q}_{\tilde{\mathbf{Y}}_{i,k}}$ are equal). Explicit direct calculation of $\hat{\epsilon}_{i,k-1}$, in terms of $\rho_{k-1}, \alpha_{i,k-1}$ and $\lambda_{i,k-1}$, results in (8). The recursive expressions in (9) are then obtained via an explicit calculation of $\mathbb{E}\{Y_{i,k} Y_{i,k-1}\}$, and the instantaneous MSEs in (10) are calculated via $\mathbb{E}\{\epsilon_{i,k}^2\}$. Finally, the instantaneous correlation coefficient is calculated via $\rho_k \triangleq \frac{\mathbb{E}\{\epsilon_{1,k} \epsilon_{2,k}\}}{\sqrt{\alpha_{1,k} \alpha_{2,k}}}$. ■

Remark 1: Fixing $\lambda_{i,k} = 0, k \geq 1$, EOL specializes to OL.

Similarly to the OL scheme, the EOL scheme has a fixed-point, which is stated in the following theorem:

Theorem 2: Consider the EOL scheme with the decoders given in (8)–(11) and encoding given in (3). Then, there exists a $(\rho, \lambda_1, \lambda_2) \in [0, 1] \times \mathfrak{R}^2$ such that if $|\rho_{k-1}| = \rho, \lambda_{i,k-1} = \lambda_i, i = 1, 2$, then $|\rho_k| = \rho, \lambda_{i,k} = \lambda_i, i = 1, 2$.

Proof: First, note that the method used to prove the fixed point for the OL scheme cannot be applied to the EOL due to the terms $\lambda_{i,k-1}$, cf. [1, pg. 669].

The fixed point is proven by applying Brouwer's fixed-point theorem, [13, Subsection 12.8.4], to the estimation scheme (8)–(11). Let $\xi_k \triangleq \text{sgn}(\rho_k) \text{sgn}(\rho_{k-1}) \in \{1, -1\}$, and define the

vector $\mathbf{V}_k \triangleq [\lambda_{1,k}, \lambda_{2,k}, \rho_k^2, \xi_k]$. Eqs. (9)–(11) imply that \mathbf{V}_{k-1} determines $\lambda_{i,k}$ and ρ_k . Let ν denote the mapping from \mathbf{V}_{k-1} to \mathbf{V}_k and let ν_1 denote the mapping from \mathbf{V}_{k-1} to \mathbf{V}_k when $\xi_k = 1, \forall k$. We prove that ν has a fixed point in two steps: First, we show that ν_1 has a fixed point. Then, we show that a fixed point of ν_1 translates into a fixed point of ν .

Fixed point of ν_1 : Assume that $\xi_k = 1, \forall k$, and define $\mathbf{V}_{1,k} \triangleq [\mathbf{V}_k]_{\xi_k=1}$. We show that $\mathbf{V}_{1,k} = \nu_1(\mathbf{V}_{1,k-1})$, i.e., knowledge of $\mathbf{V}_{1,k-1}$ and constants is sufficient to calculate $\mathbf{V}_{1,k}$. Eq. (11b) implies that Ω_{k-1}^2 is a function of $\rho_{k-1}^2, \lambda_{1,k-1}^2$ and $\lambda_{2,k-1}^2$. Similarly, from (11a) we have that T_{k-1}^2 is also a function of $\rho_{k-1}^2, \lambda_{1,k-1}^2$ and $\lambda_{2,k-1}^2$. Therefore, for $\xi_{k-1} = 1$ we have that ρ_k^2 can be obtained from $\rho_{k-1}^2, \lambda_{1,k-1}$ and $\lambda_{2,k-1}$. From (9) it follows that for $\xi_{k-1} = 1, \lambda_{i,k}$ are functions of $\rho_{k-1}^2, \lambda_{1,k-1}$ and $\lambda_{2,k-1}$.

Noting that $\lambda_{i,k}^2 < \pi_i^2, \forall k$, we conclude that for $\mathcal{A} \triangleq [-\pi_1, \pi_1] \times [-\pi_1, \pi_2] \times [0, 1]$ the mapping ν_1 obeys $\nu_1 : \mathcal{A} \mapsto \mathcal{A}$. Finally, recall Brouwer's fixed-point theorem which states that if \mathcal{D} is convex and compact and $h : \mathcal{D} \mapsto \mathcal{D}$ is a continuous function, then h has a fixed point. As \mathcal{A} is compact and convex, it follows that ν_1 has a fixed point. We denote this fixed point by $\bar{\mathbf{V}}_1 = [\bar{\lambda}_1, \bar{\lambda}_2, \bar{\rho}^2]$.

Fixed point of ν : We show that $\nu([\bar{\mathbf{V}}_1, 1]) = [\bar{\mathbf{V}}_1, 1]$. As $\bar{\mathbf{V}}_1$ is a fixed point of ν_1 , it follows that if $\rho_{k-1}^2 = \bar{\rho}^2, \lambda_{i,k-1} = \bar{\lambda}_i$, then $\rho_k^2 = \bar{\rho}^2, \lambda_{i,k} = \bar{\lambda}_i$. Therefore, as $\lambda_{1,k} = \lambda_{1,k-1}$, (9a) implies that if $\xi_{k-1} = \bar{\xi}$ then $\xi_k = \bar{\xi}$. Thus, ν has a fixed point. The proof is the same for $\xi_k = -1$. ■

Let $\bar{V} = [\bar{\lambda}_1, \bar{\lambda}_2, \bar{\rho}^2, \bar{\xi}]$ be a fixed point of ν , and let $\bar{\Psi} \triangleq \frac{P}{1+g^2+2g\bar{\rho}}$. Similarly to [1, pg. 669] the initialization procedure can be designed to guarantee $|\rho_2| = \bar{\rho} \equiv \rho_{\text{EOL}}$. Further setting $\lambda_{i,2} = \bar{\lambda}_i$ will result in $|\rho_k| = \bar{\rho}$ and $\lambda_{i,k} = \bar{\lambda}_i$ for $k \geq 3$. Therefore, the EOL scheme achieves rate pairs satisfying:

$$R_i < \frac{1}{2} \log \left(\frac{\pi_i^2 - \bar{\lambda}_i^2}{\pi_i^2 - \bar{\lambda}_i^2 - P\pi_i + \bar{\Psi}^2 g^{4-2i} (1 - \bar{\rho}^2) \pi_i} \right) \triangleq R_i^{\text{EOL}}. \quad (12)$$

IV. NUMERICAL EXAMPLES AND A DISCUSSION

A. The Achievable Rate Region

Consider the GBCF with $\sigma_1^2 = \sigma_2^2 = 1, \rho_z = 0$, and $P = 5$. Fig. 2 illustrates the achievable rate regions of the OL scheme, the EOL scheme, and the LQG scheme of [7, Thm. 1]. The regions for OL and EOL are obtained by varying g in the range [0.01, 100]. It can be observed that in this setting EOL outperforms OL, and that LQG outperforms both OL and EOL. The subfigure in Fig. 2 depicts ρ_{OL} and ρ_{EOL} versus g , for the same setting. It can be observed that $\rho_{\text{EOL}} \leq \rho_{\text{OL}}$. The intuition for this

$$\lambda_{1,k-1} = \frac{\Psi_{k-1} \Psi_{k-2} (g + |\rho_{k-2}|) \cdot g \cdot \text{sgn}(\rho_{k-1}) \text{sgn}(\rho_{k-2}) \pi_2 \sigma_2 (\sigma_2 - \rho_z \sigma_1)}{\sqrt{\pi_2^2 - \lambda_{2,k-2}^2 - \Psi_{k-2}^2 (g + |\rho_{k-2}|)^2 \pi_2^2} \sqrt{\pi_2^2 - \lambda_{2,k-2}^2}} \quad (9a)$$

$$\lambda_{2,k-1} = \frac{\Psi_{k-1} \Psi_{k-2} (1 + g \cdot |\rho_{k-2}|) \pi_1 \sigma_1 (\sigma_1 - \rho_z \sigma_2)}{\sqrt{\pi_1^2 - \lambda_{1,k-2}^2 - P\pi_1 + \Psi_{k-2}^2 \cdot g^2 (1 - \rho_{k-2}^2) \pi_1} \sqrt{\pi_1^2 - \lambda_{1,k-2}^2}}. \quad (9b)$$

$$T_{k-1} \triangleq \rho_{k-1} \cdot g \cdot \pi_1^2 \pi_2^2 - g \cdot \varphi_{k-1} \cdot \pi_1 \pi_2 \cdot \Sigma + \lambda_{2,k-1}^2 \pi_2^2 \cdot \text{sgn}(\rho_{k-1}) + g^2 \cdot \lambda_{2,k-1}^2 \pi_1^2 \cdot \text{sgn}(\rho_{k-1}) - \lambda_{1,k-1}^2 \lambda_{2,k-1}^2 \cdot \text{sgn}(\rho_{k-1}) (1 + g^2 + 2g|\rho_{k-1}|) + g \cdot \varphi_{k-1} \lambda_{1,k-1} \lambda_{2,k-1} (P + \rho_z \sigma_1 \sigma_2), \quad (11a)$$

$$\Omega_{k-1} \triangleq g \sqrt{(\pi_1^2 - \lambda_{1,k-1}^2)(\pi_2^2 - \lambda_{2,k-1}^2)} \sqrt{\pi_1^2 - \lambda_{1,k-1}^2 - P\pi_1 + \Psi_{k-1}^2 g^2 (1 - \rho_{k-1}^2) \pi_1} \sqrt{\pi_2^2 - \lambda_{2,k-1}^2 - P\pi_2 + \Psi_{k-1}^2 (1 - \rho_{k-1}^2) \pi_2}. \quad (11b)$$

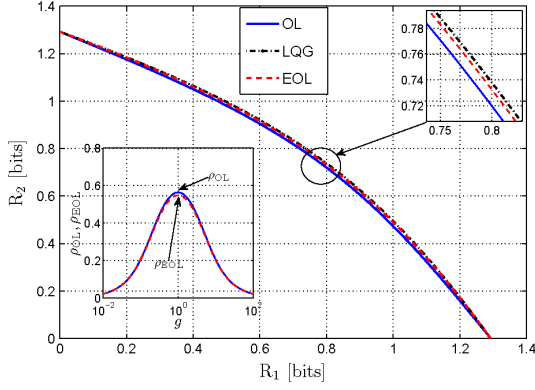


Fig. 2. Achievable rate region for $P = 5$, $\sigma_1^2 = \sigma_2^2 = 1$ and $\rho_z = 0$.

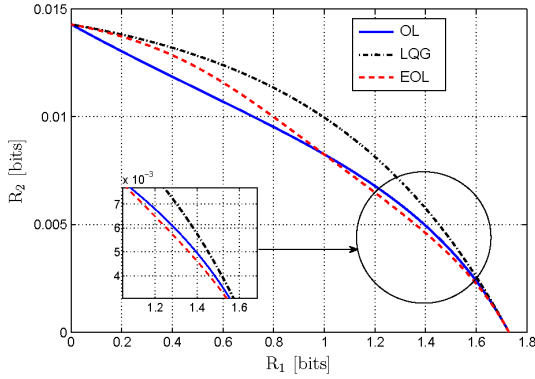


Fig. 3. Achievable rate region for $P = 1$, $\sigma_1^2 = 0.1$, $\sigma_2^2 = 50$ and $\rho_z = 0$.

relationship is as follows: since the estimator (8a) uses $Y_{1,k}$ and $Y_{1,k-1}$ for estimation, and since $Y_{1,k-1}$ is correlated with $\epsilon_{2,k-1}$, this reduces the correlation between $\epsilon_{1,k} = \epsilon_{1,k-1} - \hat{\epsilon}_{1,k-1}$ and $\epsilon_{2,k} = \epsilon_{2,k-1} - \hat{\epsilon}_{2,k-1}$, which leads to $\rho_{EOL} \leq \rho_{OL}$.

Next, note that *in some scenarios OL can outperform EOL*. The reason for this situation is that the achievable rates in the OL and EOL schemes are subject to two contradicting effects: while the subtraction of $\bar{\lambda}_i^2$ in the numerator and denominator of (12) increases R_i^{EOL} compared to R_i^{OL} (which corresponds to $\bar{\lambda}_i^2 = 0$), the fact that ρ_{EOL} can be smaller than ρ_{OL} can decrease R_i^{EOL} compared to R_i^{OL} (this follows as both R_i^{OL} and R_i^{EOL} increase with ρ_{OL} and ρ_{EOL} , respectively). This situation is illustrated in Fig. 3 which presents the achievable rate regions for $\sigma_1^2 = 0.1$, $\sigma_2^2 = 50$, $\rho_z = 0$ and $P = 1$. It can be observed in the figure that for large R_1 and small R_2 OL outperforms EOL. Finally, note that the OL and EOL schemes can be combined by applying a decoder which uses the estimator that achieves the largest R_2 at any specific R_1 .

B. Probability of Error for Finite Blocklengths

Motivated by the results of [11], in this subsection we consider the *finite blocklength* regime, which implies $P_{e,i}^{(n)} > 0$.

For independent noises with equal variances, the LQG scheme is a realization of the class of schemes presented in [8], which achieves the highest *sum-rate* among all linear-feedback schemes. Furthermore, for this setting the LQG scheme is also a realization of the class of schemes presented in [6]. In fact, [7] showed that for this setting, *in terms of achievable rates*, LQG strictly outperforms OL, as is demonstrated in Fig. 2. Recall that in the OL and in the EOL schemes, the achievable rates are determined by the scheme's steady-state (fixed point) in terms of ρ_k^2 (and $\lambda_{i,k}$). In this steady-state, at each channel use the MSE

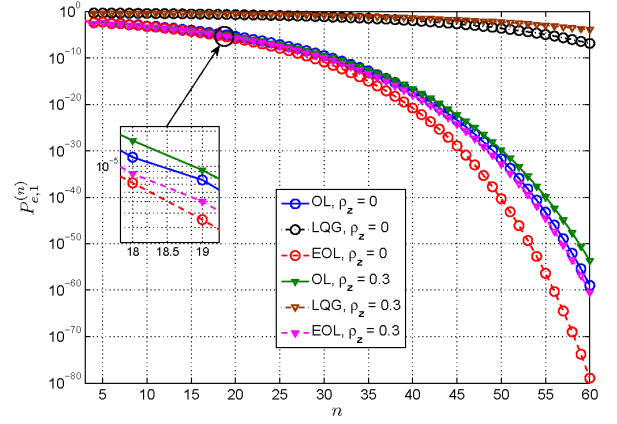


Fig. 4. $P_{e,1}^{(n)}$ vs. n , for $P = 2$, $\sigma_1^2 = \sigma_2^2 = 1$, $\rho_z = 0$, $\rho_z = 0.3$, and $g = 1$. All schemes use a transmission rate of $R = 0.9 \cdot R_1^{OL}(\rho_z)$.

$\alpha_{i,k}$ is attenuated by a *constant factor*, which determines the achievable rates, see [7, Lemma 1] on the connection between the MSEs and the achievable rates. Similarly, the achievable rates of the LQG scheme are determined by the scheme's steady-state MSE exponents. However, numerical evaluations show that the LQG scheme converges to its steady-state slower than the OL and EOL schemes. Based on this observation, [11] showed that when the codeword length is finite, the OL scheme can achieve lower MSE compared to the LQG scheme. Furthermore, it can be easily observed that if $R_i^{EOL} > R_i^{OL}$, and $\rho_{EOL} < \rho_{OL}$ (as indicated in Fig. 2), then EOL outperforms OL also in the finite blocklength regime.

Let $\beta_{i,n}$ denote the MSE achieved by a decoder of a linear-feedback transmission scheme after n channel uses, and let R_i be the transmission rate. Recall that as the scheme is linear the estimation error is a Gaussian RV [12, Subsection 10.5]. Since the data points are selected out of a PAM constellation over $[-0.5, 0.5]$, the probability of error can be computed using the standard expression for PAM [1, pg. 670]:

$$P_{e,i}^{(n)} = \frac{2^{nR_i} - 1}{2^{nR_i} + 1} Q\left(\frac{1}{2^{nR_i+1} \sqrt{\beta_{i,n}}}\right). \quad (13)$$

Let $R_1^{OL}(\rho_z)$ denote the achievable rate of the OL scheme at a specific noise correlation ρ_z , and similarly define $R_1^{EOL}(\rho_z)$ and $R_1^{LQG}(\rho_z)$. Fig. 4 depicts $P_{e,1}^{(n)}$ vs. n for the OL, EOL and LQG schemes, for $P = 2$, $\sigma_1^2 = \sigma_2^2 = 1$, and $g = 1$, for two cases: $\rho_z = 0$, and $\rho_z = 0.3$. For this setting $R_1^{OL}(0) = 0.458$, $R_1^{EOL}(0) = 0.461$, and $R_1^{LQG}(0) = 0.464$. The transmission rate, for *all the schemes*, is set to $R_1 = 0.9 \cdot R_1^{OL}(\rho_z)$, $\rho_z = 0, 0.3$. It can be observed that, for $\rho_z = 0$, the EOL scheme achieves $P_{e,1}^{(n)} = 10^{-5}$ after $n = 18$ channel uses, while the OL and LQG schemes require $n = 20$ and $n = 56$ channel uses, respectively. It can be further observed that for small n the EOL scheme and the OL scheme achieve similar $P_{e,1}^{(n)}$; however, for larger n the EOL scheme significantly improves upon the OL scheme. These observations also hold when the noises are correlated, as concluded from the curves corresponding to $\rho_z = 0.3$ in Fig. 4.

Finally, note that for $\rho_z = 0$, a fixed transmission rate $0.9 \cdot R_1^{OL}(0)$, and $\sigma_1^2 = \sigma_2^2 = 1$, the LQG scheme requires $P = 2.8$ in order to achieve $P_{e,1}^{(n)} = 10^{-5}$ after $n = 18$ channel uses. This reflects an SNR loss of 1.46 dB compared to the EOL scheme. *We conclude that in the finite blocklength regime the EOL scheme can significantly improve upon both the OL and the LQG schemes.*

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