

Broadcast Channels With Cooperating Decoders

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Abstract—We consider the problem of communicating over the general discrete memoryless broadcast channel (DMBC) with partially cooperating receivers. In our setup, receivers are able to exchange messages over noiseless conference links of finite capacities, prior to decoding the messages sent from the transmitter. In this paper, we formulate the general problem of broadcast with cooperation. We first find the capacity region for the case where the BC is physically degraded. Then, we give achievability results for the general broadcast channel, for both the two independent messages case and the single common message case.

Index Terms—Broadcast channels, channel capacity, cooperative broadcast, network information theory, relay channels.

I. INTRODUCTION

A. Motivation

IN the classic broadcast scenario the receivers decode their messages independently of each other. However, the increasing interest in networking motivates the consideration of broadcast scenarios in which each node in the network, besides decoding its own information, tries to help other nodes in decoding. This problem comes up naturally in sensor networks, where a transmitter external to the sensor network wants to download data into the network, e.g., to configure the sensor array. The concept of cooperation among receivers is also relevant to general ad-hoc networks, since such cooperation provides a method for increasing the rates without increasing the spectrum allocation. Therefore, this motivates the study of the effect of receiver cooperation on the rates for the broadcast channel.

B. The Discrete Memoryless Broadcast Channel

The broadcast channel (BC) was introduced by Cover in [1]. Following this initial work, Bergmans proved an achievability result for the degraded broadcast channel [2], and also a partial converse that holds only for the Gaussian BC [3]; in [4], Gallager established a converse that holds for any discrete memoryless degraded BC. In [5], El-Gamal generalized the capacity result for the degraded broadcast channel to the “more capable”

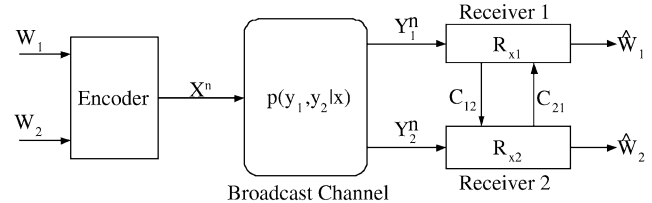


Fig. 1. Broadcast channel with two private messages and cooperating receivers.

case, and in [6] and [7] he showed that feedback does not increase the capacity region for the physically degraded case. Several other classes of BCs were studied in the following years. For example, the sum and product of two degraded BCs were considered in [8], and in [9]–[11], the deterministic BC was analyzed.

For the general BC, Cover derived an achievable rate region for the case of two independent senders in [12]. In [13], Körner and Marton considered the capacity of general broadcast channels with degraded message sets. The best achievable region and the best upper bound for the two independent senders case were derived by Marton in [14], and a simple proof of Marton’s achievable region appeared later in [15]. Another upper bound for the general BC, the so-called degraded, same-marginals (DSM) bound, was presented in [16]. This bound is weaker than the upper bound in [14] but stronger than Sato’s upper bound previously presented in [17]. We note, however, that while Marton’s upper bound is the strongest, it is valid only for the two-receiver case, while Sato’s bound and the DSM bound can be extended to more than two receivers. The effect of feedback on the capacity of the Gaussian broadcast channel was studied in [18] and [19], and in [20] the case of correlated sources was considered. A survey on the topic, with extensive references to previous work, can be found in [21]. In recent years the multiple-input multiple-output (MIMO) Gaussian BC has attracted a lot of attention. Initially, the sum-rate capacity was characterized in [22]–[25], and finally, in [26] the capacity region was obtained.

None of the early work on the DMBC considered *direct* cooperation between the receivers. In the cooperative broadcast scenario, a single transmitter sends two messages to two receivers encoded in a single-channel codeword X^n , where the superscript n denotes the length of a vector. Each of the receivers gets a noisy version of the codeword, Y_1^n at R_{x1} and Y_2^n at R_{x2} . After reception, the receivers exchange messages over noiseless conference links of finite capacities C_{12} and C_{21} , as depicted in Fig. 1. The conference messages are, in general, functions of Y_1^n (at R_{x1}), Y_2^n (at R_{x2}), and the previous messages received from the other decoder. After conferencing, each receiver decodes its own message.

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We note that in a recent work, [27], the authors consider the problem of interactive decoding of a single broadcast message over the independent broadcast channel by a group of cooperating users. In our work, we extend this scenario to the general channel and also consider the two independent senders case.

C. Cooperative Broadcast: A Combination of Broadcasting and Relaying

The scenario in which one transceiver helps a second transceiver in decoding a message is clearly a *relay* scenario. Hence, cooperative broadcast can be viewed as a generalization of the broadcast and relay scenarios into a hybrid broadcast/relay system, which better describes future communication networks.

Scenarios of this type have attracted considerable attention recently both from the practical and the theoretical aspects. From the practical aspect, new protocols are proposed for the collaborative broadcast scenario. For example, in [28] the authors present a protocol for collaborative decision making involving broadcasting and relaying. From the theoretical aspect, there is a considerable effort invested in characterizing the capacity of an entire network. This work started with [29] and recent results appear in [30] and the following work [31]–[33]. This work focuses on the Gaussian case. A complementing approach for studying the performance of a network is to combine the basic building blocks of a network, namely, multiple access, relaying, and broadcasting and study the capacity of these combinations. The recent work on relaying focuses on extending the single-relay results derived in [34] to the MIMO case (see, for example, [35]) and to the multiple-level case [36], [37]. Another recent result was introduced in [38], where joint decoding was applied to the combined decode-and-forward and estimate-and-forward scheme of [34, Theorem 7]. A third approach for studying the performance of an entire network is the network coding approach sparked by the work of [39], which focuses on encoding at the nodes for maximizing the network throughput, separately from the channel coding.

In this paper, we focus on the combination of broadcast and relay. A relevant work in this context is [40], in which the capacity of a class of independent relay channels with noiseless relay is derived. Note that the case of noiseless relay is also related to the Wyner–Ziv problem [41]. This relationship will be highlighted in the sequel. Finally, we note that a recent work, [42], presented an achievability result for the general DMBC with a single wireless cooperation channel from one receiver to the second receiver. This achievable rate region is shown to be the capacity region for the physically degraded broadcast/relay channel.

D. Main Contributions and Organization

In the following we summarize the main contributions of this work.

- We initially study a special case of the general setup formulated in Section I-B: the case of the physically degraded BC. Although the physically degraded BC is of little practical interest, it is useful in developing the coding concept for the general BC with cooperation. For the physically degraded BC, we present both an achievability result and a

converse. Together, these two results give the capacity region for this setup. Furthermore, this new region is shown to be a strict enlargement of the classical region without cooperation [21].

- Next, we give an achievability result for the general BC with cooperating receivers. This region is also greater, in general, than the classic achievable region given in [14] for the BC.
- We also consider the case where a single common message is transmitted to both receivers. We consider two different cooperation strategies and derive the achievable rates for each of them. We also derive an upper bound on the achievable rates for this scenario. Here we provide results that explicitly link the available cooperation capacity to the increase in the rate of information. Finally, we show that for a special case of the general BC, namely when one channel is distinctly better than the other, the upper and lower bounds coincide, resulting in the capacity for that case.

The rest of this paper is structured as follows: in Section II, we define the mathematical framework. In Section III, we analyze the physically degraded BC, and derive the capacity region for that case, and in Section IV, we present an achievability result for the general BC with cooperating receivers. Next, Section V presents achievability results and an upper bound on the rates for the case where only a single common message is transmitted. Concluding remarks are provided in Section VI.

II. DEFINITIONS AND NOTATIONS

First, a word about notation: in the following we use $H(\cdot)$ to denote the entropy of a discrete random variable (RV), and $I(\cdot; \cdot)$ to denote the mutual information between two discrete RVs, as defined in [43, Ch. 2]. We denote RVs with capital letters— X, Y , etc., and vectors with boldface letters, e.g., \mathbf{x}, \mathbf{y} . We denote by $A_\epsilon^{(n)}(X)$ the weakly typical set for the (possibly vector) random variable X , see [43, Ch. 3] for the definition of $A_\epsilon^{(n)}(X)$. When referring to a typical set we may omit the RVs from the notation, when these variables are clear from the context. We denote the cardinality of the finite set \mathcal{A} with $|\mathcal{A}|$. We use \mathcal{X} to denote the (discrete and finite) range of X . Finally, we denote the probability distribution of the RV X over \mathcal{X} with $p(x)$ and the conditional distribution of X given Y with $p(x|y)$.

Definition 1: A *discrete broadcast channel* is a channel with discrete input alphabet \mathcal{X} , two discrete output alphabets \mathcal{Y}_1 and \mathcal{Y}_2 , and a probability transition function, $p(y_1, y_2|x)$. We denote this channel by the triplet $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1 \times \mathcal{Y}_2)$.

Definition 2: A *memoryless broadcast channel* is a broadcast channel for which the probability transition function of a sequence of n symbols is given by

$$p(y_1^n, y_2^n | x^n) = \prod_{i=1}^n p(y_{1,i}, y_{2,i} | x_i)$$

where $y_k^n = (y_{k,1}, y_{k,2}, \dots, y_{k,n})$, $k \in \{1, 2\}$, and $x^n = (x_1, x_2, \dots, x_n)$.

We shall assume the channel to be *discrete* and *memoryless*.

Definition 3: The *physically degraded broadcast channel* is a broadcast channel in which the probability transition function can be decomposed as $p(y_1, y_2 | x) = p(y_1 | x)p(y_2 | y_1)$. Hence, for the physically degraded BC we have that $X - Y_1 - Y_2$ form a Markov chain.

Definition 4: An (R_{12}, R_{21}) -conference between R_{x_1} and R_{x_2} is defined by two conference message sets

$$\mathcal{W}_{12} = \{1, 2, \dots, 2^{nR_{12}}\}, \quad \mathcal{W}_{21} = \{1, 2, \dots, 2^{nR_{21}}\}$$

and two mapping functions, h_{12} and h_{21} , which map the received sequence of n symbols and the conference messages at one receiver into a message transmitted to the other receiver

$$\begin{aligned} h_{12} : \mathcal{Y}_1^n \times \mathcal{W}_{21} &\mapsto \mathcal{W}_{12} \\ h_{21} : \mathcal{Y}_2^n \times \mathcal{W}_{12} &\mapsto \mathcal{W}_{21}. \end{aligned}$$

We note that this is not the most general definition of a conference, see, for example, [44], [45] for a more general form. In this paper, we consider only conferences in which each receiver sends at most one message to the other receiver. Note that there are cases where a single conference message is enough to achieve capacity: for example, in Section III, a single conference step achieves capacity for the physically degraded broadcast channel, and in [45], a single conference step achieves capacity for the discrete memoryless multiple access channel counterpart of the setup discussed here.

Definition 5: A (C_{12}, C_{21}) -admissible conference is a conference for which $R_{12} \leq C_{12}$ and $R_{21} \leq C_{21}$.

Definition 6: A $((2^{nR_1}, 2^{nR_2}), n, (C_{12}, C_{21}))$ code for the broadcast channel with cooperating receivers having conference links of capacities C_{12} and C_{21} between them, consists of two sets of integers

$$\mathcal{W}_1 = \{1, 2, \dots, 2^{nR_1}\}, \quad \mathcal{W}_2 = \{1, 2, \dots, 2^{nR_2}\}$$

called *message sets*, an encoding function

$$f : \mathcal{W}_1 \times \mathcal{W}_2 \mapsto \mathcal{X}^n,$$

a (C_{12}, C_{21}) -admissible conference

$$\begin{aligned} h_{12} : \mathcal{Y}_1^n \times \mathcal{W}_{21} &\mapsto \mathcal{W}_{12} \\ h_{21} : \mathcal{Y}_2^n \times \mathcal{W}_{12} &\mapsto \mathcal{W}_{21}, \end{aligned}$$

and two decoding functions

$$\begin{aligned} g_1 : \mathcal{W}_{21} \times \mathcal{Y}_1^n &\mapsto \mathcal{W}_1 & (1) \\ g_2 : \mathcal{W}_{12} \times \mathcal{Y}_2^n &\mapsto \mathcal{W}_2. & (2) \end{aligned}$$

Definition 7: The *average probability of error* is defined as the probability that the decoded message pair is different from the transmitted message pair

$$P_e^{(n)} = \Pr(g_1(W_{21}, Y_1^n) \neq W_1 \text{ or } g_2(W_{12}, Y_2^n) \neq W_2).$$

We also define the average probability of error for each receiver as

$$P_{e1}^{(n)} = \Pr(g_1(W_{21}, Y_1^n) \neq W_1) \quad (3)$$

$$P_{e2}^{(n)} = \Pr(g_2(W_{12}, Y_2^n) \neq W_2) \quad (4)$$

where we assume transmission of n symbols for each codeword. By the union bound we have that

$$\max\{P_{e1}^{(n)}, P_{e2}^{(n)}\} \leq P_e^{(n)} \leq P_{e1}^{(n)} + P_{e2}^{(n)}.$$

Hence, $P_e^{(n)} \rightarrow 0$ implies that both $P_{e1}^{(n)} \rightarrow 0$ and $P_{e2}^{(n)} \rightarrow 0$, and when both individual error probabilities go to zero then $P_e^{(n)}$ goes to zero as well.

In the analysis that follows, we assume that user 1 and user 2 select their respective messages W_1 and W_2 independently and uniformly over their respective message sets.

Definition 8: A rate pair (R_1, R_2) is said to be *achievable*, if there exists a sequence of $((2^{nR_1}, 2^{nR_2}), n, (C_{12}, C_{21}))$ codes with $P_e^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. Obviously, this is satisfied if both $P_{e1}^{(n)} \rightarrow 0$ and $P_{e2}^{(n)} \rightarrow 0$ as n increases.

Definition 9: The *capacity region* for the discrete memoryless BC with cooperating receivers is the convex hull of all achievable rates.

III. CAPACITY REGION OF THE PHYSICALLY DEGRADED BROADCAST CHANNEL WITH COOPERATING RECEIVERS

We consider the physically degraded BC with three independent messages: a private message to each receiver and a common message to both. We note that for the physically degraded channel, following the argument in [43, Theorem 14.6.4], we can incorporate a common rate to both receivers by replacing R_2 , the private rate to the bad receiver, obtained for the two private messages case with $R_0 + R_2$, where R_0 denotes the rate of the common information. Without cooperation, the capacity region for the physically degraded BC $X - Y_1 - Y_2$ given in [43, Theorem 14.6.4], is the convex hull of all the rate triplets (R_0, R_1, R_2) that satisfy

$$R_1 \leq I(X; Y_1 | U) \quad (5)$$

$$R_0 + R_2 \leq I(U; Y_2) \quad (6)$$

for some joint distribution $p(u)p(x|u)p(y_1|x)p(y_2|y_1)$, where

$$\|U\| \leq \min\{\|\mathcal{X}\|, \|\mathcal{Y}_1\|, \|\mathcal{Y}_2\|\}. \quad (7)$$

Next, consider cooperation between receivers over the physically degraded BC. First note that for this case, the link from R_{x_2} to R_{x_1} does not contribute to increasing the rates due to cooperation, and that only the link from R_{x_1} to R_{x_2} does. This is due to the data processing inequality (see [43, Theorem 2.8.1]): since $X - Y_1 - Y_2$ form a Markov chain, any information about X contained in Y_2 will also be contained in Y_1 , and thus conferencing cannot help:

$$I(X; Y_1, Y_2) = I(X; Y_1) + \underbrace{I(X; Y_2 | Y_1)}_{=0} = I(X; Y_1).$$

For the rest of this section then, we shall consider only a communication link from the good receiver R_{x1} , to the bad receiver R_{x2} (i.e., we set $C_{21} = 0$). This implies that W_{21} is a constant and we can thus omit it from the analysis. We begin with a statement of the theorem.

Theorem 1: The capacity region for sending independent information over the discrete memoryless physically degraded BC $X - Y_1 - Y_2$, with cooperating receivers having a noiseless conference link of capacity C_{12} , as defined in Section II, is the convex hull of all rate triplets (R_0, R_1, R_2) that satisfy

$$R_1 \leq I(X; Y_1 | U) \quad (8)$$

$$R_0 + R_2 \leq \min(I(U; Y_1), I(U; Y_2) + C_{12}) \quad (9)$$

for some joint distribution $p(u)p(x|u)p(y_1, y_2|x)$, where the auxiliary random variable U has cardinality bounded by $|\mathcal{U}| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|\}$.

We note that this result presented in [46] was simultaneously derived in [42] for the case of a wireless relay.

A. Achievability Proof

In this subsection, we show that the rate triplets of Theorem 1 are indeed achievable. We will show that the region defined by (8) and (9) with $R_0 = 0$ is achievable. Incorporating $R_0 > 0$ easily follows as explained earlier.

1) *Overview of Coding Strategy:* The coding strategy is a combination of a broadcast code as an “outer” code used to split the rate between R_{x1} and R_{x2} , and an “inner” code for R_{x2} , using the code construction for the physically degraded relay channel, described in [34, Theorem 1]. We first generate codewords U^n for R_{x2} , according to the relay channel code construction. Then, the codewords for R_{x2} are used as “cloud centers” for the codewords transmitted to R_{x1} (which are also the output to the channel). Upon reception, R_{x1} decodes both its own message and the message for R_{x2} , and then uses the relay code selection to select the message relayed to R_{x2} . R_{x2} uses its received signal Y_2^n to generate a list of possible U^n candidates, and then uses the information from R_{x1} to resolve for the correct codeword.

2) Details of Coding Strategy:

a) Code generation:

- 1) Consider first the set of $M_R = 2^{nC_{12}}$ relay messages. These are the messages that the relay R_{x1} transmits to R_{x2} through the noiseless finite capacity conference link between the two receivers. Index these messages by s , where $s \in \{1, 2, \dots, M_R\}$. Next, fix $p(u)$ and $p(x|u)$.
- 2) For each index $s \in [1, M_R]$, generate 2^{nR_2} conditionally independent codewords $\mathbf{u}(w_2|s) \sim \prod_{i=1}^n p(u_i)$, where $w_2 \in \{1, 2, \dots, 2^{nR_2}\}$.
- 3) For each codeword $\mathbf{u}(w_2|s)$ generate 2^{nR_1} conditionally independent codewords

$$\mathbf{x}(w_1, w_2|s) \triangleq \mathbf{x}(w_1|\mathbf{u}(w_2|s)) \sim \prod_{i=1}^n p(x_i|u_i(w_2|s))$$

where $w_1 \in \{1, 2, \dots, 2^{nR_1}\}$.

- 4) Randomly partition the message set for R_{x2} , $\{1, 2, \dots, 2^{nR_2}\}$, into M_R sets $\{S_1, S_2, \dots, S_{M_R}\}$, by independently and uniformly assigning to each message an index in $[1, M_R]$.

b) *Encoding procedure:* Consider transmission of B blocks, each block transmitted using n channel symbols. Here we use nB symbol transmissions to transmit $B - 1$ message pairs

$$(w_{1,i}, w_{2,i}) \in [1, 2^{nR_1}] \times [1, 2^{nR_2}], \quad i = 1, 2, \dots, B - 1.$$

As $B \rightarrow \infty$, we have that the rate $(R_1, R_2) \frac{B-1}{B} \rightarrow (R_1, R_2)$. Hence, any rate pair achievable without blocking can be approached arbitrarily close with blocking as well. Let $w_{1,i}$ and $w_{2,i}$ be the messages intended for R_{x1} and R_{x2} , respectively, at the i th block, and also assume that $w_{2,i-1} \in S_{s_i}$. R_{x1} has an estimate $\hat{w}_{2,i-1}$ of the message sent to R_{x2} at block $i - 1$. Let $\hat{w}_{2,i-1} \in S_{\hat{s}_i}$. At the i th block, the transmitter outputs the codeword $\mathbf{x}(w_{1,i}, w_{2,i}|s_i)$, and R_{x1} sends the index \hat{s}_i to R_{x2} through the noiseless conference link.

c) *Decoding procedure:* Assume first that up to the end of the $(i - 1)$ th block there was no decoding error. Hence, at the end of the $(i - 1)$ th block, R_{x1} knows $(w_{1,1}, w_{1,2}, \dots, w_{1,i-1})$, $(w_{2,1}, w_{2,2}, \dots, w_{2,i-1})$, and (s_1, s_2, \dots, s_i) , and R_{x2} knows $(w_{2,1}, w_{2,2}, \dots, w_{2,i-2})$, and $(s_1, s_2, \dots, s_{i-1})$. The decoding at block i proceeds as follows:

- 1) R_{x1} knows s_i from $w_{2,i-1}$. Hence, R_{x1} determines uniquely $(\hat{w}_{1,i}, \hat{w}_{2,i})$ s.t.

$$(\mathbf{u}(\hat{w}_{2,i}|s_i), \mathbf{x}(\hat{w}_{1,i}, \hat{w}_{2,i}|s_i), \mathbf{y}_1(i)) \in A_\epsilon^{(n)}.$$

If there is none or there is more than one, an error is declared.

- 2) R_{x2} receives s_i from R_{x1} . From knowledge of s_{i-1} and $\mathbf{y}_2(i - 1)$, R_{x2} forms a list of possible messages

$$\mathcal{L}(i - 1) = \{w_2 : (\mathbf{y}_2(i - 1), \mathbf{u}(w_2|s_{i-1})) \in A_\epsilon^{(n)}\}.$$

Now, R_{x2} uses s_i to find a unique $\hat{w}_{2,i-1} \in S_{s_i} \cap \mathcal{L}(i - 1)$. If there is none or there is more than one, an error is declared.

3) *Analysis of the Probability of Error:* The achievable rate to R_{x2} can be proved using the same technique as in [34, Theorem 1]. For the ease of description, assume that R_{x1} is connected via an orthogonal channel to R_{x2} and let X' denote the channel input from R_{x1} and Y' the corresponding channel output to R_{x2} . Thus, R_{x2} has combined input (Y_2, Y') . The overall transition matrix is given by

$$p(y_1, y_2, y' | x, x') = p(y_1, y_2 | x)p(y' | x'). \quad (10)$$

Additionally, we select the transition matrix $p(y' | x')$ and the input and output alphabets $\mathcal{X}', \mathcal{Y}'$ such that the capacity of the orthogonal channel $X' - Y'$ is C_{12} . An example for such a selection is letting $\mathcal{X}' = \mathcal{Y}' = \{0, 1, \dots, 2^{\lceil C_{12} \rceil} - 1\}$, where $\lceil \cdot \rceil$ denotes the ceil function. Letting $[a]$ denotes the integer part of the real number a , we set the channel transition function to be

$$p(Y' | X') = \begin{cases} 1 - \alpha, & Y' = X' \\ \alpha, & Y' = \text{mod}(X' + 2^{\lceil C_{12} \rceil}, 2^{\lceil C_{12} \rceil}) \end{cases}$$

with α selected such that $H(Y'|X') = \lceil C_{12} \rceil - C_{12}$. The capacity of this channel is C_{12} and is achieved by letting $p(x') = \frac{1}{2^{\lceil C_{12} \rceil}}, \forall x' \in \mathcal{X}'$. This setup is equivalent to the original setup described in Section I-B, when $C_{21} = 0$.

Now consider the rate to R_{x2} . The Markov chain $U - X - (Y_1, Y_2)$ combined with the condition in (10) implies the following probability distribution function (pdf)

$$p(u, y_1, y_2, y', x') = p(y_1, y_2 | u)p(y' | x')p(u, x').$$

Now, applying [34, Theorem 1], with $p(u, x') = p(u)p(x')$, we have that (see also [32])

$$\begin{aligned} R_2 &\leq \min\{I(U, X'; Y_2, Y'), I(U; Y_1 | X')\} \\ &= \min\{I(U, X'; Y') + I(U, X'; Y_2 | Y'), I(U; Y_1)\} \\ &= \min\{I(X'; Y') + I(U; Y' | X') + I(U; Y_2 | Y') \\ &\quad + I(X'; Y_2 | Y', U), I(U; Y_1)\} \\ &= \min\{C_{12} + I(U; Y_2), I(U; Y_1)\}. \end{aligned}$$

Next, consider the rate to R_{x1} . From the proof of [34, Theorem 1] we have that R_{x1} decodes W_2 . Therefore, R_{x1} can now use successive decoding similar to the decoding at R_{x1} in [43, Ch. 14.6.2], which imply that the achievable rate to R_{x1} is given by $R_1 \leq I(X; Y_1 | U)$. Combining both bounds we get the rate constraints of Theorem 1.

B. Converse Proof

In this subsection, we prove that for $P_e^{(n)} \rightarrow 0$, the rates must satisfy the constraints in Theorem 1. First, note that for the case of the physically degraded BC with cooperating receivers we have the following Markov chain

$$X^n - Y_1^n - (W_{12}(Y_1^n), Y_2^n). \quad (11)$$

Considering the definition of the decoders in (1) and (2), and the definition of the probability of error for each of the receivers in (3) and (4), we have from Fano's inequality ([43, Ch. 2.11]) that

$$\begin{aligned} H(W_1 | Y_1^n) &\leq P_{e1}^{(n)} \log_2(2^{nR_1} - 1) + h(P_{e1}^{(n)}) \quad (12) \\ &\triangleq n\delta(P_{e1}^{(n)}) \end{aligned}$$

$$\begin{aligned} H(W_2 | Y_2^n, W_{12}(Y_1^n)) &\leq P_{e2}^{(n)} \log_2(2^{nR_2} - 1) + h(P_{e2}^{(n)}) \quad (13) \\ &\triangleq n\delta(P_{e2}^{(n)}) \end{aligned}$$

where $h(P)$ is the entropy of a Bernoulli RV with parameter P . Note that when $P_{e1}^{(n)} \rightarrow 0$ then $\delta(P_{e1}^{(n)}) \rightarrow 0$ and when $P_{e2}^{(n)} \rightarrow 0$ then $\delta(P_{e2}^{(n)}) \rightarrow 0$.

Now, for R_{x1} we have that

$$nR_1 = H(W_1) = I(W_1; Y_1^n) + H(W_1 | Y_1^n).$$

Applying inequality (12), and then proceeding as in [4] we get the bound on R_1 as

$$nR_1 \leq \sum_{k=1}^n I(X_k; Y_{1,k} | U_k) + n\delta(P_{e1}^{(n)})$$

where $U_k \triangleq (Y_{1,1}, Y_{1,2}, \dots, Y_{1,k-1}, W_2)$.

For R_{x2} , we can write

$$\begin{aligned} nR_2 &= H(W_2) \\ &\stackrel{(a)}{\leq} I(W_2; Y_2^n, W_{12}(Y_1^n)) + n\delta(P_{e2}^{(n)}) \quad (14) \\ &= I(W_2; Y_2^n) + I(W_2; W_{12}(Y_1^n) | Y_2^n) + n\delta(P_{e2}^{(n)}) \end{aligned}$$

where the inequality in (a) is due to (13). Proceeding as in [4], we bound $I(W_2; Y_2^n) \leq \sum_{k=1}^n I(U_k; Y_{2,k})$. Next, we bound $I(W_2; W_{12}(Y_1^n) | Y_2^n)$ as follows:

$$\begin{aligned} I(W_{12}(Y_1^n); W_2 | Y_2^n) &\leq H(W_{12}(Y_1^n) | Y_2^n) \\ &\leq H(W_{12}(Y_1^n)) \\ &\leq nC_{12} \quad (15) \end{aligned}$$

where the first inequality follows from the definition of mutual information, the second is due to removing the conditioning, and the third is due to the admissibility of the conference. Combining both bounds we get that

$$nR_2 \leq \sum_{k=1}^n I(U_k; Y_{2,k}) + nC_{12} + n\delta(P_{e2}^{(n)}). \quad (16)$$

The bound on R_2 can be developed in an alternative way. Begin with (14):

$$\begin{aligned} nR_2 &\leq I(W_2; Y_2^n, W_{12}(Y_1^n)) + n\delta(P_{e2}^{(n)}) \\ &\stackrel{(a)}{\leq} I(W_2; Y_2^n, Y_1^n) + n\delta(P_{e2}^{(n)}) \\ &= \sum_{k=1}^n I(W_2; Y_{1,k}, Y_{2,k} | Y_1^{k-1}, Y_2^{k-1}) + n\delta(P_{e2}^{(n)}) \quad (17) \end{aligned}$$

where (a) follows from the fact that $(W_1, W_2) - (Y_1^n, Y_2^n) - (W_{12}, Y_2^n)$ is a Markov relation and from the data processing inequality. Next, we can write

$$\begin{aligned} &I(W_2; Y_{1,k}, Y_{2,k} | Y_1^{k-1}, Y_2^{k-1}) \\ &\stackrel{(a)}{\leq} I(W_2; Y_{1,k} | Y_1^{k-1}, Y_2^{k-1}) \\ &= H(Y_{1,k} | Y_1^{k-1}, Y_2^{k-1}) \\ &\quad - H(Y_{1,k} | Y_1^{k-1}, Y_2^{k-1}, W_2) \\ &\stackrel{(b)}{\leq} H(Y_{1,k}) - H(Y_{1,k} | Y_1^{k-1}, Y_2^{k-1}, W_2) \\ &\stackrel{(c)}{\leq} H(Y_{1,k}) - H(Y_{1,k} | Y_1^{k-1}, W_2) \\ &= I(Y_{1,k}; Y_1^{k-1}, W_2) \\ &= I(Y_{1,k}; U_k) \quad (18) \end{aligned}$$

where the equality in (a) is due to the physical degradedness and memorylessness of the channel, (b) is due to removing the conditioning, and (c) is because the Markov chain makes $Y_{1,k}$ independent of Y_2^{k-1} given Y_1^{k-1} . Plugging this into (17), we get a second bound on R_2 :

$$nR_2 \leq \sum_{k=1}^n I(U_k; Y_{1,k}) + n\delta(P_{e2}^{(n)}).$$

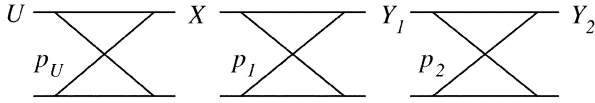


Fig. 2. The physically degraded BSBC. p_U , p_1 , and p_2 are the transition probabilities at the left, middle, and right segments, respectively.

Collecting the three bounds we have

$$R_1 \leq \frac{1}{n} \sum_{k=1}^n I(X_k; Y_{1,k} | U_k) + \delta(P_{e1}^{(n)}) \quad (19)$$

$$R_2 \leq \frac{1}{n} \sum_{k=1}^n I(U_k; Y_{2,k}) + C_{12} + \delta(P_{e2}^{(n)}) \quad (20)$$

$$R_2 \leq \frac{1}{n} \sum_{k=1}^n I(U_k; Y_{1,k}) + \delta(P_{e2}^{(n)}) \quad (21)$$

Using the standard time-sharing argument as in [43, Ch. 14.3], we can write the averages in (19)–(21) by introducing an appropriate time-sharing variable, with cardinality upper-bounded by 4. Therefore, if $P_{e1}^{(n)} \rightarrow 0$ and $P_{e2}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$, the convex hull of this region can be shown to be equivalent to the convex hull of the region defined by

$$R_1 \leq I(X; Y_1 | U) \quad (22)$$

$$R_2 \leq I(U; Y_2) + C_{12} \quad (23)$$

$$R_2 \leq I(U; Y_1). \quad (24)$$

Finally, the bound on the cardinality of \mathcal{U} follows from the same arguments as in the converse for the noncooperative case in [4]. Note, however, that $|\mathcal{Y}_2|$ is absent from the minimization on the cardinality (cf. (7) for the noncooperative case). The reason is that even when $|\mathcal{Y}_2| = 1$, information to R_{x2} (represented by the random variable U), can be sent through the conference link between the two receivers. \square

C. Discussion

To illustrate the implications of Theorem 1, consider the physically degraded binary symmetric broadcast channel (BSBC) depicted in Fig. 2. For this channel, Theorem 1 implies that $|\mathcal{U}| = 2$. Due to the symmetry of the channel, the probability distribution of U which maximizes the rates, is a symmetric binary distribution $\Pr(U = 0) = \Pr(U = 1) = \frac{1}{2}$. The resulting capacity region for this case is depicted in Fig. 3 for the case where $R_0 = 0$. In the figure, the bottom line (dashed) is the noncooperative capacity region, and the top line (dash-dotted) is the maximum possible sum rate, which requires that $C_{12} \geq h(p_{12}) - h(p_1)$, where

$$h(p) = -p \log_2(p) - (1-p) \log_2(1-p),$$

$$p_{12} = p_1(1-p_2) + p_2(1-p_1).$$

This maximum sum-rate of $I(X; Y_1)$ is obtained by summing the rate to R_{x1} given by (22) and the maximum possible rate for R_{x2} given by (24), and using the Markov chain relation $U - X - Y_1$. The middle line (solid) is the capacity region for the partial cooperation case where $0 < C_{12} < h(p_{12}) - h(p_1)$.

As can be seen from this example, the capacity region derived in this section is strictly larger than the capacity region for

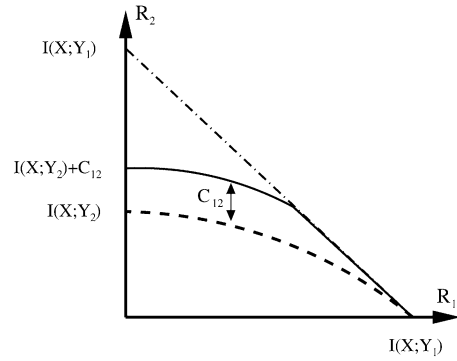


Fig. 3. The capacity region for the physically degraded BSBC. Top, middle, and bottom lines correspond to maximum possible cooperation, partial cooperation, and no-cooperation scenarios, respectively.

the noncooperation case. Indeed, summing the constraints on R_0 , R_1 and R_2 without cooperation ((5), (6)), results in a maximum achievable sum-rate of

$$R_0 + R_1 + R_2 \leq I(X; Y_1) - (I(U; Y_1) - I(U; Y_2)) \quad (25)$$

where the second term is always positive due to the Markov chain $U - X - Y_1 - Y_2$ (assuming the degrading channel is noninvertible).¹ In this setup, the maximum possible sum-rate $I(X; Y_1)$ is achieved only when U is a constant, and thus no information is sent to R_{x2} . When $R_0 + R_2 > 0$, because of the relationship $R_0 + R_2 \leq I(U; Y_2) < I(U; Y_1)$, we cannot achieve the maximum sum-rate of $I(X; Y_1)$ to R_{x1} . However, summing (23) or (24) with (22), results in a maximum achievable sum-rate with cooperating receivers of

$$R_0 + R_1 + R_2 \leq I(X; Y_1) + \min\{0, C_{12} - (I(U; Y_1) - I(U; Y_2))\}. \quad (26)$$

Comparing this to noncooperative sum-rate given by (25), it is clear that cooperation allows a net increase in the sum-rate, by at most C_{12} .

IV. ACHIEVABLE RATES FOR THE GENERAL BROADCAST CHANNEL WITH COOPERATING RECEIVERS

For the classic general BC scenario, the best achievability result was derived by Marton in [14]. This result states that for the general BC, any rate pair (R_1, R_2) satisfying

$$R_1 \leq I(U; Y_1) \quad (27)$$

$$R_2 \leq I(V; Y_2) \quad (28)$$

$$R_1 + R_2 \leq I(U; Y_1) + I(V; Y_2) - I(U; V) \quad (29)$$

for some joint distribution $p(u, v, x, y_1, y_2) = p(u, v, x)p(y_1, y_2 | x)$, is achievable.

We note that Marton's largest region contains three auxiliary RVs, (W, U, V) , where W represents information decoded by both receivers. Here we use a simplified version, where W is set to a constant.

¹It can be shown that $I(U; Y_1) - I(U; Y_2) = 0$ for the degraded channel setup implies that if $R_0 + R_2 > 0$ then $H(Y_1 | Y_2) = 0$, i.e., the channel from R_{x1} to R_{x2} is invertible. Under these circumstances, this setup can be replaced by an equivalent setup in which both receivers get Y_1 , but such a degenerate setup is not interesting.

We now consider cooperation between the receivers. We begin with a statement of the theorem.

Theorem 2: Let $(\mathcal{X}, p(y_1, y_2 | x), \mathcal{Y}_1 \times \mathcal{Y}_2)$ be any discrete memoryless BC, with cooperating receivers having noiseless conference links of finite capacities C_{12} and C_{21} , as defined in Section II. Then, for sending independent information, any rate pair (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq R(U), \\ R_2 &\leq R(V), \\ R_1 + R_2 &\leq R(U) + R(V) - I(U; V) \end{aligned}$$

subject to

$$C_{21} \geq I(\hat{U}; Y_2) - I(\hat{U}; Y_1), \quad (30)$$

$$C_{12} \geq I(\hat{V}; Y_1) - I(\hat{V}; Y_2) \quad (31)$$

where

$$R(U) = I(U; Y_1, \hat{U}), \quad (32)$$

$$R(V) = I(V; Y_2, \hat{V}) \quad (33)$$

for some joint distribution

$$p(u, v, x, y_1, y_2, \hat{u}, \hat{v}) = p(u, v, x) p(y_1, y_2 | x) p(\hat{u} | y_2) p(\hat{v} | y_1)$$

is achievable, with $u \in \mathcal{U}, v \in \mathcal{V}, \hat{u} \in \hat{\mathcal{U}}, \hat{v} \in \hat{\mathcal{V}}, \|\hat{\mathcal{U}}\| \leq \|\mathcal{Y}_2\| + 1$ and $\|\hat{\mathcal{V}}\| \leq \|\mathcal{Y}_1\| + 1$.

In the following subsections, we provide the proof of this theorem.

A. Overview of Coding Strategy

As in the achievability part of Theorem 1, the proposed code is a hybrid broadcast–relay code. Here, we combine the relay code construction of [34, Theorem 6] and the broadcast code construction of [15]. The fact that in these two theorems the channel encoding and the relay operation are performed independently, allows to easily combine them into a hybrid coding scheme. The encoder generates broadcast codewords, each selected from a codebook constructed similarly to the construction of [15]. This codebook splits the rate between the two users. Next, each relay (R_{x1} acts as a relay for R_{x2} and vice versa) generates its codebook according to the construction of [34, Theorem 6]. In the decoding step, using the received signal (Y_1^n at R_{x1} and Y_2^n at R_{x2}), each receiver generates a list of the possible transmitted relay messages and uses the conference message from the next time interval to resolve for the relay message. Then, each receiver uses the decoded relay message and its received channel output to decode its own message.

B. Encoding at the Transmitter

- 1) Let $\epsilon > 0$ and $n \geq 1$ be given. Fix $p(u, v, x), p(\hat{u} | y_2)$ and $p(\hat{v} | y_1)$, and let $\delta > 0$ be a positive number, whose selection is described in the next item. Let $A_\delta^{*(n)}(U)$ denote the set of strongly typical independent and identically distributed (i.i.d.) sequences of length $n, \mathbf{u} \in \mathcal{U}^n$, as defined in [43, Ch. 13.6]. Let $A_\delta^{*(n)}(V)$ denote the set of strongly typical i.i.d. sequences of length $n, \mathbf{v} \in \mathcal{V}^n$. Let

$S_{[U]\delta}^{(n)}$ denote the set of all sequences $\mathbf{u} \in A_\delta^{*(n)}(U)$, such that $A_\delta^{*(n)}(V | \mathbf{u})$ is nonempty as defined in [47, Corollary 5.11], and similarly define $S_{[V]\delta}^{(n)}$ for the sequences $\mathbf{v} \in A_\delta^{*(n)}(V)$.

- 2) Select $2^{n(R(U)-\epsilon)}$ strongly typical sequences \mathbf{u} in an i.i.d. manner, according to the probability

$$p(\mathbf{u}) = \begin{cases} \frac{1}{|S_{[U]\delta}^{(n)}|}, & \mathbf{u} \in S_{[U]\delta}^{(n)} \\ 0, & \text{otherwise.} \end{cases}$$

Label these sequences by $\mathbf{u}(k), k \in [1, 2^{n(R(U)-\epsilon)}]$. Select $2^{n(R(V)-\epsilon)}$ strongly typical sequences \mathbf{v} in an i.i.d. manner, according to the probability

$$p(\mathbf{v}) = \begin{cases} \frac{1}{|S_{[V]\delta}^{(n)}|}, & \mathbf{v} \in S_{[V]\delta}^{(n)} \\ 0, & \text{otherwise.} \end{cases}$$

Label these sequences by $\mathbf{v}(l), l \in [1, 2^{n(R(V)-\epsilon)}]$. Note that from [47, Corollary 5.11] we have that

$$\|S_{[U]\delta}^{(n)}\| \geq (1 - \delta)2^{n(H(U)-\psi)}$$

where $\psi \rightarrow 0$ as $n \rightarrow \infty$ and $\delta \rightarrow 0$, so for any $\epsilon > 0$ we can always find $0 < \delta \leq \epsilon$ such that for n large enough we obtain

$$\|S_{[U]\delta}^{(n)}\| > 2^{n(I(U; Y_1, \hat{U})-\epsilon)}$$

and

$$\|S_{[V]\delta}^{(n)}\| > 2^{n(I(V; Y_2, \hat{V})-\epsilon)}.$$

- 3) Define the cells

$$B_i = [(i - 1)2^{n(R(U)-R_1-\epsilon)} + 1, i2^{n(R(U)-R_1-\epsilon)}], \\ i \in [1, 2^{nR_1}].$$

This is a partition of the \mathbf{u} sequences into 2^{nR_1} sets. Define the cells

$$C_j = [(j - 1)2^{n(R(V)-R_2-\epsilon)} + 1, j2^{n(R(V)-R_2-\epsilon)}], \\ j \in [1, 2^{nR_2}]$$

- which form a partition of the \mathbf{v} sequences into 2^{nR_2} sets.
- 4) For every pair of integers $(w_1, w_2) \in [1, 2^{nR_1}] \times [1, 2^{nR_2}]$, define the set $\mathcal{D}_{w_1, w_2} = \{(\mathbf{u}(k), \mathbf{v}(l)) : k \in B_{w_1}, l \in C_{w_2}, (\mathbf{u}(k), \mathbf{v}(l)) \in A_\epsilon^{*(n)}(U, V)\}$. Here, $A_\epsilon^{*(n)}(U, V)$ denotes the strongly typical set for the random variables U and V as defined in [43, Ch. 13.6]. In the following, we may omit the random variables when referring to the strongly typical set, when these variables are clear from the context. We now have the following (slightly modified) lemma from [15].

Lemma 1: For any two–dimensional (2-D) cell $B_i \times C_j, \epsilon > 0$, and n large enough, we have that $\Pr(\|\mathcal{D}_{ij}\| = 0) \leq \epsilon$, provided that

$$R_1 + R_2 < R(U) + R(V) - I(U; V) - 2\epsilon - \epsilon_1 \quad (34)$$

where $\epsilon_1 \rightarrow 0$ as $\epsilon \rightarrow 0$ and $n \rightarrow \infty$.

Proof: The proof of this lemma is obtained by direct application of the technique used to prove [15, Lemma on p. 121], and therefore will not be repeated here. \square

5) For each message pair (w_1, w_2) , select one pair $(\mathbf{u}(k_{w_1, w_2}), \mathbf{v}(l_{w_1, w_2})) \in \mathcal{D}_{w_1, w_2}$. For each of the selected pairs (one pair for each message pair), generate a codeword according to

$$\mathbf{x}(w_1, w_2) \sim \prod_{i=1}^n p(x_i | u_i(k_{w_1, w_2}), v_i(l_{w_1, w_2})).$$

6) To transmit the message pair (w_1, w_2) the transmitter outputs $\mathbf{x}(w_1, w_2)$.

C. Encoding the Relay Messages

Consider first the relay encoding at R_{x2} , which acts as a relay for R_{x1} .

- 1) R_{x2} -relay has a set of $2^{nC_{21}}$ relay messages indexed by $s' \in [1, 2^{nC_{21}}]$. For each index s' , generate $2^{nR'}$ i.i.d. sequences $\hat{\mathbf{u}}$, each with probability

$$p(\hat{\mathbf{u}}) = \prod_{i=1}^n p(\hat{u}_i)$$

$$p(\hat{u}) = \sum_{\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2} p(\hat{u} | y_2) p(y_1, y_2 | x) p(x)$$

and

$$p(x) = \sum_{U, V} p(u, v, x).$$

Label these codewords $\hat{\mathbf{u}}(z' | s')$, $s' \in [1, 2^{nC_{21}}]$, $z' \in [1, 2^{nR'}]$.

- 2) Randomly and uniformly partition the message set $[1, 2^{nR'}]$ into $2^{nC_{21}}$ sets $S'_{s'}$, $s' \in [1, 2^{nC_{21}}]$.
- 3) **Encoding:** Assume that after receiving $\mathbf{y}_2(i-1)$ we have at R_{x2} that $(\hat{\mathbf{u}}(z'_{i-1} | s'_{i-1}), \mathbf{y}_2(i-1)) \in A_\epsilon^{*(n)}$, and that $z'_{i-1} \in S'_{s'_{i-1}}$ (s'_{i-1} is known from the previous transmission of z'_{i-2}). Then, at the i th transmission interval the relay transmits the index s'_i to R_{x1} .

Relay encoding at R_{x1} is performed in a symmetric manner to the relay encoding at R_{x2} . The corresponding variables for R_{x1} are $S''_{s''}$ and $\hat{\mathbf{v}}(z'' | s'')$, $s'' \in [1, 2^{nC_{12}}]$, $z'' \in [1, 2^{nR''}]$.

D. Decoding the Relay Messages at the Relays

Consider decoding the relay message at R_{x2} . The relay decoder at R_{x2} uses its channel input $\mathbf{y}_2(i)$, and its previously decoded s'_i to generate the relay message z'_i as follows: upon receiving $\mathbf{y}_2(i)$, the relay R_{x2} decides that the message z'_i was received at time i if $(\hat{\mathbf{u}}(z'_i | s'_i), \mathbf{y}_2(i)) \in A_\epsilon^{*(n)}$. Following the argument in [34, Theorem 6] (see also the proof in [43, Ch. 13.6]), there exists such z'_i with probability that is arbitrarily close to one as long as

$$R' \geq I(\hat{U}; Y_2) \quad (35)$$

and n is sufficiently large. Relay decoding at R_{x1} is done in a symmetric manner to the relay decoding at R_{x2} .

E. Decoding at the Receivers

We first find the rate constraint for decoding at R_{x1} . R_{x1} decodes its message $w_{1, i-1}$ based on its channel input $\mathbf{y}_1(i-1)$ and the relay indices s'_i and s'_{i-1} .

- 1) From knowledge of s'_{i-1} and $\mathbf{y}_1(i-1)$, R_{x1} , calculates the set $\mathcal{L}_1(i-1)$ such that

$$\mathcal{L}_1(i-1) = \left\{ z' \in [1, 2^{nR'}] : (\hat{\mathbf{u}}(z' | s'_{i-1}), \mathbf{y}_1(i-1)) \in A_\epsilon^{*(n)} \right\}.$$

- 2) At the time interval of the i th codeword, R_{x1} receives the relayed s'_i . Since s'_i is selected from a set of $2^{nC_{21}}$ possible messages, it can be transmitted over the noiseless conference link without error.
- 3) R_{x1} now chooses z'_{i-1} as the relay message at time $i-1$ if and only if there exists a unique $z'_{i-1} \in S'_{s'_i} \cap \mathcal{L}_1(i-1)$. Again, following the reasoning in [34, Theorem 6], this can be done with an arbitrarily small probability of error as long as

$$R' \leq I(\hat{U}; Y_1) + C_{21} \quad (36)$$

and n is large enough. Combining this with inequality (35) we get the constraint on the relay information rate

$$C_{21} \geq I(\hat{U}; Y_2) - I(\hat{U}; Y_1). \quad (37)$$

This expression is similar to the Wyner–Ziv expression for the rate required to transmit Y_2 to receiver R_{x1} up to a given distortion, determined by $p(\hat{u} | y_2)$ and a decoder. Here the performance of the decoder is implied in the mutual information $I(\hat{U}; Y_1, \hat{U})$. The compressed Y_2^n is then used by R_{x1} to assist in decoding W_1 .

- 4) Finally, R_{x1} decodes $w_{1, i-1}$ (or, equivalently $\mathbf{u}(k_{w_{1, i-1}, w_{2, i-1}})$) by choosing $\mathbf{u}(k_{\hat{w}_{1, i-1}, \hat{w}_{2, i-1}})$ such that

$$(\mathbf{u}(k_{\hat{w}_{1, i-1}, \hat{w}_{2, i-1}}), \mathbf{y}_1(i-1), \hat{\mathbf{u}}(z'_{i-1} | s'_{i-1})) \in A_\epsilon^{*(n)}.$$

From the point-to-point channel coding theorem (see [15]) we have that $\hat{w}_{1, i-1} = w_{1, i-1}$ with probability that is arbitrarily close to one, as long as z'_{i-1} was correctly decoded at R_{x1} and

$$R_1 \leq R(U) \triangleq I(U; Y_1, \hat{U}) \quad (38)$$

for sufficiently large n . Combining this with (37) yields the rate constraint on R_1

$$R_1 \leq R(U), \quad (39)$$

$$\text{as long as } C_{21} \geq I(\hat{U}; Y_2) - I(\hat{U}; Y_1). \quad (40)$$

Using symmetric arguments to those presented for decoding at R_{x1} we find the rate constraint for R_{x2} to be

$$R_2 \leq R(V), \quad (41)$$

$$\text{as long as } C_{12} \geq I(\hat{V}; Y_1) - I(\hat{V}; Y_2). \quad (42)$$

Combining (34), (39), (40), (41), and (42), gives the conditions in Theorem 2.

F. Error Events

In the scheme described above we have to account for the following error events for decoding $(w_{1,i-1}, w_{2,i-1})$.

1) Encoding at the transmitter fails:

$$E_{D,i} = \{ \|\mathcal{D}_{w_{1,i-1}, w_{2,i-1}}\| = 0 \}.$$

2) Joint typicality decoding fails:

$$E_{0,i} = \left\{ \left(\mathbf{u}(k_{w_{1,i-1}, w_{2,i-1}}), \mathbf{v}(l_{w_{1,i-1}, w_{2,i-1}}), \mathbf{x}(w_{1,i-1}, w_{2,i-1}), \mathbf{y}_1(i-1), \mathbf{y}_2(i-1) \right) \notin A_\epsilon^{*(n)} \right\}.$$

3) Decoding at the relays fails: $E_{1,i} = E_{11,i} \cup E_{12,i}$

$$E_{11,i} = \left\{ \exists z' \in [1, 2^{nR'}] \text{ s.t. } \left(\hat{\mathbf{u}}(z' | s'_{i-1}), \mathbf{y}_2(i-1) \right) \in A_\epsilon^{*(n)} \right\}$$

$$E_{12,i} = \left\{ \exists z'' \in [1, 2^{nR''}] \text{ s.t. } \left(\hat{\mathbf{v}}(z'' | s''_{i-1}), \mathbf{y}_1(i-1) \right) \in A_\epsilon^{*(n)} \right\}.$$

4) Decoding the relay message at the receivers fails: $E_{2,i} = E_{21,i} \cup E_{22,i}$, where $E_{21,i} = E'_{21,i} \cup E''_{21,i}$ and $E_{22,i} = E'_{22,i} \cup E''_{22,i}$

$$E'_{21,i} = \left\{ z'_{i-1} \notin S'_{s'_i} \cap \mathcal{L}_1(i-1) \right\}$$

$$E''_{21,i} = \left\{ \exists \tilde{z}' \neq z'_{i-1} \text{ s.t. } \tilde{z}' \in S'_{s'_i} \cap \mathcal{L}_1(i-1) \right\}$$

$$E'_{22,i} = \left\{ z''_{i-1} \notin S''_{s''_i} \cap \mathcal{L}_2(i-1) \right\}$$

$$E''_{22,i} = \left\{ \exists \tilde{z}'' \neq z''_{i-1} \text{ s.t. } \tilde{z}'' \in S''_{s''_i} \cap \mathcal{L}_2(i-1) \right\}$$

$$\mathcal{L}_2(i-1) \triangleq \left\{ z'' \in [1, 2^{nR''}] : \left(\hat{\mathbf{v}}(z'' | s''_{i-1}), \mathbf{y}_2(i-1) \right) \in A_\epsilon^{*(n)} \right\}.$$

5) Final decoding at the receivers fails:

$E_{3,i} = E_{31,i} \cup E_{32,i}$, where

$$E_{31,i} = \left\{ \left(\mathbf{u}(k_{w_{1,i-1}, w_{2,i-1}}), \mathbf{y}_1(i-1), \hat{\mathbf{u}}(z'_{i-1} | s'_{i-1}) \right) \notin A_\epsilon^{*(n)} \right\}$$

$$\cup \left\{ \exists w_1 \neq w_{1,i-1} \text{ s.t. } \left(\mathbf{u}(k_{w_1, w_2}), \mathbf{y}_1(i-1), \hat{\mathbf{u}}(z'_{i-1} | s'_{i-1}) \right) \in A_\epsilon^{*(n)} \right\}$$

$$E_{32,i} = \left\{ \left(\mathbf{v}(l_{w_{1,i-1}, w_{2,i-1}}), \mathbf{y}_2(i-1), \hat{\mathbf{v}}(z''_{i-1} | s''_{i-1}) \right) \notin A_\epsilon^{*(n)} \right\}$$

$$\cup \left\{ \exists w_2 \neq w_{2,i-1} \text{ s.t. } \left(\mathbf{v}(l_{w_1, w_2}), \mathbf{y}_2(i-1), \hat{\mathbf{v}}(z''_{i-1} | s''_{i-1}) \right) \in A_\epsilon^{*(n)} \right\}.$$

We now bound the probability of the error events at time i . Note that at time i both R_{x1} and R_{x2} share the same s'_{i-1} and s''_{i-1} irrespective whether the decoding at the relays was correct at time $i-1$. Hence, a decoding error at time $i-1$ does not affect the decoding at time i . Now, from Lemma 1 it follows that by

taking n large enough the probability of $E_{D,i}$ can be made arbitrarily small, as long as (34) is satisfied. Additionally, by taking n large enough, the probability $\Pr(E_{0,i} \cap E_{D,i}^c)$ can be made arbitrarily small by the properties of strongly typical sequences, see [43, Lemma 13.6.2]. The probability $\Pr(E_{1,i})$ can be made arbitrarily small as long as (40) and (42) are satisfied, as explained in Section IV-D. Next, the Markov lemma [50, Lemma 4.2] and the Markov chains $Y_1 - Y_2 - \hat{U}$ and $Y_2 - Y_1 - \hat{V}$, imply that $\Pr(E'_{21,i} \cap E_{1,i}^c \cap E_{0,i}^c)$ and $\Pr(E'_{22,i} \cap E_{1,i}^c \cap E_{0,i}^c)$ can be made arbitrarily small by taking n large enough, and $\Pr(E''_{21,i} \cap E_{1,i}^c)$ and $\Pr(E''_{22,i} \cap E_{1,i}^c)$ can be made arbitrarily small by taking n large enough as long as (40) and (42) are satisfied. Finally, $\Pr(E_{31,i} \cap E_{2,i}^c \cap E_{1,i}^c \cap E_{0,i}^c \cap E_{D,i}^c)$ and $\Pr(E_{32,i} \cap E_{2,i}^c \cap E_{1,i}^c \cap E_{0,i}^c \cap E_{D,i}^c)$ can be made arbitrarily small by taking n large enough by the Markov lemma and the chains $U, Y_1 - Y_2 - \hat{U}$ and $V, Y_2 - Y_1 - \hat{V}$, and as long as (39) and (41) are satisfied. \square

This concludes the proof of Theorem 2. \square

G. An Upper Bound

Proposition 1: Assume the broadcast channel setup of Theorem 2. Then, for sending independent information, any achievable rate pair (R_1, R_2) must satisfy

$$R_1 \leq I(X; Y_1) + C_{21}$$

$$R_2 \leq I(X; Y_2) + C_{12}$$

$$R_1 + R_2 \leq I(X; Y_1, Y_2)$$

for some distribution $p(x)$ on \mathcal{X} .

Proof: The proof uses the cut-set bound [43, Theorem 14.10.1]. First, we define an equivalent system by introducing two orthogonal channels $X'_2 - Y'_1$ from R_{x2} to R_{x1} and $X'_1 - Y'_2$ from R_{x1} to R_{x2} . The joint probability distribution function then becomes

$$p((y_1, y'_1), (y_2, y'_2) | x, x'_1, x'_2) = p(y_1, y_2 | x) p(y'_1 | x'_2) p(y'_2 | x'_1)$$

where the signal received at R_{x1} is (Y_1, Y'_1) and the signal received at R_{x2} is (Y_2, Y'_2) . As in the proof in Section III-A.3, we select $\mathcal{X}'_1, \mathcal{X}'_2, \mathcal{Y}'_1, \mathcal{Y}'_2, p(x'_1), p(x'_2), p(y'_1 | x'_2)$, and $p(y'_2 | x'_1)$ such that the capacities of the channels $X'_2 - Y'_1$ and $X'_1 - Y'_2$ are C_{21} and C_{12} , respectively. Additionally, the codewords for the conference transmissions are determined independently from the source codebook so we set $p(x, x'_1, x'_2) = p(x)p(x'_1)p(x'_2)$. Now, from the cut-set bound, letting the transmitter and R_{x2} form one group and R_{x1} the second group, we have

$$R_{11} \leq I(X, X'_2; Y_1, Y'_1 | X'_1)$$

$$= I(X'_2; Y_1, Y'_1 | X'_1) + I(X; Y_1, Y'_1 | X'_1, X'_2)$$

$$= I(X'_2; Y'_1 | X'_1) + I(X'_2; Y_1 | X'_1, Y'_1)$$

$$+ I(X; Y'_1 | X'_1, X'_2) + I(X; Y_1 | X'_1, X'_2, Y'_1)$$

$$= I(X'_2; Y'_1) + I(X; Y_1)$$

$$= C_{21} + I(X; Y_1)$$

where $I(X'_2; Y_1 | X'_1, Y'_1) = I(X; Y'_1 | X'_1, X'_2) = 0$ follows from direct application of the distribution function. Similarly, we obtain the rate constraint on R_2 . Finally, for the sum-rate

consider the transmitter in one group and the receivers in the second. Then, the cut-set bound results in

$$\begin{aligned} R_1 + R_2 &\leq I(X; Y_1, Y_2, Y'_1, Y'_2 | X'_1, X'_2) \\ &= I(X; Y_1, Y_2 | X'_1, X'_2) \\ &\quad + I(X; Y'_1, Y'_2 | X'_1, X'_2, Y_1, Y_2) \\ &= I(X; Y_1, Y_2) \end{aligned}$$

yielding the last constraint in the proposition. \square

H. Remarks

Comment 4.1: We note that although we present a single-letter characterization of the rates, we are not able to apply standard cardinality bounding techniques such as those used in [48] or [49] for bounding $\|\mathcal{U}\|$ and $\|\mathcal{V}\|$. The method of [48] cannot be applied since it relies on the fact that the auxiliary random variables are independent, which is not the case here. The method of [49] cannot be applied as explained in the comment for Theorem 2 in [20]. The cardinality bounds on $\|\hat{\mathcal{U}}\|$ and $\|\hat{\mathcal{V}}\|$ are trivial since they are transmitted over noiseless links.

Comment 4.2: The relay strategies can be divided into two general classes. The first class is referred to as *decode-and-forward* (DAF). In this strategy, the relay first decodes the message intended for the destination and then generates a relay message based on the decoded information. The second class is referred to as *estimate-and-forward* (EAF). In this class, the relay does not decode the message intended for the destination but transmits an estimate of its channel input to the destination. For the physically degraded BC we used DAF, based on [34, Theorem 1], to derive Theorem 1, and for the general BC we used the EAF scheme of [34, Theorem 6], to derive Theorem 2. Of course, one can also combine both strategies and perform partial decoding at each receiver of the other receiver's message before conferencing, following [34, Theorem 7]. This combination will, in general, result in an increased achievable rate region.

I. Special Cases

1) *No Cooperation:* $C_{12} = C_{21} = 0$: Consider first cooperation from R_{x2} to R_{x1} . Setting $C_{21} = 0$ in Theorem 2 implies that

$$H(\hat{U} | Y_1) = H(\hat{U} | Y_2). \quad (43)$$

From (32), the constraint on R_1 can be written in the form

$$R_1 \leq I(U; Y_1) + I(U; \hat{U} | Y_1).$$

Now we find $I(U; \hat{U} | Y_1)$

$$\begin{aligned} I(U; \hat{U} | Y_1) &= H(\hat{U} | Y_1) - H(\hat{U} | Y_1, U) \\ &\stackrel{(a)}{=} H(\hat{U} | Y_2) - H(\hat{U} | Y_1, U) \\ &\stackrel{(b)}{=} H(\hat{U} | Y_2, Y_1, U) - H(\hat{U} | Y_1, U) \quad (44) \\ &= -I(\hat{U}; Y_2 | Y_1, U) \end{aligned}$$

where (a) is due to (43), and (b) is due to the Markov chain $U - (U, V) - X - (Y_1, Y_2) - Y_2 - \hat{U}$, which implies that given Y_2, \hat{U} is independent of Y_1 and U . Now, since mutual

information is nonnegative, we conclude that $I(U; \hat{U} | Y_1) = 0$. Hence, the rate constraint on R_1 becomes

$$R_1 \leq I(U; Y_1).$$

Similarly, the maximum rate R_2 is given by $I(V; Y_2)$, and in conclusion when $C_{12} = C_{21} = 0$ we resort back to the rate region without cooperation derived in [14] (with a constant W).

2) *Full Cooperation:* $C_{12} = H(Y_1 | Y_2), C_{21} = H(Y_2 | Y_1)$: When $C_{12} = H(Y_1 | Y_2)$, we get from (31) that

$$\begin{aligned} H(Y_1 | Y_2) &= C_{12} \\ &\geq I(\hat{V}; Y_1) - I(\hat{V}; Y_2) \\ &= H(\hat{V} | Y_2) - H(\hat{V} | Y_1) \end{aligned}$$

which is satisfied when $\hat{V} = Y_1$. Plugging this into (33), we get that when full cooperation from R_{x1} to R_{x2} is available, the rate constraint for R_{x2} becomes

$$R_2 \leq I(V; Y_2, Y_1).$$

Using the same reasoning, we conclude that when full cooperation from R_{x2} to R_{x1} is available, the rate constraint for R_{x1} becomes $R_1 \leq I(U; Y_1, Y_2)$.

3) *Partial Cooperation:* When $0 < C_{12} < H(Y_1 | Y_2)$ and $0 < C_{21} < H(Y_2 | Y_1)$, we get that

$$\begin{aligned} C_{21} &\geq H(\hat{U} | Y_1) - H(\hat{U} | Y_2) \\ \Rightarrow H(\hat{U} | Y_1) &\leq C_{21} + H(\hat{U} | Y_2). \quad (45) \end{aligned}$$

Hence, the achievable rate to R_{x1} is upper-bounded by

$$\begin{aligned} R_1 &\leq I(U; Y_1, \hat{U}) \\ &= I(U; Y_1) + I(U; \hat{U} | Y_1) \\ &= I(U; Y_1) + H(\hat{U} | Y_1) - H(\hat{U} | U, Y_1) \\ &\stackrel{(a)}{\leq} I(U; Y_1) + H(\hat{U} | Y_2) - H(\hat{U} | U, Y_1) + C_{21} \\ &\stackrel{(b)}{=} I(U; Y_1) + H(\hat{U} | Y_2, Y_1, U) - H(\hat{U} | U, Y_1) + C_{21} \\ &= I(U; Y_1) + C_{21} - I(\hat{U}; Y_2 | U, Y_1) \quad (46) \end{aligned}$$

where (a) is due to (45) and (b) follow from the same reasoning leading to (44). Similarly, $R_2 \leq I(V; Y_2) + C_{12} - I(\hat{V}; Y_1 | V, Y_2)$.

Note that there exist negative terms $-I(\hat{U}; Y_2 | U, Y_1)$ and $-I(\hat{V}; Y_1 | V, Y_2)$ in the achievable rate upper bounds. This can be explained as follows: the mutual information $I(\hat{U}; Y_2 | U, Y_1)$ can be considered as a type of ‘‘ancillary’’ information that \hat{U} contains, since this information is contained in \hat{U} while U and Y_1 are already known—therefore, this information is a ‘‘noise’’ part of Y_2 which does not include any helpful information for decoding U at R_{x1} . Thus, for cooperating in the optimal way, \hat{U} has to be a type of ‘‘sufficient and complete’’ cooperation information.

V. THE GENERAL BROADCAST CHANNEL WITH A SINGLE COMMON MESSAGE

We now consider the case where only a single message, rather than two independent messages, is transmitted to both receivers.

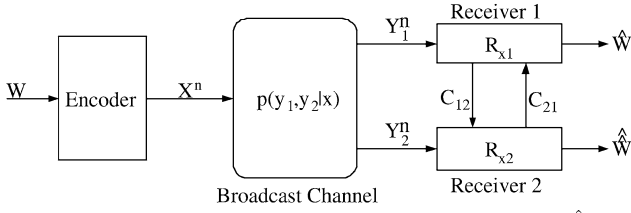


Fig. 4. The single-message BC with cooperating receivers. \hat{W} and $\hat{\hat{W}}$ are the estimates of W at R_{x1} and R_{x2} , respectively.

The main motivation for considering this case is that in the two independent messages case it is difficult to specify an explicit cooperation scheme, and we therefore have to represent cooperation through auxiliary random variables. Hence, we cannot identify directly the gain from cooperation, except in the case of full cooperation, and we also cannot evaluate the achievable region. For the single common message case, we are able to derive results for partial cooperation without auxiliary variables, which make this region explicitly computable. This scenario is depicted in Fig. 4.

For this scenario, we need to specialize the definitions of a code and the average probability of error as follows:

- A $(2^{nR}, n, (C_{12}, C_{21}))$ code for sending a common message over the BC with cooperating receivers having conference links of capacities C_{12} and C_{21} between them, is defined in a similar manner to Definition 6 with $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_1 \times \mathcal{W}_2$ all replaced with $\mathcal{W} = \{1, 2, \dots, 2^{nR}\}$.
- The average probability of error is defined similarly to Definition 7 with W_1 and W_2 replaced with W .

The capacity for the noncooperative single-message scenario is given in [5] by

$$C = \sup_{p(x)} \{\min\{I(X; Y_1), I(X; Y_2)\}\}. \quad (47)$$

In the following, we consider two cooperation schemes, referred to as a single-step scheme and a two-step scheme. These schemes are described in Fig. 5. In the single-step scheme, after reception each receiver generates a single cooperation message based on its channel input. In the two-step scheme, after reception one receiver generates a cooperation message based only on its channel input, as in the previous case, but the second receiver generates its cooperation message only after decoding (which is done with the help of the conference message from the first receiver). In both cases, each receiver generates a single conference message, however, in the single-step conference the emphasis is on low delay, while in the two-step conference we sacrifice delay in order to gain rate.

A. Decoding With a Single-Step Cooperation

In this subsection, we constrain both decoders to output their decoded messages after a conference that consists of a single message from each receiver, based only on its received channel input. For this case, we can specialize the derivation of Theorem 2 and get the following achievable rate for the BC with partially cooperating receivers:

Theorem 3: Let $(\mathcal{X}, p(y_1, y_2 | x), \mathcal{Y}_1 \times \mathcal{Y}_2)$ be any discrete memoryless broadcast channel, with cooperating receivers

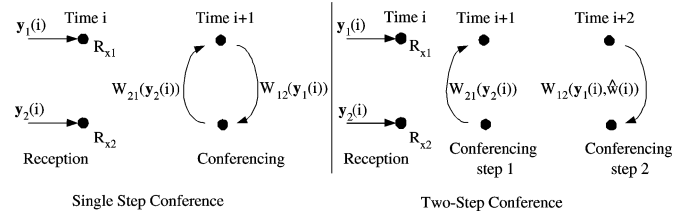


Fig. 5. Schematic description of the single-step and the two-step conference schemes.

having noiseless conference links of finite capacities C_{12} and C_{21} , as defined in Section II. Then, for sending a common message to both receivers, any rate R satisfying

$$R \leq \sup_{p(x)} \left\{ \min\{I(X; Y_1, \hat{U}), I(X; Y_2, \hat{V})\} \right\}$$

subject to

$$\begin{aligned} C_{21} &\geq I(\hat{U}; Y_2) - I(\hat{U}; Y_1), \\ C_{12} &\geq I(\hat{V}; Y_1) - I(\hat{V}; Y_2) \end{aligned}$$

for some joint distribution

$$p(x, y_1, y_2, \hat{u}, \hat{v}) = p(x)p(y_1, y_2 | x)p(\hat{u} | y_2)p(\hat{v} | y_1)$$

is achievable, with $\|\hat{\mathcal{U}}\| \leq \|\mathcal{Y}_2\| + 1$ and $\|\hat{\mathcal{V}}\| \leq \|\mathcal{Y}_1\| + 1$.

The proof of Theorem 3 follows the same lines of the proof of Theorem 2 and will not be repeated here. We next show how we can increase the rates by introducing the two-step conference.

B. Decoding With a Two-Step Cooperation

We consider a two-step conference: at the first step, only one receiver decodes the message. The second receiver decodes after the second step. Therefore, after the first receiver decodes the message, relaying to the second receiver reduces to the DAF relay situation of [34, Theorem 1]. The rates achievable with a two-step conference are given in the following theorem.

Theorem 4: Assume the broadcast channel setup of Theorem 3. Then, for sending a common message to both receivers, any rate R satisfying

$$\begin{aligned} R &\leq \sup_{p(x)} [\max\{R^{12}(p(x)), R^{21}(p(x))\}] \\ R^{12}(p(x)) &\triangleq \min \left(I(X; Y_1) + C_{21}, I(X; Y_2) - I(\hat{V}; Y_1 | Y_2, X) \right. \\ &\quad \left. + \min(C_{12}, H(\hat{V} | Y_2) - H(\hat{V} | Y_1)) \right) \\ R^{21}(p(x)) &\triangleq \min \left(I(X; Y_2) + C_{12}, I(X; Y_1) - I(\hat{U}; Y_2 | Y_1, X) \right. \\ &\quad \left. + \min(C_{21}, H(\hat{U} | Y_1) - H(\hat{U} | Y_2)) \right) \end{aligned}$$

for some joint distribution

$$p(x, y_1, y_2, \hat{u}, \hat{v}) = p(x)p(y_1, y_2 | x)p(\hat{u} | y_2)p(\hat{v} | y_1)$$

is achievable, with $\|\hat{\mathcal{U}}\| \leq \|\mathcal{Y}_2\| + 1$ and $\|\hat{\mathcal{V}}\| \leq \|\mathcal{Y}_1\| + 1$, and with the appropriate $C_{12} \geq I(\hat{V}; Y_1 | Y_2, X)$ or $C_{21} \geq I(\hat{U}; Y_2 | Y_1, X)$ (the one used for the first cooperation step).

Proof:

1) *Overview of Coding Strategy:* The scheme described in Theorem 3 uses a single-step conference for both decoders. However, if we let one receiver use a two-step conference, then that receiver, instead of using conference information derived from the raw input of the other receiver, can use information generated by the second receiver after it already decoded the message. This conference information is less noisy, and thus the rate to the first receiver can be increased.

To put this in more concrete terms, assume that at time $i + 1$, R_{x1} sends to R_{x2} the index s'_{i+1} of the partition into which its relay message at time i , denoted $z_{\hat{v},i}$, belongs. In Appendix B, we show that R_{x2} can decode the message $w_{0,i}$ with an arbitrarily small probability of error as long as n is large enough and

$$R \leq I(X; Y_2) - I(\hat{V}; Y_1 | Y_2, X) + \min(C_{12}, H(\hat{V} | Y_2) - H(\hat{V} | Y_1)) \quad (48)$$

and

$$C_{12} \geq I(\hat{V}; Y_1 | Y_2, X). \quad (49)$$

We now introduce the following modifications to the scheme used in Theorem 3.

2) *Relay Sets Generation at R_{x2} :* R_{x2} partitions the message set \mathcal{W} into $2^{nC_{21}}$ subsets in a uniform and independent manner. Denote these subsets with $\tilde{s}''_{s''}, s'' \in [1, 2^{nC_{21}}]$.

3) *Relay Encoding at R_{x2} :* R_{x2} has an estimate $\hat{w}_{0,i}$ of the message $w_{0,i}$. Now, R_{x2} looks for the partition into which $\hat{w}_{0,i}$ belongs and sends the index of this partition, denoted \tilde{s}''_{i+2} , to R_{x1} at time $i + 2$.

4) *Decoding at R_{x1} :* Upon reception of $\mathbf{y}_1(i)$, R_{x1} generates the set $\mathcal{L}_1(i) = \{w \in \mathcal{W} : (\mathbf{x}(w), \mathbf{y}_1(i)) \in A_e^{*(n)}(X, Y_1)\}$. At time $i + 2$, upon reception of \tilde{s}''_{i+2} , R_{x1} looks for an index w such that $w \in \mathcal{L}_1(i) \cap \tilde{s}''_{i+2}$. If a unique such w exists then R_{x1} sets $\hat{w}_{0,i} = w$, otherwise an error is declared.

5) *Bounding the Probability of Error:* Using the proof technique in [34, Theorem 1], it can be easily shown that assuming correct decoding at R_{x2} , then any rate $R \leq I(X; Y_1) + C_{21}$ is achievable to R_{x1} .

Combining the bounds derived above, we conclude that with a two-step conference at R_{x1} , any rate satisfying

$$R \leq \min(I(X; Y_1) + C_{21}, I(X; Y_2) - I(\hat{V}; Y_1 | Y_2, X) + \min(C_{12}, H(\hat{V} | Y_2) - H(\hat{V} | Y_1))),$$

$$C_{12} \geq I(\hat{V}; Y_1 | Y_2, X)$$

is achievable. Repeating the same derivation when R_{x2} uses a two-step conference, and combining with the previous case proves Theorem 4. \square

Setting $\hat{U} = Y_2, \hat{V} = Y_1$ in Theorem 4 we obtain the following achievable region.

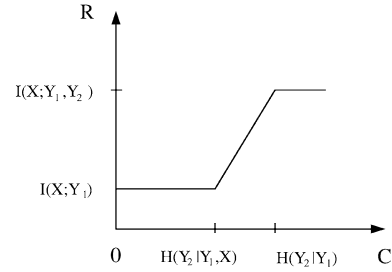


Fig. 6. Achievable rate versus C , for the two independent, identical, BSBCs with a single common message, resulting from Corollary 1.

Corollary 1: Assume the broadcast channel setup of Theorem 3. Then, for sending a common message to both receivers, any rate R satisfying

$$R \leq \sup_{p(x)} [\max\{R^{12}(p(x)), R^{21}(p(x))\}]$$

$$R^{12}(p(x)) \triangleq \min(I(X; Y_1) + C_{21}, I(X; Y_2) - H(Y_1 | Y_2, X) + \min(C_{12}, H(Y_1 | Y_2)))$$

$$R^{21}(p(x)) \triangleq \min(I(X; Y_2) + C_{12}, I(X; Y_1) - H(Y_2 | Y_1, X) + \min(C_{21}, H(Y_2 | Y_1)))$$

with the appropriate $C_{12} > H(Y_1 | Y_2, X)$ or $C_{21} > H(Y_2 | Y_1, X)$ (the one used for the first cooperation step), is achievable.

This gives a partial cooperation result without auxiliary random variables.

C. An Example for Corollary 1

Consider two independent, identical, BSBCs with transition probability p , and cooperation links of capacities $C_{12} = C_{21} = C$. For this case, Corollary 1 gives the following maximum achievable rate:

$$R = \sup_{p_0} \{ \min[H(Y_1) - h(p) + C, \min(H(Y_1) + C, H(Y_1, Y_2)) - 2h(p)] \}$$

$$= \sup_{p_0} \{ \min[H(Y_1) - 2h(p) + C, H(Y_1, Y_2) - 2h(p)] \}$$

for $C \geq h(p)$, where $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{X} = \{0, 1\}$, $p_0 = \Pr(X = 0)$, and

$$\Pr(y_1, y_2) = \begin{cases} (1-p)^2 p_0 + p^2(1-p_0), & y_1 = y_2 = 0 \\ p(1-p), & y_1 \neq y_2 \\ p^2 p_0 + (1-p)^2(1-p_0), & y_1 = y_2 = 1 \end{cases}$$

$$\Pr(y_1) = \begin{cases} (1-p)p_0 + p(1-p_0), & y_1 = 0 \\ pp_0 + (1-p)(1-p_0), & y_1 = 1. \end{cases}$$

Solving for the supremum for each value of C , we get the achievable rates depicted in Fig. 6. Note the linear increase in the achievable rate for $H(Y_2 | Y_1, X) < C < H(Y_2 | Y_1)$.

D. An Upper Bound

The upper bound for the single common message case can be obtained from the bound for the two independent messages case in Proposition 1.

Corollary 2: Let $(\mathcal{X}, p(y_1, y_2 | x), \mathcal{Y}_1 \times \mathcal{Y}_2)$ be any discrete memoryless BC, with cooperating receivers having noiseless conference links of finite capacities C_{12} and C_{21} , as defined in Section II. Then, for sending a common message to both receivers, any rate R must satisfy

$$R \leq \sup_{p(x)} \{\min(I(X; Y_1) + C_{21}, I(X; Y_2) + C_{12}, I(X; Y_1, Y_2))\}.$$

Proof: Follows directly from Proposition 1 by noting that the common rate has to satisfy all three constraints: the individual rates and the sum rate. \square

E. Remarks

Comment 5.1: Note that there are special cases where the lower bound of Corollary 1 coincides with the upper bound of Corollary 2, yielding the capacity for these cases. For example, assume a strong version of the “more capable” condition of [5]: $I(X; Y_1) \gg I(X; Y_2)^2$ for all input distributions $p(x)$ on \mathcal{X} . Assume also that $H(Y_2 | Y_1, X) < C_{21} < H(Y_2 | Y_1)$ and $H(Y_1 | Y_2, X) < C_{12} < H(Y_1 | Y_2)$. Under these conditions, we have that $I(X; Y_1) + C_{21} > I(X; Y_2) + C_{12} - H(Y_1 | Y_2, X)$. Thus, if R_{x1} is helping R_{x2} first, the achievable rate is $I(X; Y_2) + C_{12} - H(Y_1 | Y_2, X)$. If R_{x2} is helping R_{x1} first, then the achievable rate is $I(X; Y_2) + C_{12}$. Since $C_{12} - H(Y_1 | Y_2, X) < C_{12}$, this cooperation scheme achieves the upper bound $R = \sup_{p(x)} \{I(X; Y_2) + C_{12}\}$.²

Comment 5.2: Note that the capacity region for the deterministic BC with cooperating receivers follows from Corollaries 1 and 2. This region was derived in [51]. For this case, we have that $H(Y_1 | X) = H(Y_2 | X) = 0$ hence $I(X; Y_i) = H(Y_i)$, $i = 1, 2$. The achievable rate (from Corollary 1) is given by

$$R \leq \min\{H(Y_2) + C_{12}, H(Y_1) + \min(C_{21}, H(Y_2 | Y_1))\} \\ = \min\{H(Y_2) + C_{12}, H(Y_1) + C_{21}, H(Y_1, Y_2)\},$$

and the same from Corollary 2.

Comment 5.3: We note that although the expressions in (48) and (49) seem different from the EAF expression of [34, Theorem 6], given in Theorem 3 (cf. $R \leq I(X; Y_2, \hat{V})$, subject to $C_{12} \geq I(\hat{V}; Y_1) - I(\hat{V}; Y_2)$), this does not improve on the achievable rate of the standard EAF. The reason is that every rate achievable according to (48)–(49) can also be achieved with the standard EAF using the same mapping of the auxiliary RV and an appropriate time sharing.³ However, when considering a *specific, fixed assignment of the auxiliary random variable* (such as in Corollary 1) then the rate achievable with (48)–(49) is indeed greater than the classic EAF with the same assignment.

VI. CONCLUSION

In this paper, we investigated the effect of cooperation between receivers on the rates for the BC. As communication networks evolve, it can be expected that in future networks, nodes that are close enough to be able to communicate directly, will use this ability to help each other in reception. Accommodating

²The precise condition requires that $I(X; Y_1) > I(X; Y_2) + C_{12} - C_{21} + H(Y_2 | Y_1, X)$ for all input distributions $p(x)$.

³This observation is due to Shlomo Shamai and Gerhard Kramer.

this characteristic, we extended the traditional broadcast scenario, in which each decoder is assumed to operate independently, into a scenario where the receivers have finite capacity links used for cooperation. We analyzed three related scenarios: the physically degraded BC—for which we derived the capacity region; the general BC—for which we presented an achievability result; and the single common message case. For the last case, we identified a special case where capacity can be achieved. We note that it is not trivial to extend these results to more than two steps, since the intermediate steps need to extract information from partial relay information. Although this can be done by introducing additional auxiliary variables, obtaining a computable region is not a simple task. This study is an initial step in this investigation and future work includes several extensions: a natural first extension is to consider a fully wireless system, and extend the analysis to the Gaussian case. Another extension is to consider the interaction between the Wyner–Ziv compression and the achievable rates for the general channel.

APPENDIX A

BACKGROUND RESULTS

Consider the construction in Section III-A. Let

$$\mathcal{L}(i-1) = \{w_2 : (\mathbf{y}_2(i-1), \mathbf{u}(w_2 | s_{i-1})) \in A_\epsilon^{(n)}\}.$$

We bound $E_{\mathbf{y}_2} \{\|\mathcal{L}(i-1)\|\}$. Let

$$\psi(w_2 | \mathbf{y}_2(i-1)) = \begin{cases} 1, & (\mathbf{u}(w_2 | s_{i-1}), \mathbf{y}_2(i-1)) \in A_\epsilon^{(n)} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, as in [34, Theorem 1], we can write the random variable $\|\mathcal{L}(i-1)\|$ as a sum of random variables

$$\|\mathcal{L}(i-1)\| = \sum_{w_2=1}^{2^{nR_2}} \psi(w_2 | \mathbf{y}_2(i-1))$$

and therefore

$$E_{\mathbf{y}_2} \{\|\mathcal{L}(i-1)\|\} = E_{\mathbf{y}_2} \{\psi(w_2, i-1 | \mathbf{y}_2(i-1))\} \\ + \sum_{\substack{w_2=1 \\ w_2 \neq w_2, i-1}}^{2^{nR_2}} E_{\mathbf{y}_2} \{\psi(w_2 | \mathbf{y}_2(i-1))\}.$$

When $w_2 \neq w_2, i-1$, we get from the properties of independent sequence ([43, Theorem 8.6.1]) that

$$E_{\mathbf{y}_2} \{\psi(w_2 | \mathbf{y}_2(i-1))\} = \Pr\{\psi(w_2 | \mathbf{y}_2(i-1)) = 1\} \\ \leq 2^{-n(I(U; Y_2) - 3\epsilon)}$$

thus

$$E_{\mathbf{y}_2} \{\|\mathcal{L}(i-1)\|\} \leq 1 + 2^{nR_2} 2^{-n(I(U; Y_2) - 3\epsilon)}. \quad (\text{A1})$$

Note that this result holds also when considering the strongly typical set rather than the weakly typical set.

APPENDIX B

PROOF OF THE ACHIEVABLE RATE TO THE FIRST DECODER IN THEOREM 4 ((48) AND (49))

A. Overview of Coding Strategy

The encoder generates a single codebook in a random and independent manner. Next, the first relay partitions its collection

of relay codewords ($\mathcal{Z}(\hat{V})$ for R_{x1}) into disjoint sets. When a channel input is received, the first relay finds the index of the partition set which contains a relay codeword jointly typical with its channel input, and transmits it over the noiseless conference link to the second receiver. Then, the second receiver looks for a unique source codeword that is jointly typical with its channel input, and with at least one of the relay codewords in the set of possible codewords received from the first relay.

In the following analysis, we assume that R_{x1} is the first relay and R_{x2} decodes first.

B. Codebook Generation and Encoding at the Transmitter

Fix $p(x)$ and generate 2^{nR} i.i.d. codewords \mathbf{x} , with

$$p(\mathbf{x}(w)) = \prod_{i=1}^n p(x_i(w)), \quad w \in \mathcal{W} = \{1, 2, \dots, 2^{nR}\}.$$

For transmitting the message $w_{0,i}$ at time i , the transmitter outputs $\mathbf{x}(w_{0,i})$ to the channel.

C. Relay Sets Generation

Fix $p(\hat{v}|y_1)$.

- Consider the pdf

$$p(\hat{v}) = \sum_{\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2} p(\hat{v}|y_1)p(y_1, y_2 | x)p(x)$$

on $\hat{\mathcal{V}}$.

- R_{x1} generates $2^{nR'_1}$ $\hat{\mathbf{v}}$ sequences in an i.i.d. manner according to

$$p(\hat{\mathbf{v}}(z_{\hat{v}})) = \prod_{i=1}^n p(\hat{v}_i(z_{\hat{v}})), \quad z_{\hat{v}} \in \mathcal{Z}(\hat{\mathcal{V}}) = \{1, 2, \dots, 2^{nR'_1}\}.$$

- R_{x1} partitions the message set $\mathcal{Z}(\hat{\mathcal{V}})$ into $2^{nC_{12}}$ sets, by assigning an index between $[1, 2^{nC_{12}}]$ to each $z_{\hat{v}} \in \mathcal{Z}(\hat{\mathcal{V}})$, in a random, independent and uniform manner over $[1, 2^{nC_{12}}]$. Denote these sets by $S'_{s'}$, $s' \in [1, 2^{nC_{12}}]$.

D. Decoding and Encoding at the Relay (R_{x1})

- Upon reception of $\mathbf{y}_1(i)$, the relay R_{x1} decides that $z_{\hat{v},i} \in \mathcal{Z}(\hat{\mathcal{V}})$ was received if $(\hat{\mathbf{v}}(z_{\hat{v},i}), \mathbf{y}_1(i)) \in A_{\epsilon}^{*(n)}(\hat{\mathcal{V}}, Y_1)$. Now, R_{x1} finds the index s'_{i+1} of the set $S'_{s'_{i+1}}$ s.t. $z_{\hat{v},i} \in S'_{s'_{i+1}}$. Then, at time $i+1$, R_{x1} transmits s'_{i+1} to R_{x2} through the finite capacity noiseless conference link. If there is no $z_{\hat{v}} \in \mathcal{Z}(\hat{\mathcal{V}})$ such that $\hat{\mathbf{v}}(z_{\hat{v}})$ is jointly typical with $\mathbf{y}_1(i)$, an error is declared.

E. Decoding the Source Message at R_{x2}

At the i th transmission interval R_{x2} generates the set

$$\mathcal{L}_2(i) = \{w \in \mathcal{W} : (\mathbf{x}(w), \mathbf{y}_2(i)) \in A_{\epsilon}^{*(n)}(X, Y_2)\}.$$

At the $(i+1)$ th transmission interval, R_{x2} receives s'_{i+1} from R_{x1} through the noiseless conference link. R_{x2} then looks for a

unique \hat{w}_0 s.t. $\hat{w}_0 \in \mathcal{L}_2(i)$ and $\exists z_{\hat{v}} \in S'_{s'_{i+1}}$, for which

$$(\mathbf{x}(\hat{w}_0), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v}})) \in A_{\epsilon}^{*(n)}(X, Y_2, \hat{\mathcal{V}}).$$

If such unique \hat{w}_0 exists, then \hat{w}_0 is the decoded message at time i . If there is none, or there is more than one, an error is declared.

F. Analysis of the Probability of Error

1) *Error Events:* The error events for the scheme described above, for decoding the message $w_{0,i}$ are as follows.

- 1) Relay decoding fails:

$$E_{0,i} = \left\{ \nexists z_{\hat{v}} \in \mathcal{Z}(\hat{\mathcal{V}}) \text{ s.t. } (\hat{\mathbf{v}}(z_{\hat{v}}), \mathbf{y}_1(i)) \in A_{\epsilon}^{*(n)}(\hat{\mathcal{V}}, Y_1) \right\}.$$

- 2) Joint typicality decoding fails: Let $E_{1,i} = E'_{1,i} \cup E''_{1,i}$, where

$$E'_{1,i} = \left\{ (\mathbf{x}(w_{0,i}), \mathbf{y}_1(i), \mathbf{y}_2(i)) \notin A_{\epsilon}^{*(n)}(X, Y_1, Y_2) \right\}$$

$$E''_{1,i} = \left\{ (\mathbf{x}(w_{0,i}), \hat{\mathbf{v}}(z_{\hat{v},i}), \mathbf{y}_2(i)) \notin A_{\epsilon}^{*(n)}(X, \hat{\mathcal{V}}, Y_2) \right\}.$$

- 3) Decoding at R_{x2} fails: $E_{2,i} = E'_{2,i} \cup E''_{2,i}$

$$E'_{2,i} = \left\{ \nexists z_{\hat{v}} \in S'_{s'_{i+1}} \text{ for which}$$

$$(\mathbf{x}(w_{0,i}), \hat{\mathbf{v}}(z_{\hat{v}}), \mathbf{y}_2(i)) \in A_{\epsilon}^{*(n)}(X, \hat{\mathcal{V}}, Y_2) \right\}$$

$$E''_{2,i} = \left\{ \exists w \neq w_{0,i}, w \in \mathcal{L}_2(i) \text{ s.t. } \exists z_{\hat{v}} \in S'_{s'_{i+1}},$$

$$(\mathbf{x}(w), \hat{\mathbf{v}}(z_{\hat{v}}), \mathbf{y}_2(i)) \in A_{\epsilon}^{*(n)}(X, \hat{\mathcal{V}}, Y_2) \right\}.$$

Next, applying the union bound we get that

$$P_e^{(n)} = \Pr \left(\bigcup_{k=0}^2 E_{k,i} \right)$$

$$= \Pr(E_{0,i}) + \Pr \left(E_{1,i} \cap E_{0,i}^c \right)$$

$$+ \Pr \left(E_{2,i} \cap E_{1,i}^c \cap E_{0,i}^c \right).$$

2) *Bounding the Probabilities of the Error Events:* Following the same argument as in Section IV-D, $R'_1 \geq I(\hat{\mathcal{V}}; Y_1)$ implies that taking n large enough, we can make $\Pr(E_{0,i}) \leq \epsilon$. Next, from the properties of strongly typical sequences (see [43, Lemma 13.6.1]), by taking n large enough, we can make $\Pr(E'_{1,i}) \leq \frac{\epsilon}{2}$. Additionally, the Markov lemma, [50, Lemma 4.2] implies that we can make $\Pr(E''_{1,i} \cap E'_{1,i} \cap E_{0,i}^c) \leq \frac{\epsilon}{2}$ for any arbitrary $\epsilon > 0$ by taking n large enough. Therefore, by the union bound, $\Pr(E_{1,i} \cap E_{0,i}^c) \leq \epsilon$. We also have that $\Pr(E'_{2,i} \cap E_{1,i}^c \cap E_{0,i}^c) = 0$ because under $E_{1,i}^c \cap E_{0,i}^c$ we have that $\mathbf{x}(w_{0,i}), \mathbf{y}_2(i)$ and $\hat{\mathbf{v}}(z_{\hat{v},i})$ are jointly typical, and by construction, $z_{\hat{v},i} \in S'_{s'_{i+1}}$. Hence, we need to show that the probability $\Pr(E''_{2,i} \cap E_{1,i}^c \cap E_{0,i}^c)$ can be made arbitrarily small. Note that due to the symmetry of the construction, the probability of error does not depend on the specific message $w_{0,i}$ transmitted.

3) *Bounding* $\Pr(E''_{2,i} \cap E^c_{1,i} \cap E^c_{0,i})$: The probability of $E''_{2,i} \cap E^c_{1,i} \cap E^c_{0,i}$ can be written as

$$\begin{aligned} & \Pr\left(E''_{2,i} \cap E^c_{1,i} \cap E^c_{0,i}\right) \\ &= \Pr\left(\left\{\exists z_{\hat{v}} \in S'_{s'_{i+1}}, \exists w \neq w_{0,i}, w \in \mathcal{L}_2(i), \right. \right. \\ & \quad \left. \left. \begin{aligned} & (\mathbf{x}(w), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v}})) \\ & \in A_\epsilon^{*(n)}(X, Y_2, \hat{V}) \end{aligned} \right\} \cap E^c_{1,i} \cap E^c_{0,i}\right) \\ &\stackrel{(a)}{=} \Pr\left(\left\{\exists w \neq w_{0,i}, w \in \mathcal{L}_2(i), \right. \right. \\ & \quad \left. \left. \begin{aligned} & (\mathbf{x}(w), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v},i})) \\ & \in A_\epsilon^{*(n)}(X, Y_2, \hat{V}) \end{aligned} \right\} \cap E^c_{1,i} \cap E^c_{0,i}\right) \\ &+ \Pr\left(\left\{\exists w \neq w_{0,i}, w \in \mathcal{L}_2(i), \right. \right. \\ & \quad \left. \left. \begin{aligned} & \exists z_{\hat{v}} \in S'_{s'_{i+1}}, z_{\hat{v}} \neq z_{\hat{v},i}, \\ & (\mathbf{x}(w), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v}})) \\ & \in A_\epsilon^{*(n)}(X, Y_2, \hat{V}) \end{aligned} \right\} \cap E^c_{1,i} \cap E^c_{0,i}\right) \\ &\triangleq \Pr(E''_{2,1,i}) + \Pr(E''_{2,2,i}) \end{aligned}$$

where (a) is because the elements of $S'_{s'_{i+1}}$ are selected in an independent manner.

We first bound $\Pr(E''_{2,1,i})$ as follows:

$$\begin{aligned} & \Pr(E''_{2,1,i}) \\ &= \sum_{\mathcal{L}_2(i)} \Pr\left(\left\{\exists w \neq w_{0,i}, w \in \mathcal{L}_2(i), \right. \right. \\ & \quad \left. \left. \begin{aligned} & (\mathbf{x}(w), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v},i})) \in A_\epsilon^{*(n)}(X, Y_2, \hat{V}) \\ & \cap E^c_{1,i} \cap E^c_{0,i} \mid \mathcal{L}_2(i) \end{aligned} \right\} \Pr(\mathcal{L}_2(i))\right) \\ &\stackrel{(a)}{\leq} E_{\mathbf{y}_2} \left\{ \sum_{\substack{w \in \mathcal{L}_2(i) \\ w \neq w_{0,i}}} \Pr\left(\left\{(\mathbf{x}(w), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v},i})) \right. \right. \right. \\ & \quad \left. \left. \left. \in A_\epsilon^{*(n)}(X, Y_2, \hat{V}) \right\} \cap E^c_{1,i} \cap E^c_{0,i} \mid \mathbf{y}_2(i)\right) \right\} \\ &= E_{\mathbf{y}_2} \left\{ \sum_{\substack{w \in \mathcal{L}_2(i) \\ w \neq w_{0,i}}} \sum_{\substack{\hat{\mathbf{v}} \in \\ A_\epsilon^{*(n)}(\hat{V} \mid \mathbf{y}_2(i), \mathbf{x}(w))}} \Pr(\hat{\mathbf{v}} \mid \mathbf{y}_2(i), \mathbf{x}(w)) \right\} \\ &\stackrel{(b)}{=} E_{\mathbf{y}_2} \left\{ \sum_{\substack{w \in \mathcal{L}_2(i) \\ w \neq w_{0,i}}} \sum_{\hat{\mathbf{v}} \in A_\epsilon^{*(n)}(\hat{V} \mid \mathbf{y}_2(i), \mathbf{x}(w))} \Pr(\hat{\mathbf{v}} \mid \mathbf{y}_2(i)) \right\} \\ &\leq E_{\mathbf{y}_2} \left\{ \sum_{\substack{w \in \mathcal{L}_2(i) \\ w \neq w_{0,i}}} \left\| A_\epsilon^{*(n)}(\hat{V} \mid \mathbf{y}_2(i), \mathbf{x}(w)) \right\| \times \right. \\ & \quad \left. \max_{\substack{\hat{\mathbf{v}}: \\ (\mathbf{y}_2(i), \hat{\mathbf{v}}) \in A_\epsilon^{*(n)}(Y_2, \hat{V})}} \{\Pr(\hat{\mathbf{v}} \mid \mathbf{y}_2(i))\} \right\} \\ &\stackrel{(c)}{\leq} E_{\mathbf{y}_2} \left\{ \sum_{\substack{w \in \mathcal{L}_2(i) \\ w \neq w_{0,i}}} 2^{n(H(\hat{V} \mid Y_2, X) + 2\eta)} 2^{-n(H(\hat{V} \mid Y_2) - 2\eta)} \right\} \\ &\leq E_{\mathbf{y}_2} \{ \|\mathcal{L}_2(i)\| \} 2^{-n(H(\hat{V} \mid Y_2) - H(\hat{V} \mid Y_2, X) - 4\eta)}, \end{aligned}$$

where (a) is because $\mathcal{L}_2(i)$ is a deterministic function of $\mathbf{y}_2(i)$ and we also applied the union bound and (b) is because $\hat{\mathbf{v}}(z_{\hat{v},i})$ is independent of $\mathbf{x}(w)$ for $w \neq w_{0,i}$. The bounds in (c) on the size of the conditionally typical set and the maximum conditional probability follow from [47, Theorem 5.2] with $\eta \rightarrow 0$ as $\epsilon \rightarrow 0$, assuming that n is large enough. Finally, we note that here

$$\Pr(\mathbf{y}_2(i)) \triangleq \Pr(\mathbf{y}_2(i) \text{ received} \mid \mathbf{x}(w_{0,i}) \text{ transmitted}).$$

Next, applying the same technique to bound the expectation of $\|\mathcal{L}_2(i)\|$ as in [34, Theorem 1] (see also derivation of (A1)), we get that for n large enough

$$E_{\mathbf{y}_2} \{ \|\mathcal{L}_2(i)\| \} \leq 1 + 2^{n(R - I(X; Y_2) + 3\eta)}. \quad (\text{B1})$$

Plugging this back into the bound on $\Pr(E''_{2,1,i})$ we get that

$$\begin{aligned} \Pr(E''_{2,1,i}) &\leq 2^{-n(I(X; \hat{V} \mid Y_2) - 4\eta)} \\ &\quad + 2^{n(R - I(X; Y_2) - H(\hat{V} \mid Y_2) + H(\hat{V} \mid Y_2, X) + 7\eta)} \quad (\text{B2}) \end{aligned}$$

which can be made less than any arbitrary $\epsilon > 0$ by taking n large enough, as long as⁴

$$R \leq I(X; Y_2) - H(\hat{V} \mid Y_2, X) + H(\hat{V} \mid Y_2). \quad (\text{B3})$$

For bounding $\Pr(E''_{2,2,i})$, we begin essentially in the same manner and get that

$$\begin{aligned} & \Pr(E''_{2,2,i}) \\ &\stackrel{(a)}{\leq} E_{\mathbf{y}_2, \hat{\mathbf{v}}} \left\{ \Pr\left(\left\{\exists w \neq w_{0,i}, w \in \mathcal{L}_2(i), \exists z_{\hat{v}} \in S'_{s'_{i+1}}, \right. \right. \right. \\ & \quad \left. \left. \left. \begin{aligned} & z_{\hat{v}} \neq z_{\hat{v},i}, (\mathbf{x}(w), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v}})) \\ & \in A_\epsilon^{*(n)}(X, Y_2, \hat{V}) \end{aligned} \right\} \mid \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v},i})\right) \right\} \\ &\stackrel{(b)}{\leq} E_{\mathbf{y}_2, \hat{\mathbf{v}}} \left\{ \sum_{\substack{z_{\hat{v}} \in S'_{s'_{i+1}} \\ z_{\hat{v}} \neq z_{\hat{v},i}}} \sum_{\substack{w \in \mathcal{L}_2(i) \\ w \neq w_{0,i}}} \Pr\left(\left\{(\mathbf{x}(w), \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v}})) \right. \right. \right. \\ & \quad \left. \left. \left. \in A_\epsilon^{*(n)}(X, Y_2, \hat{V}) \mid \mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v},i})\right) \right\} \right\} \\ &\stackrel{(c)}{=} E_{\mathbf{y}_2, \hat{\mathbf{v}}} \left\{ \sum_{\substack{z_{\hat{v}} \in S'_{s'_{i+1}} \\ z_{\hat{v}} \neq z_{\hat{v},i}}} \sum_{\substack{w \in \mathcal{L}_2(i) \\ w \neq w_{0,i}}} \sum_{\hat{\mathbf{v}} \in A_\epsilon^{*(n)}(\hat{V} \mid \mathbf{y}_2(i), \mathbf{x}(w))} \Pr(\hat{\mathbf{v}}) \right\} \\ &\stackrel{(d)}{\leq} E_{\hat{\mathbf{v}}} \left\{ \left\| S'_{s'_{i+1}} \right\| \right\} E_{\mathbf{y}_2} \{ \|\mathcal{L}_2(i)\| \} 2^{-n(H(\hat{V}) - H(\hat{V} \mid Y_2, X) - 3\eta)} \\ &\stackrel{(e)}{\leq} (1 + 2^{n(R'_1 - C_{12})}) (1 + 2^{n(R - I(X; Y_2) + 3\eta)}) \\ &\quad \times 2^{-n(H(\hat{V}) - H(\hat{V} \mid Y_2, X) - 3\eta)} \\ &\leq 2^{-n(C_{12} + H(\hat{V}) - R'_1 - H(\hat{V} \mid Y_2, X) - 3\eta)} \\ &\quad + 2^{n(R - I(X; Y_2) - I(\hat{V}; Y_2, X) + 6\eta)} + 2^{-n(I(\hat{V}; Y_2, X) - 3\eta)} \\ &\quad + 2^{n(R - I(X; Y_2) - C_{12} + R'_1 - H(\hat{V}) + H(\hat{V} \mid Y_2, X) + 6\eta)} \end{aligned}$$

⁴We assume that $I(X; \hat{V} \mid Y_2) > 0$, otherwise the relay message does not help decoding the source message at R_{x2} .

where (a) is because we dropped the intersection with $E_{1,i}^c \cap E_{0,i}^c$, (b) is due to the union bound, (c) is because $\hat{\mathbf{v}}(z_{\hat{v}})$ is independent of $\mathbf{x}(w)$ and $\mathbf{y}_2(i)$ when $z_{\hat{v}} \neq z_{\hat{v},i}$, and (d) is because

$$\begin{aligned} & E_{\mathbf{y}_2, \hat{\mathbf{v}}} \left\{ \|\mathcal{L}_2(i)\| \cdot \left\| S'_{s'_{i+1}} \right\| \right\} \\ &= E_{\mathbf{y}_2} \left\{ E_{\hat{\mathbf{v}} | \mathbf{y}_2} \left\{ \|\mathcal{L}_2(i)\| \cdot \left\| S'_{s'_{i+1}} \right\| \right\} \right\} \\ &\stackrel{(f)}{=} E_{\mathbf{y}_2} \left\{ \|\mathcal{L}_2(i)\| E_{\hat{\mathbf{v}} | \mathbf{y}_2} \left\{ \left\| S'_{s'_{i+1}} \right\| \right\} \right\} \\ &\stackrel{(g)}{=} E_{\mathbf{y}_2} \left\{ \|\mathcal{L}_2(i)\| E_{\hat{\mathbf{v}}} \left\{ \left\| S'_{s'_{i+1}} \right\| \right\} \right\} \\ &= E_{\mathbf{y}_2} \left\{ \|\mathcal{L}_2(i)\| \right\} E_{\hat{\mathbf{v}}} \left\{ \left\| S'_{s'_{i+1}} \right\| \right\} \end{aligned}$$

where (f) is because the average size of $\mathcal{L}_2(i)$ does not depend on $\hat{\mathbf{v}}(z_{\hat{v},i})$ when $\mathbf{y}_2(i)$ is given, and (g) is because the average size of $S'_{s'_{i+1}}$ does not depend of $\mathbf{y}_2(i)$. The bounds on $\Pr(\hat{\mathbf{v}})$ and $\|A_{\epsilon}^{*(n)}(\hat{\mathbf{V}} | \mathbf{y}_2, \mathbf{x})\|$ in (d) follow from [47, Ch. 5]. The bound on $E_{\mathbf{y}_2} \left\{ \|\mathcal{L}_2(i)\| \right\}$ in (e) follows from (B1). We note that here

$$\begin{aligned} & \Pr(\mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v},i})) \\ & \triangleq \Pr((\mathbf{y}_2(i), \hat{\mathbf{v}}(z_{\hat{v},i})) \text{ received} | \mathbf{x}(w_{0,i}) \text{ transmitted}). \end{aligned}$$

We conclude that $\Pr(E_{2,2,i}^{\prime\prime})$ can be made smaller than any $\epsilon > 0$ by taking n large enough, as long as

$$R \leq I(X; Y_2) - H(\hat{V} | Y_2, X) + C_{12} - R'_1 + H(\hat{V}) \quad (\text{B4})$$

$$R'_1 \leq C_{12} - H(\hat{V} | Y_2, X) + H(\hat{V}) \quad (\text{B5})$$

$$R \leq I(X; Y_2) + I(\hat{V}; Y_2, X) \quad (\text{B6})$$

$$R'_1 \geq I(\hat{V}; Y_1) \quad (\text{B7})$$

where (B7) follows from Appendix B-F.2.

Now note that making $\Pr(E_{2,i}^{\prime\prime} \cap E_{1,i}^c \cap E_{0,i}^c)$ arbitrarily small requires making both $\Pr(E_{2,1,i}^{\prime\prime})$ and $\Pr(E_{2,2,i}^{\prime\prime})$ arbitrarily small. Thus, we also need to satisfy (B3). Combining with (B6) we see that (B3) guarantees (B6) and we are left with (B3), (B4), (B5), and (B7).

The maximum rate is achieved for the minimal R'_1 , therefore, we plug $R'_1 = I(\hat{V}; Y_1)$ in (B4) and combining with (B3) we obtain the following achievable rate:

$$\begin{aligned} R &\leq I(X; Y_2) - H(\hat{V} | Y_2, X) \\ &\quad + \min(C_{12} + H(\hat{V} | Y_1), H(\hat{V} | Y_2)). \quad (\text{B8}) \end{aligned}$$

From the combination of (B5) and (B7), we conclude that this is achievable as long as

$$\begin{aligned} C_{12} &\geq I(\hat{V}; Y_1) + H(\hat{V} | Y_2, X) - H(\hat{V}) \\ &= H(\hat{V} | Y_2, X) - H(\hat{V} | Y_1) \\ &= I(\hat{V}; Y_1 | X, Y_2). \quad (\text{B9}) \end{aligned}$$

Equations (B8) and (B9) give the conditions for the message W to be decoded at R_{x2} with an arbitrarily small probability of error by taking n large enough. Note that the requirement in (B9) implies that when $C_{12} < I(\hat{V}; Y_1 | Y_2, X)$, R_{x1} cannot use this cooperation scheme, and the rate to R_{x2} is simply $I(X; Y_2)$. Combining this with (B8) yields the rate expression in (48) and (49).

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