Active Anomaly Detection in Heterogeneous Processes

Boshuang Huang, Kobi Cohen, Qing Zhao

***Abstract*— An active inference problem of detecting an anomalous process among M heterogeneous processes is considered. At each time, a subset of processes can be probed. The objective is to design a sequential probing strategy that dynamically determines which processes to observe at each time and when to terminate the search so that the expected detection time is minimized under a constraint on the probability of misclassifying any process. This problem falls into the general setting of sequential design of experiments pioneered by Chernoff in 1959, in which a randomized strategy, referred to as the Chernoff test, was proposed and shown to be asymptotically optimal as the error probability approaches zero. For the problem considered in this paper, a low-complexity deterministic test is shown to enjoy the same asymptotic optimality while offering significantly better performance in the finite regime and faster convergence to the optimal rate function, especially when the number of processes is large. Furthermore, the proposed test offers considerable reduction in implementation complexity. Extensions to detecting multiple anomalous processes are also discussed.**

***Index Terms*—** Active hypothesis testing, sequential design of experiments, anomaly detection, dynamic search.

I. Introduction

We consider the problem of detecting an anomalous process (referred to as the target) among M heterogeneous processes (referred to as the cells). At each time, K $(1 \leq K < M)$ cells can be probed simultaneously to search for the target. Each search of cell i generates a noisy observation drawn i.i.d. over time from two different distributions f_i and g_i, depending on whether the target is absent or present. The objective is to designa sequential search strategy that dynamically determines which cells to probe at each time and when to terminate the search so that the expected detection time is minimized under a constraint on the probability of declaring a wrong location of the target.

The above problem is prototypical of searching for rare events in a large number of data streams or a large system. The rare events could be opportunities (e.g., financial trading opportunities or transmission opportunities in dynamic spectrum access [1]), unusual activities in surveillance feedings, frauds in financial transactions, attacks and intrusions in communication and computer networks, anomalies in infrastructures such as bridges, buildings, and the power grid that may indicate catastrophes. Depending on the application, a cell may refer to an autonomous data stream with a continuous data flow or a system component that only generates data when probed.

A. Main Results

The anomaly detection problem considered in this paper is a special case of active hypothesis testing originated from Chernoff's seminal work on sequential design of experiments in 1959 [2]. Compared with the classic passive sequential hypothesis testing pioneered by Wald [3], where the observation model under each hypothesis is predetermined, active hypothesis testing has a control aspect that allows the decision maker to choose the experiment to be conducted at each time. Different experiments generate observations from different distributions under each hypothesis. Intuitively, as more observations are gathered, the decision maker becomes more certain about the true hypothesis, which in turn leads to better choices of experiments.

In [2], Chernoff proposed a *randomized* strategy, referred to as the Chernoff test, and established its asmyptotic (as the error probability diminishes) optimality[1]. This randomized test chooses, at each time, a probability distribution that governs the selection of the experiment to be carried out at this time. This distribution is obtained by solving a minimax problem so that the next observation generated under the random action can best differentiate the current maximum likelihood estimate of the true hypothesis (using all past observations) from its closest alternative, where the closeness is measured by the Kullback-Liebler (KL) divergence. Due to the complexity in solving this minimax problem at each time, the Chernoff test can be expensive to compute and cumbersome to implement, especially when the number of hypotheses or the number of experiments is large.

It is not difficult to see that the problem at hand is a special case of the general active hypothesis testing problem. Specically, the available experiments are in the form of different subsets of K cells to probe, and the number of experiments is $\binom{M}{K}$. Under each hypothesis that cell m $(m = 1, ..., M)$ is the target, the distribution of the next observation (a vector of dimension K) depends on which K cells are chosen. The Chernoff test thus directly apply. Unfortunately, with the large number of hypotheses and the large number of experiments, it can be computationally prohibitive to obtain the Chernoff test.

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[1]The asymptotic optimality of the Chernoff test was shown under the assumption that the hypotheses are distinguishable under every experiment.

In this paper, we show that the anomaly detection problem considered here exhibits sufficient structures to admit a low-complexity *deterministic* policy with strong performance. In particular, we develop a deterministic test that *explicitly* specifies which K cells to search at each given time and show that this test enjoys the same asymptotic optimality as the Chernoff test[2]. Furthermore, extensive simulation examples have demonstrated a significant performance gain over the Chernoff test in the finite regime and faster convergence to the optimal rate function, especially when M is large. In contrast to the Chernoff test, the proposed test requires little offline or online computation. The test can also be extended to cases with multiple targets as discussed in Section V. Its asymptotic optimality is preserved for $K = 1$.

We point out that proving the asymptotic optimality of the deterministic policy is much more involved comparing with the Chernoff test, due to the time dependency in the test statistics, namely, the log-likelihood ratios (LLRs), introduced by deterministic actions. In particular, since the distribution of the random action chosen by the Chernoff test depends only on the current maximimum likelihood estimate of the underlying hypothesis which becomes time-invariant after an initial phase with a bounded duration, the stochastic behaviors of the LLRs are independent over time, resulting in a much easier analysis of the detection delay. The deterministic actions of the proposed test, however, lead to complex time dependencies in LLRs that make the analysis much more involved.

B. Related Work

Chernoff's pioneering work on sequential design of experiments focuses on sequential binary composite hypothesis testing [2]. Variations and extensions of the problem were studied in [4]–[9], where the problem was referred to as controlled sensing for hypothesis testing in [5]–[7] and active hypothesis testing in [8], [9]. As variants of the Chernoff test, the tests developed in [4]–[9] are all randomized tests.

There is an extensive literature on dynamic search and target whereabout problems under various scenarios, most of them focusing on homogeneous processes. We discuss here existing studies within the sequential inference setting, which is the most relevant to this work. In [10], the problem of searching among Gaussian signals with rare mean and variance values was studied and an adaptive group sampling strategy was developed. In [11], searching over homogeneous Poisson point processes with unknown rates was investigated and an asymptotically optimal randomized test was developed. In [12], the problem of tracking a target that moves as a Markov Chain in a finite discrete environment is studied and a search strategy that provides the most confident estimate is developed. In [13], an important case of multichannel sequential change detection is studied and an asymptotic framework in which the number of sensors tends to infinity was proposed. Asymptotically optimal search policies over homogeneous processes were established in [14] under a non-parametric setting with finite discrete distributions and in [15] under a parametric composite hypothesis setting with continuous distributions. In [16], [17], the problem of quickly detecting anomalous components under the objective of minimizing system-wide cost incurred by all anomalous components was studied. The objective of minimizing operational cost as opposed to detection delay led to a different problem from the one considered in this paper. Other related work on quickest search over multiple processes under various models and formulations includes [18]–[21] and references therein. Sequential spectrum sensing within both the passive and active hypothesis testing frameworks has also received extensive attention in the application domain of cognitive radio networks (see, for example, [22]–[25] and references therein). The readers are also referred to [26] for a comprehensive survey on the problem of detecting outlying sequences.

A prior study by Cohen and Zhao considered the problem for homogeneous processes (i.e., $f_i \equiv f$ and $g_i \equiv g$) [27]. This work builds upon this prior work and addresses the problem in heterogeneous systems where the absence distribution f_i and the presence distribution g_i are different across processes. Allowing heterogeneity significantly complicates the design of the test and the establishment of asymptotic optimality. Specifically, since each process has different observation distributions, the rate at which the state of a cell can be inferred is different across processes. Hence, the decision maker must balance the search time effectively among the observed processes, which makes both the algorithm design and the performance analysis much more involved under the heterogeneous case. In terms of algorithm design, when dealing with homogeneous processes, the search strategy is often static in nature [11], [14], [18], [27]. In contrast, the asymptotically optimal search strategy developed here for heterogeneous processes dynamically changes based on the current belief about the location of the target. In terms of performance analysis, handling heterogeneity adds new challenges and difficulties for establishing asymptotic optimality. When searching over homogeneous processes, the resulting rate function (which is inversely proportional to the search time) always obeys a certain averaging over the KL divergences between normal and abnormal distributions of all process. This observation follows from the fact that the decision maker completes gathering the required information from all the processes at approximately the same time due to the homogeneity. In contrast, when searching over heterogeneous processes, the overall rate function does not always obey a simple averaging across the KL divergences of all processes. In Section IV, we show that the search time can be analyzed by considering two separate scenarios, referred to as the balanced and the unbalanced cases. The balanced case holds when the a judicious allocation of probing resources can ensure the information gathering from all the processes be completed at approximately the same time, in which case the rate function is a weighted average among the heterogeneous processes. The unbalanced case occurs when there is a process with a sufficiently small KL divergence that it dominates the overall rate function of the search. We establish asymptotic optimality by analyzing the sum LLR dynamics of the heterogeneous processes under these two cases which adds significant technical difficulties as

[2]The asymptotic optimality of the proposed test holds for all but at most three singular values of K (see Lemma 1 and Theorem 1).

compared to the homogeneous case as detailed in Section IV.

Besides the active inference approach to anomaly detection considered in this paper, there is a growing body of literature on various approaches to the general problem of anomaly detection. We refer the readers to [28], [29] for comprehensive surveys on this topic.

II. Problem Formulation

We consider the problem of detecting a single target located in one of M cells. If the target is in cell m, we say that hypothesis H_m is true. The *a priori* probability that H_m is true is denoted by π_m, where $\sum_{m=1}^{M} \pi_m = 1$. To avoid trivial solutions, it is assumed that $0 < \pi_m < 1$ for all m.

When cell m is observed at time n, an observation $y_m(n)$ is drawn, independent of previous observations. If cell m contains a target, $y_m(n)$ follows distribution $g_m(y)$. Otherwise, $y_m(n)$ follows distribution $f_m(y)$. Let $\mathbf{P}_m$ be the probability measure under hypothesis H_m and $\mathbf{E}_m$ the operator of expectation with respect to the measure $\mathbf{P}_m$.

An active search strategy Γ consists of a stopping rule τ governing when to terminate the search, a decision rule δ for determining the location of the target at the time of stopping, and a sequence of selection rules $\{\phi(n)\}_{n\geq 1}$ governing which K cells to probed at each time n. Let $\mathbf{y}(n)$ be the set of all cell selections and observations up to time n. A deterministic selection rule $\phi(n)$ at time n is a mapping from $\mathbf{y}(n-1)$ to $\{1, 2, ..., M\}^K$. A randomized selection rule $\phi(n)$ is a mapping from $\mathbf{y}(n-1)$ to probability mass functions over $\{1, 2, ..., M\}^K$.

The error probability under policy Γ is defined as $P_e(\Gamma) = \sum_m \pi_m \alpha_m(\Gamma)$, where $\alpha_m(\Gamma) = \mathbf{P}_m(\delta \neq m|\Gamma)$ is the probability of declaring $\delta \neq m$ when H_m is true. Let $\mathbf{E}(\tau|\Gamma) = \sum_{m=1}^{M} \pi_m \mathbf{E}_m(\tau|\Gamma)$ be the average detection delay under Γ.

We adopt a Bayesian approach as in Chernoff's original study [2] by assigning a cost of c for each observation and a loss of 1 for a wrong declaration. Note that c represents the ratio of the sampling cost to the cost of wrong detections. The Bayes risk under strategy Γ when hypothesis H_m is true is given by:

$$R_m(\Gamma) \triangleq \alpha_m(\Gamma) + c\mathbf{E}_m(\tau|\Gamma). \tag{1}$$

The average Bayes risk is given by:

$$R(\Gamma) = \sum_{m=1}^{M} \pi_m R_m(\Gamma) = P_e(\Gamma) + c\mathbf{E}(\tau|\Gamma). \tag{2}$$

The objective is to find a strategy Γ that minimizes the Bayes risk $R(\Gamma)$:

$$\inf_{\Gamma} \quad R(\Gamma). \tag{3}$$

A strategy Γ^* is *asymptotically optimal* if

$$\lim_{c\to 0} \frac{R(\Gamma^*)}{\inf_\Gamma R(\Gamma)} = 1, \tag{4}$$

which is denoted as

$$R(\Gamma^*) \sim \inf_{\Gamma} R(\Gamma). \tag{5}$$

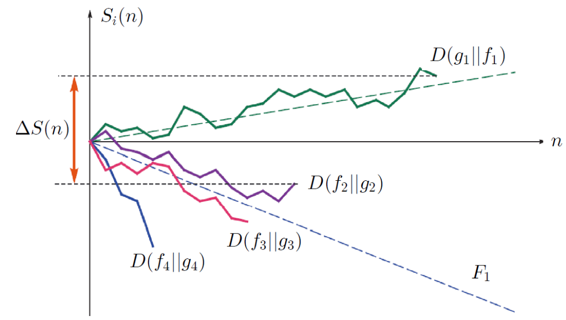

Fig. 1: Typical sample paths of sum LLRs.

III. The Deterministic DGFi Policy

In this section we propose a deterministic policy, referred to as the DGFi policy.

A. *DGFi policy for* $K = 1$

We first consider the case where only a single process can be observed at a time, i.e., $K = 1$.

Let $\mathbf{1}_m(n)$ be the indicator function, where $\mathbf{1}_m(n) = 1$ if cell m is observed at time n, and $\mathbf{1}_m(n) = 0$ otherwise. Let

$$\ell_m(n) \triangleq \log \frac{g_m(y_m(n))}{f_m(y_m(n))} \ , \tag{6}$$

and

$$S_m(n) \triangleq \sum_{t=1}^{n} \ell_m(t)\mathbf{1}_m(t) \tag{7}$$

be the log-likelihood ratio (LLR) and the observed sum LLRs of cell m at time n, respectively. Let $D(g||f)$ denote the KL divergence between two distributions g and f given by

$$D(g||f) \triangleq \int_{-\infty}^{\infty} \log \frac{g(x)}{f(x)} g(x)\, dx. \tag{8}$$

Illustrated in Fig. 1 are typical sample paths of the sum LLRs of $M = 4$ cells, where, without loss of generality, we assume that cell 1 is the target. Note that the sum LLR of cell 1 is a random walk with a positive expected increment $D(g_1||f_1)$, whereas the sum LLR of cell i is a random walk with a negative expected increment $-D(f_i||g_i)$ for $i = 2, 3, 4$. Thus, when the gap between the largest sum LLR and the second largest sum LLR is sufficiently large, we can declare with sufficient accuracy that the cell with the largest sum LLR is the target. This is the intuition behind the stopping rule and the decision rule. Specifically, we define $m^{(i)}(n)$ as the index of the cell with the i^{th} largest observed sum LLRs at time n. Let

$$\Delta S(n) \triangleq S_{m^{(1)}(n)}(n) - S_{m^{(2)}(n)}(n) \tag{9}$$

denote the difference between the largest and the second largest observed sum LLRs at time n. The stopping rule and the decision rule under the DGFi policy are given by:

$$\tau = \inf\left\{n \ : \ \Delta S(n) \geq -\log c\right\} \ , \tag{10}$$

and

$$\delta = m^{(1)}(\tau) \ . \tag{11}$$

We now specify the selection rule of the DGFi policy. The intuition behind the selection rule is to select a cell from which the observation can increase $\Delta S(n)$ at the fastest rate. The selection rule is thus given by comparing the rate at which $S_{m^{(1)}}(n)$ increases with the rate at which $S_{m^{(2)}(n)}(n)$ decreases. If $S_{m^{(1)}(n)}$ is expected to increase faster than $S_{m^{(2)}(n)}(n)$ decreases, cell $m^{(1)}(n)$ is chosen. Otherwise, cell $m^{(2)}(n)$ is chosen. This leads to the following selection rule:

$$\phi(n) = \begin{cases} m^{(1)}(n) & \text{if } D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)}) \geq F_{m^{(1)}(n)} \\ m^{(2)}(n) & \text{otherwise} \end{cases}, \tag{12}$$

where

$$F_m \triangleq \frac{1}{\sum_{j\neq m} \frac{1}{D(f_j||g_j)}}. \tag{13}$$

The selection rule in (12) can be intuitively understood by noticing that $D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)})$ is the asymptotic increasing rate of $S_{m^{(1)}}(n)$ when cell $m^{(1)}$ is probed at each time. This is due to the fact that $m^{(1)}(n)$ is the true target after an initial phase (defined by the last passage time that $m^{(1)}(n)$ is an empty cell) which can be shown to have a bounded expected duration. Similarly, even though much more involved to prove, $F_{m^{(1)}(n)}$ is the asymptotic rate at which $S_{m^{(2)}(n)}(n)$ decreases when cell $m^{(2)}(n)$ is probed at each time. To see the expression of F_m for any m as given in (13), consider the following analogy. Consider $M-1$ cars being driven by a single driver from 0 to $-\infty$. Car j ($j = 1, \ldots, M$, $j \neq m$) has a constant speed of $D(f_j||g_j)$. At each time, the car closest to the origin is chosen by the driver and driven by one unit of time. We are interested in the average moving speed of the position of the closest car to the origin. It is not difficult to see that it is given by F_m in (13). This analogy, concerned with deterministic processes, only serves as an intuitive explanation for the expression of F_m. As detailed in Sec. IV, proving $F_{m^{(1)}(n)}$ to be the asymptotic decreasing rate of $S_{m^{(1)}(n)}(n)$ requires analyzing the trajectories of the M sum LLRs $\{S_m(n)\}_{m=1}^M$, which are stochastic processes with complex dependencies both in time and across processes.

B. DGFi under multiple simultaneous observations

Next we extend the DGFi policy to the case where multiple simultaneous observations are allowed, i.e., $K > 1$.

The stopping rule and the decision rule remains the same as given in (10), (11), whereas the selection rule requires significant modification. The main reason is that when K cells can be observed simultaneously, the asymptotic increasing rate of $S_{m^{(1)}(n)}(n)$ and the asymptotic decreasing rate of $S_{m^{(2)}(n)}(n)$ are much more involved to analyze.

The selection rule is as follows. At each time n, the selection rule $\phi(n)$, as given in (14), chooses either the K cells with the top K largest sum LLRs or those with the second to the $(K+1)^{th}$ largest sums LLRs, where

$$F_m(K) \triangleq \min\{\frac{K}{\sum_{j\neq m} \frac{1}{D(f_j||g_j)}},\ \min_{j\neq m} D(f_j||g_j)\}. \tag{15}$$

Similar to the case with $K = 1$, the intuition behind the selection rule is to select K cells from which the observations increase $\Delta S(n)$ at the fastest rate. Specifically, $F_{m^{(1)}(n)}(K)$ is the asymptotic decreasing rate of $S_{m^{(2)}(n)}(n)$ when K cells with the second largest to the $(K+1)$th largest sum LLRs are probed each time. The expression of $F_m(K)$ for any m as given in (15) can be explained with the same car analogy, except now there are $K > 1$ drivers. It is not difficult to see that $F_m(K)$ is upper bounded by the speed $\min_{j\neq m} D(f_j||g_j)$ of the slowest car among the $M-1$ cars. In particular, when the speed of the slowest car is sufficiently small, this car always lags behind even with a dedicated driver. We refer to this case as the unbalanced case, which presents the most challenge in proving the asymptotic optimality of DGFi (see Theorem 1 and Appendix B). With this intuitive understanding of $F_m(K)$, we can see that the asymptotic increasing rate of $\Delta S(n)$ is $D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)}) + F_{m^{(1)}(n)}(K-1)$ when the cells with the top K largest sum LLRs are probed each time, where $D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)})$ is the asymptotic increasing rate of $S_{m^{(1)}(n)}(n)$ and $F_{m^{(1)}(n)}(K-1)$ is the asymptotic decreasing rate of $S_{m^{(2)}(n)}(n)$.

It is easy to see that when $K = 1$, the policy reduces to the one described in section III-A.

IV. Performance Analysis

In this section, we establish the asymptotic optimality of the DGFi policy. While the intuitive exposition of DGFi given in Sec. III may make its asymptotic optimality seem expected, constructing a proof is much more involved. In particular, bounding the detection time of DGFi requires analyzing the trajectories of the M stochastic processes $\{S_m(n)\}_{m=1}^M$ which exhibit complex dependencies both over time and across processes as induced by the deterministic selection rule.

We first state the following assumption.

Assumption 1: Under hypothesis H_m, assume that

$$u_m^* \triangleq \arg\max_{u\in[0,1]} uD(g_m||f_m) + F_m(K-u) \tag{16}$$

takes value of either 0 or 1, where we allow the domain of $F_m(\cdot)$ defined in (15) to be all real numbers.

In the following lemma, we give an explicit characterization on when Assumption 1 is violated and at what values of K.

Lemma 1: For any given $\{D(g_i||f_i), D(f_i||g_i)\}_{i=1}^M$, we have the following statements.

$$\phi(n) = \begin{cases} \left(m^{(1)}(n), m^{(2)}(n), ..., m^{(K)}(n)\right) & \text{if } D(g_{m^{(1)}(n)}||f_{m^{(1)}(n)}) + F_{m^{(1)}(n)}(K-1) \geq F_{m^{(1)}(n)}(K) \\ \left(m^{(2)}(n), m^{(3)}(n), ..., m^{(K+1)}(n)\right) & \text{otherwise} \end{cases} \tag{14}$$

1) For each $m = 1, \dots, M$, Assumption 1 holds if at least one of the following two statements is true:
   (a) $\sum_{j \neq m} \frac{\min_{j \neq m} D(f_j||g_j)}{D(f_j||g_j)}$ is an integer,
   (b) $D(g_m||f_m) \geq \frac{1}{\sum_{j \neq m} \frac{1}{D(f_j||g_j)}}$.

   If neither is true, then Assumption 1 does not hold for a single value of K, denoted as $\tilde{K}_m$, as given below.

$$\tilde{K}_m = \left\lceil \sum_{j \neq m} \frac{\min_{j \neq m} D(f_j||g_j)}{D(f_j||g_j)} \right\rceil \tag{17}$$

2) All $\{\tilde{K}_m\}_{m=1}^M$ take at most three distinct values.

Proof: See Appendix A. ■

From Lemma 1, we conclude that for every given M and $\{D(g_i||f_i), D(f_i||g_i)\}_{i=1}^M$, Assumption 1 always holds for $K = 1$. For the general case and with an arbitrarily large M, Assumption 1 holds for all but at most three values of $K \in \{1, 2, \dots, M\}$.

Next, we establish the asymptotic optimality of DGFi under Assumption 1. Define

$$I_m \triangleq \max\{D(g_m||f_m) + F_m(K-1), F_m(K)\}, \tag{18}$$

which is the increasing rate of $\Delta S(n)$ under hypothesis H_m. For a given *a priori* distribution $\{\pi_m\}_{m=1}^M$ of the true hypothesis, define

$$I^* \triangleq \frac{1}{\sum_{m=1}^M \frac{\pi_m}{I_m}}. \tag{19}$$

As shown in Theorem 1 below, I^* is the optimal rate function of the Bayes risk.

Theorem 1: Let R^* and $R(\Gamma)$ be the Bayes risks under the DGFi policy and an arbitrary policy Γ, respectively. Assume that Assumption 1 holds for all $m = 1, \dots, M$. Then,

$$R^* \sim \frac{-c \log c}{I^*} \sim \inf_{\Gamma} R(\Gamma) \tag{20}$$

Proof: Here we provide a sketch of the proof. The detailed proof can be found in Appendix B.

We first show that the proposed DGFi policy achieves a Bayes risk $-c \log c / I^*$ asymptotically. First, we show that when $\Delta S(\tau)$ is large, the probability of error is small, i.e. $P_e = O(c)$. As a result, by the definition of the Bayes risk, it suffices to show that the detection time is upper bounded by $-\log c / I^*$. By the definition of I^* in (19), it suffices to show that the detection time is upper bounded by $-\log c / I_m$ under hypothesis H_m. Since the decision maker might not complete to gather the required information from all the cells at the same time, we carry out the analysis by considering the balanced and the unbalanced cases separately. In particular, if $K / \sum_{j \neq m} \frac{1}{D(f_j||g_j)} < \min_{j \neq m} D(f_j||g_j)$, we refer to this case as the *balanced case*. Otherwise, we refer to this case as the *unbalanced case*.

The balanced case is when $K / \sum_{j \neq m} \frac{1}{D(f_j||g_j)} \leq \min_{j \neq m} D(f_j||g_j)$. The key to bounding the detection time in this case is to show that the dynamic range of the $M-1$ sum LLRs corresponding to the $M-1$ empty cells are sufficiently small such that the increasing rate of $\Delta S(n)$ is given by a certain averaging among the heterogeneous processes.

The unbalanced case is when $K / \sum_{j \neq m} \frac{1}{D(f_j||g_j)} > \min_{j \neq m} D(f_j||g_j)$. In this case, there is a process with a sufficiently small information acquisition rate $D(f_j||g_j)$ such that it becomes the bottleneck of the detection process and determines the asymptotic increasing rate of $\Delta S(n)$. Directly bounding the dynamic range of all sum LLR trajectories is no longer tractable. Instead, the proof is built upon the analysis of the trajectory of the sum LLR with the smallest expected increment. In particular, we recognize that the key in handling the imbalance in the information acquisition rates among empty cells is to define a last passage time as the last time at which the empty cell with the smallest $D(f_j||g_j)$ is not probed and then analyze, separately, the detection process before and after this last passage time.

Next, we show that $\frac{-c \log c}{I^*}$ is an asymptotic lower bound on the Bayes risk. This is done by first proving that if the Bayes risk is sufficiently small under strategy Γ, i.e., $R(\Gamma) = O(-c \log c)$, the difference between the largest sum LLRs and the second largest sum LLRs must be sufficiently large when the test terminates, i.e. $\Delta S(\tau) = \Omega(-\log c)$. Otherwise, it is not possible to achieve a risk $O(-c \log c)$ due to a large error probability. We then show that in order to make $\Delta S(n)$ sufficiently large, the sample size must be large enough, i.e., $E[\tau|\Gamma] \geq \frac{-\log c}{I^*}$. Since each sample costs c, the total risk will be lower bounded by $\frac{-c \log c}{I^*}$ as desired. ■

V. Extension to Detecting Multiple Targets

In this section we extend the DGFi policy to the case with $L > 1$ targets. The number of hypotheses in this case is $\binom{M}{L}$. We consider first $K = 1$. The DGFi policy can be extended to detect multiple targets as follows. The stopping rule and decision rule are given below, similar in principle to those for $L = 1$ as described in Section III:

$$\tau = \inf\{n \ : \ \Delta S_L(n) \geq -\log c\}, \tag{21}$$

$$\delta = \{m^{(1)}(\tau), m^{(2)}(\tau), ..., m^{(L)}(\tau)\}, \tag{22}$$

where

$$\Delta S_L(n) \triangleq S_{m^{(L)}(n)}(n) - S_{m^{(L+1)}(n)}(n) \tag{23}$$

denotes the difference between the L^{th} and the $(L+1)^{th}$ largest observed sum LLRs at time n.

For the selection rule, define, for a given set $\mathcal{D} \subset \{1, 2, \dots, M\}$ with $|\mathcal{D}| = L$,

$$F_{\mathcal{D}} \triangleq \frac{1}{\sum_{j \notin \mathcal{D}} \frac{1}{D(f_j||g_j)}}. \tag{24}$$

Similar to F_m defined in (13), $F_{\mathcal{D}}$ can be viewed as the asymptotic increasing rate of $\Delta S_L(n)$ when the L targets are given by set $\mathcal{D}$ and we probe the cell with the $(L+1)^{th}$ largest sum LLR. We also define

$$G_{\mathcal{D}} \triangleq \frac{1}{\sum_{j \in \mathcal{D}} \frac{1}{D(g_j||f_j)}}, \tag{25}$$

which can be viewed as the asymptotic increasing rate for $\Delta S_L(n)$ when we probe the cell with the L^{th} largest sum LLR.

The selection rule follows the same design principle of maximizing the asymptotic increasing rate of $\Delta S_L(n)$, and is given by

$$\phi(n) = \begin{cases} m^{(L)}(n) & \text{if } G_{\mathcal{D}(n)} \geq F_{\mathcal{D}(n)} \\ m^{(L+1)}(n) & \text{otherwise} \end{cases}, \tag{26}$$

where

$$\mathcal{D}(n) = \{m^{(1)}(n), m^{(2)}(n), ..., m^{(L)}(n)\}. \tag{27}$$

It is not difficult to see that when $L = 1$, the policy reduces to the one described in Section III.

Next, we establish the asymptotic optimality of the DGFi policy for $L > 1$ and $K = 1$. Let $\mathcal{D}$ denote a subset of L cells and $\pi_{\mathcal{D}}$ the prior probability of hypothesis $H_{\mathcal{D}}$ (i.e, the target cells are given by $\mathcal{D}$). Define

$$\begin{aligned} I_{\mathcal{D}} &\triangleq \max\{F_{\mathcal{D}}, G_{\mathcal{D}}\}, \\ I_L^* &\triangleq \frac{1}{\sum_{\mathcal{D}} \frac{\pi_{\mathcal{D}}}{I_{\mathcal{D}}}}, \end{aligned} \tag{28}$$

where I_L^* is again the optimal rate function of the Bayes risk as shown in the theorem below, and reduces to the one defined in (19) when $L = 1$.

Theorem 2: Let R_L^* and $R_L(\Gamma)$ be the Bayes risks under the DGFi policy and an arbitrary policy Γ, respectively. Then, for $K = 1$,

$$R^* \sim \frac{-c \log c}{I_L^*} \sim \inf_{\Gamma} R(\Gamma) . \tag{29}$$

Proof: See Appendix C. ∎

When $K > 1$, the stopping rule and the decision rule remain the same. For the selection rule, define

$$F_{\mathcal{D}}(K) \triangleq \min\{\frac{K}{\sum_{j \notin \mathcal{D}} \frac{1}{D(f_j||g_j)}}, \min_{j \notin \mathcal{D}} D(f_j||g_j)\}. \tag{30}$$

Similar to $F_m(K)$ defined in (15), $F_{\mathcal{D}}(K)$ can be viewed as the asymptotic increasing rate of $\Delta S_L(n)$ when the L targets are given by set $\mathcal{D}$ and we probe those K cells with the $(L+1)^{th}$ to the $(L+K)^{th}$ largest sum LLR. Similarly,

$$G_{\mathcal{D}}(K) \triangleq \min\{\frac{K}{\sum_{j \in \mathcal{D}} \frac{1}{D(g_j||f_j)}}, \min_{j \in \mathcal{D}} D(g_j||f_j)\}, \tag{31}$$

which can be viewed as the asymptotic increasing rate of $\Delta S_L(n)$ when we probe the cells with the $(L-K+1)^{th}$ to the L^{th} largest sum LLR.

Let

$$k_{\mathcal{D}}^* \triangleq \arg \max_{0 \leq k \leq K} F_{\mathcal{D}}(K-k) + G_{\mathcal{D}}(k), \tag{32}$$

which can be interpreted as the optimal number of target cells that should be probed at each time for maximizing the asymptotic increasing rate of $\Delta S_L(n)$. The selection rule of DGFi is thus given by

$$\phi(n) = \{m^{(L-k^*_{\mathcal{D}(n)}+1)}(n), \cdots, m^{(L-k^*_{\mathcal{D}(n)}+K)}(n)\}, \tag{33}$$

where

$$\mathcal{D}(n) = \{m^{(1)}(n), m^{(2)}(n), ..., m^{(L)}(n)\}. \tag{34}$$

The asymptotic optimality of DGFi for $L > 1$ and $K > 1$ remains open. We have, however, strong belief of the following conjecture.

Conjecture 1: The DGFi policy preserves its asymptotic optimality if

$$u_{\mathcal{D}}^* \triangleq \arg \max_{u \in [0,K]} F_{\mathcal{D}}(K-u) + G_{\mathcal{D}}(u) \tag{35}$$

is an integer for all $\mathcal{D}$, where we allow the domain of $F_{\mathcal{D}}(\cdot)$ and $G_{\mathcal{D}}(\cdot)$ to be real numbers.

VI. Comparison with the Chernoff Test

In this section, we compare the performance of the proposed DGFi policy and the Chernoff test in terms of both computational complexity and sample complexity.

A. The Chernoff Test

The Chernoff test has a randomized selection rule. Specifically, let $q = (q_1, ..., q_\kappa)$ be a probability mass function over a set of κ available experiments $\{u_i\}_{i=1}^{\kappa}$ that the decision maker can choose from, where q_i is the probability of choosing experiment u_i. Note that in our case, $\kappa = \binom{M}{K}$. For a general M-ary active hypothesis testing problem, the action at time n under the Chernoff test is drawn from a distribution $q^*(n) = (q_1^*(n), ..., q_\kappa^*(n))$ that depends on the past actions and observations:

$$q^*(n) = \arg \max_{q} \min_{j \in \mathcal{M} \setminus \{\hat{i}(n)\}} \sum_{u_i} q_i D(p_{\hat{i}(n)}^{u_i} || p_j^{u_i}) , \tag{36}$$

where $\mathcal{M}$ is the set of the M hypotheses, $\hat{i}(n)$ is the ML estimate of the true hypothesis at time n based on past actions and observations, and $p_j^{u_i}$ is the observation distribution under hypothesis j when action u_i is taken. The stopping rule and the decision rule are the same as in (10), (11).

B. Comparison in computational complexity

Here we compare the computational complexity of the proposed DGFi policy with the Chernoff test. We show that the Chernoff test can be expensive to compute especially when the number of processes or the number of experiments is large. In contrast to the Chernoff test, the DGFi policy requires little computaion.

For the case of detecting a single target ($L = 1$), computing the selection rule of Chernoff test defined in (36) requires solving M minimax problems, each corresponding to a particular value of the ML estimate $\hat{i}(n) \in \{1, \ldots, M\}$. One efficient way of solving minimax problems is through linear programming which takes polynomial time with respect to the number of variables and constraints. For this problem, however, the number of variables is $\binom{M}{K}$, which is not polynomial and can be exponential in M in the worst case.

The only computation involved in the selection rule of DGFi is (15), which requires M summations each with $M-1$ elements. As a result, the compuational time is $O(M^2)$, which is polynomial in M and independent of K.

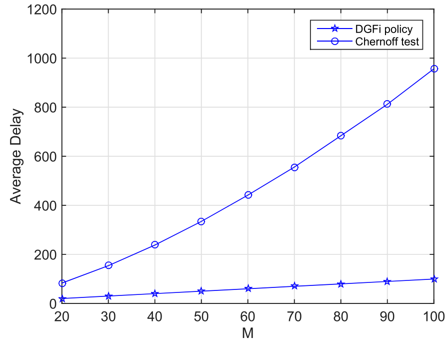

Fig. 2: Performance comparison ($K = 1, \lambda_g^{(m)} = 9 + m, \lambda_f^{(m)} = 0.0188, c = 10^{-5}$).

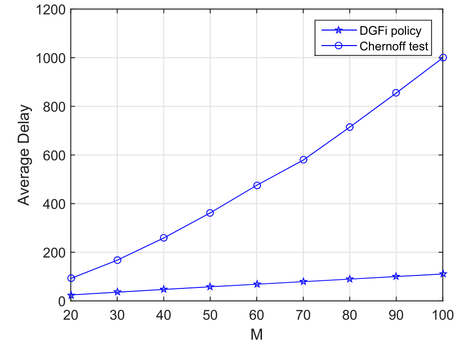

Fig. 4: Performance comparison ($L = 2, K = 1, \lambda_g^{(m)} = 9 + m, \lambda_f^{(m)} = 0.0188, c = 10^{-5}$).

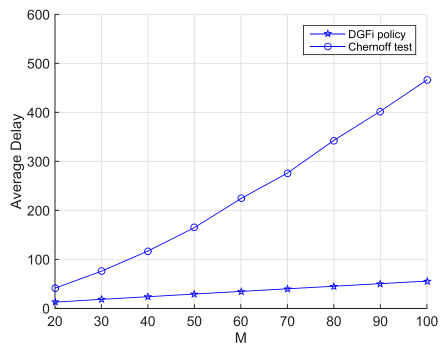

Fig. 3: Performance comparison ($K = 2, \lambda_g^{(m)} = 9 + m, \lambda_f^{(m)} = 0.0188, c = 10^{-5}$).

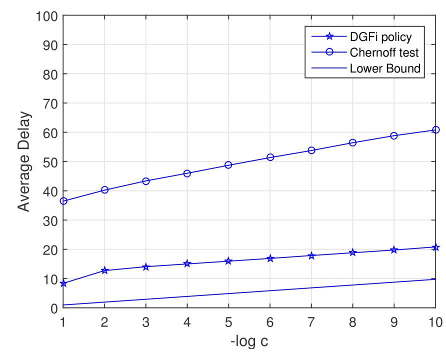

Fig. 5: Performance comparison ($M = 20, L = 2, K = 2, \lambda_g^{(m)} = 50 + 0.1m, \lambda_f^{(m)} = 2$).

C. Comparison in sample complextity

Although both the Chernoff test and the DGFi policy are asymptotically optimal[3], we show below via simulation examples the significant performance gain of DGFi over the Chernoff test in the finite regime (i.e., when the sample cost c is bounded away from 0).

Consider a uniform prior and exponentially distributed observations: $f_m \sim \exp(\lambda_f^{(m)})$ and $g_m \sim \exp(\lambda_g^{(m)})$. The KL divergences can be easily computed as follows.

$$D(g_m||f_m) = \log(\lambda_g^{(m)}) - \log(\lambda_f^{(m)}) + \frac{\lambda_f^{(m)}}{\lambda_g^{(m)}} - 1 ,$$

$$D(f_m||g_m) = \log(\lambda_f^{(m)}) - \log(\lambda_g^{(m)}) + \frac{\lambda_g^{(m)}}{\lambda_f^{(m)}} - 1 .$$

Shown in Fig. 2 is the performance comparison between DGFi policy and Chernoff test for $L = 1$ and $K = 1$. The figure clearly demonstrates the significant reduction in detection delay offered by the DGFi policy as compared with the Chernoff test. The performance gain increases drastically as M increases. A similar comparison is observed in Fig. 3 with $K = 2$. The performance comparison for a case with multiple targets is shown in Fig. 4 with $L = 2, K = 1$.

In Fig. 5, we consider a case of $L = 2$ and $K = 2$ and examine the performance of DGFi as c approaches 0. Also plotted in Fig. 5 is the asymptotic lower bound $\frac{-\log c}{I_L^*}$ on the detection delay which increases linearly with $-\log c$ with rate $1/I_L^*$ as given in (28). We observe that the increasing rate of the detection delay offered by DGFi quickly converges to that of the lower bound, which supports the conjecture that DGFi preserves its asymptotic optimality for the case with $L > 1$ and $K > 1$. Besides showing the same level of reduction in the finite regime compared with the Chernoff test, Fig. 5 also reveals a significantly faster convergence to the optimal rate function I_L^* with the detection delay of the Chernoff test increasing at a faster rate even at $c = 10^{-10}$.

Next, we provide an intuition argument for the better finite-time performance of DGFi. Consider a special case where $K = 1$ and all f_i and g_i are identical, i.e., $f_i \equiv f$ and $g_i \equiv g$ and we assume $D(f||g) > (M-1)D(g||f)$. In this case, the DGFi policy chooses, at each time, the cell with the second largest sum LLR whereas the Chernoff test randomly and uniformly chooses a cell from all but the one with the largest sum LLR at each time. Consider a short horizon scenario where the sampling cost c is sufficiently high such that $D(f||g) > -\log c$. This means each empty cell only need one observation (with high probability) to distinguish

[3] While the assumption of positive KL divergence between every pair of hypotheses under every probing action as required in Chernoff's proof of asymptotic optimality does not hold here, it can be shown that Chernoff test preserves its asymptotic optimality for the problem at hand.

from the true cell. We can formulate this as coupon collectors problem, where each empty cell is a coupon and the goal is to collect all $M-1$ coupons.

Since Chernoff test employs a randomized strategy that chooses empty cells with equal probability, based on results in coupon collectors problem, the expected probing time will be roughly $M \log M$. However, the proposed DGFi policy is deterministic and guaranteed to collect a new coupon at each time, therefore the expected probing time will only be M.

VII. Conclusion

The problem of detecting anomalies among a large number of heterogeneous processes was considered. A low-complexity deterministic test was developed and shown to be asymptotically optimal. Its finite-time performance and computational complexity were shown to be superior to the classic Chernoff test for active hypothesis testing, especially when the problem size is large.

Appendix A: proof of Lemma 1

Define

$$h_m(u) = uD(g_m||f_m) + F_m(\tilde{K}_m - u). \tag{37}$$

If u_m^* takes value other than 0 or 1, i.e., $u_m^* \in (0,1)$, then $h'_m(u) > 0$ for $u \in (0, u_m^*)$ and $h'_m(u) < 0$ for $u \in (u_m^*, 1)$. By taking the derivative of $h_m(u)$, we have

$$h'_m(u) = D(g_m||f_m) - F'_m(\tilde{K}_m - u), \tag{38}$$

where

$$F'_m(u) = \begin{cases} \frac{1}{\sum_{j\neq m} \frac{1}{D(f_j||g_j)}}, & \text{if } u < \sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)} \\ 0, & \text{if } u > \sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)}. \end{cases} \tag{39}$$

Since $F'_m(u)$ is piecewise constant with a breakpoint $\sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)}$, $h'_m(u)$ is piecewise constant with a breakpoint $\tilde{K}_m - \sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)}$. Therefore,

$$\tilde{K}_m = u_m^* + \sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)}. \tag{40}$$

Since $h'_m(u) < 0$ for $u \in (u_m^*, 1)$,

$$D(g_m||f_m) < \frac{1}{\sum_{j\neq m} \frac{1}{D(f_j||g_j)}}. \tag{41}$$

Note that $u_m^* \in (0,1)$ and $\tilde{K}_m$ is an integer. Such $\tilde{K}_m$ exists only if neither (a) or (b) holds and we have

$$\tilde{K}_m = \left\lceil \sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)} \right\rceil. \tag{42}$$

Next we show that there are only three possible values of K. Let $j' = \arg\min_j D(f_j||g_j)$. Since there is only one possible $\tilde{K}_{j'}$ as proved above. It remains to show that there are only two possible values of K_m when $m \neq j'$. Let

$$V \triangleq \sum_{j=1}^{M} \frac{D(f_{j'}||g_{j'})}{D(f_j||g_j)}.$$

Since $0 \leq \frac{D(f_{j'}||g_{j'})}{D(f_m||g_m)} \leq 1$, we have

$$\sum_{j\neq m} \frac{\min_{j\neq m} D(f_j||g_j)}{D(f_j||g_j)} = V - \frac{D(f_{j'}||g_{j'})}{D(f_m||g_m)} \in [V-1, V]$$

for all $m \neq j'$. Combining (42) implies that $\tilde{K}_m, m \neq j'$ can only take 2 possible integers as desired.

Appendix B: proof of Theorem 1

The structure of the proof is as follows. In subsection A, we show that $-c\log c/I^*$ is an asymptotic upper bound on the Bayes risk that DGFi achieves. Specifically, the asymptotic optimality property of DGFi is based on Lemma 8, showing that the asymptotic expected search time is upper bounded by $-\log c/I^*$, while the error probability is $O(c)$ following Lemma 2. In subsection B, we provide the sum LLR analysis of the heterogeneous empty cells. The analysis is based on studying two cases, referred to as balanced and unbalanced cases. For the balanced case, the decision maker can balance the remaining information required to be gathered among the processes. For the unbalanced case, there is a process with a very small KL divergence so that it dominates the increasing rate. Finally, in subsection C we show that the asymptotic Bayes risk that can be achieved by any policy is lower bounded by $-c\log c/I^*$, in which together with Appendix A completes the proof.

Throughout the this section, we use the following notations. Let

$$N_j(n) \triangleq \sum_{t=1}^{n} \mathbf{1}_j(t) \tag{43}$$

be the number of times that cell j has been observed up to time n. Let

$$\Delta S_{m,j}(n) \triangleq S_m(n) - S_j(n) \tag{44}$$

be the difference between the observed sum of LLRs of cells m and j. We also define

$$\Delta S_m(n) \triangleq \min_{j\neq m} \Delta S_{m,j}(n) \ . \tag{45}$$

As a result, we have:

$$\Delta S(n) = S_{m^{(1)}(n)}(n) - S_{m^{(2)}(n)}(n) = \max_m \Delta S_m(n) \ . \tag{46}$$

Without loss of generality we prove the theorem when hypothesis m is true. We define

$$\tilde{\ell}_k(i) = \begin{cases} \ell_k(i) - D(g_k||f_k) \ , \text{ if } k = m, \\ \ell_k(i) + D(f_k||g_k) \ , \text{ if } k \neq m, \end{cases} \tag{47}$$

which is a zero-mean r.v under hypothesis H_m.

A. The Asymptotic Upper Bound on the Bayes Risk under DGFi

In this subsection we show that the Bayes risk obtained by DGFi policy is upper bounded by $-c\log c/I^*$ as c approaches zero..

Lemma 2: If DGFi policy is used, then the error probability is upper bounded by:

$$P_e \leq (M-1)c \; . \tag{48}$$

Proof: Let $\alpha_{m,j} = \mathbf{P}_m(\delta = j)$ for all $j \neq m$. Thus, $\alpha_m = \sum_{j\neq m} \alpha_{m,j}$. By the definition of the stopping rule under DGFi (see (10)), accepting H_j is done when $\Delta S_j(n) \geq -\log c$ which implies $\Delta S_{j,m} \geq -\log c$. Hence, for all $j \neq m$ we have:

$$\begin{aligned} \alpha_{m,j} &= \mathbf{P}_m\left(\delta = j\right) \\ &\leq \mathbf{P}_m\left(\Delta S_{j,m}(\tau) \geq -\log c\right) \\ &\leq c\mathbf{P}_j\left(\Delta S_{j,m}(\tau) \geq -\log c\right) \leq c \; , \end{aligned} \tag{49}$$

where changing the measure in the second inequality follows by the fact that $\Delta S_{j,m}(\tau) \geq -\log c$. As a result,

$$\alpha_m = \sum_{j\neq m} \alpha_{m,j} \leq (M-1)c \; .$$

Hence, (48) follows. ■

Lemma 3: There exist constants $C > 0$ and $\gamma > 0$ such that for any fixed $0 < q < 1$, under any arbitrary policy, the following statements hold:

$$\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) \geq qn\right) \leq Ce^{-\gamma n} \; , \tag{50}$$

and

$$\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_m(n) \geq qn\right) \leq Ce^{-\gamma n} \; , \tag{51}$$

for $m = 1, 2, ..., M$ and $j \neq m$.

Proof: We start with proving (50). Note that $N_j(n), N_m(n)$ can take integer values $N_j(n) = \lceil qn \rceil, \lceil qn \rceil + 1, ...n$, and $N_m(n) = 0, ..., n$. Applying the Chernoff bound and using the i.i.d. property of the observations across time yield:

$$\begin{aligned} &\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) \geq qn\right) \\ &\leq \sum_{r=\lceil qn \rceil}^{n} \sum_{k=0}^{n} \mathbf{P}_m\left(\sum_{i=1}^{r} \ell_j(i) + \sum_{i=1}^{k} -\ell_m(i) \geq 0\right) \\ &\leq \sum_{r=\lceil qn \rceil}^{n} \sum_{k=0}^{n} \left[\mathbf{E}_m\left(e^{s\ell_j(1)}\right)\right]^r \left[\mathbf{E}_m\left(e^{s(-\ell_m(1))}\right)\right]^k \end{aligned} \tag{52}$$

for all $s > 0$.

Since a moment generating function (MGF) is equal to one at $s = 0$ and $\mathbf{E}_m(\ell_j(1)) = -D(f_j||g_j) < 0$, $\mathbf{E}_m(-\ell_m(1)) = -D(g_m||f_m) < 0$ are strictly negative, differentiating the MGFs of $\ell_j(1), \ell_m(1)$ with respect to s yields strictly negative derivatives at $s = 0$. As a result, there exist $s > 0$ and $\gamma_1 > 0$ such that $\mathbf{E}_m\left(e^{s\ell_j(1)}\right)$, $\mathbf{E}_m\left(e^{s(-\ell_m(1))}\right)$ are strictly less than $e^{-\gamma_1} < 1$. Hence, there exist $C > 0$ and $\gamma = \gamma_1 \cdot q > 0$ such that

$$\begin{aligned} &\mathbf{P}_m\left(S_j(n) - S_m(n) \geq 0, N_j(n) \geq qn\right) \\ &\leq \sum_{r=\lceil qn \rceil}^{n} e^{-\gamma_1 r} \sum_{k=0}^{n} e^{-\gamma_1 k} \leq Ce^{-\gamma n} \; . \end{aligned} \tag{53}$$

For proving (51) we can use the Chernoff bound with minor modifications. ■

Definition 1: τ_1 is defined as the smallest integer such that $S_m(n) > S_j(n)$ for all $j \neq m$ for all $n \geq \tau_1$.

Note that τ_1 is not a stopping time since it depends on the future. τ_1 is a last passage time. It is the last time in which $S_m(n)$ crosses $S_j(n)$ for all $j \neq m$. In Lemma 4 we show that the probability that τ_1 is greater than n decreases exponentially with n. This result will be used when evaluating the asymptotic expected search time to show that it is not affected by τ_1.

Remark 1: We often analyze the dynamic of the sum LLRs according to the selection rule of DGfi in the asymptotic regime. Thus, when we say that the selection rule of *DGFi policy is implemented indefinitely* we mean that we probe the cells according to the selection rule of DGFi as given in (14) indefinitely, while the stopping rule is disregarded.

Lemma 4: If the selection rule of DGFi is implemented indefinitely, there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m\left(\tau_1 > n\right) \leq Ce^{-\gamma n} \; , \tag{54}$$

for $m = 1, 2, ..., M$.

Proof: We focus on proving for $M > 2$. Proving for $M = 2$ is straightforward. Note that the event $\tau_1 > n$ implies that there exists a time instant t for $t \geq n$ in which for some $j \neq m$, $S_j(t) > S_m(t)$. Hence,

$$\begin{aligned} \mathbf{P}_m\left(\tau_1 > n\right) &\leq \mathbf{P}_m\left(\max_{j\neq m} \sup_{t\geq n} \left(S_j(t) - S_m(t)\right) \geq 0\right) \\ &\leq \sum_{j\neq m} \sum_{t=n}^{\infty} \mathbf{P}_m\left(S_j(t) \geq S_m(t)\right) \; . \end{aligned} \tag{55}$$

Following (55), it suffices to show that there exist $C > 0$ and $\gamma > 0$ such that $\mathbf{P}_m\left(S_j(n) \geq S_m(n)\right) \leq Ce^{-\gamma n}$.

We next establish the required exponential decay. Let

$$\begin{aligned} k_m &= \frac{\max_{j\neq m} D(f_j||g_j)}{\min_{j\neq m} D(f_j||g_j)}, \\ \underline{j_m} &= \arg\min_{j\neq m} D(f_j||g_j), \\ \rho_m &= \frac{1}{8(k_m+1)(M-2)}. \end{aligned} \tag{56}$$

Note that $0 < \rho_m \leq 1/16$. Thus, we can write

$$\begin{aligned} &\mathbf{P}_m\left(S_j(n) \geq S_m(n)\right) \\ &\leq \mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) < \rho_m n, N_m(n) < \rho_m n\right) \\ &+\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_j(n) \geq \rho_m n\right) \\ &+\mathbf{P}_m\left(S_j(n) \geq S_m(n), N_m(n) \geq \rho_m n\right) \end{aligned} \tag{57}$$

The second and the third terms on the RHS of (57) decay exponentially with n by Lemma 3. Thus, it remains to show that the first term decays exponentially with n as well. Note that the event $(N_j(n) < \rho_m n, N_m(n) < \rho_m n)$ implies that at least $\tilde{n} = n - N_j(n) - N_m(n) \geq n\left(1 - 2\rho_m\right)$ times cells j, m

are not probed. We define $\widetilde{N}_r(n)$ as the number of times in which cell $r \neq j, m$ has been probed and cells j, m have not been probed by time n. There exists a cell $r \neq j, m$ such that $\widetilde{N}_r(n) \geq \frac{\tilde{n}}{M-2} = \frac{n(1-2\rho_m)}{M-2}$. Hence, we can upper bound (57) as follows:

$$\begin{aligned}&\mathbf{P}_m\left(S_j(n) \geq S_m(n)\right)\\&\leq \sum_{r\neq j,m} \mathbf{P}_m\bigg(\tilde{N}_r(n) > \frac{n(1-2\rho_m)}{M-2}, N_j(n) < \rho_m n,\\&\qquad\qquad N_m(n) < \rho_m n\bigg) + 2De^{-\gamma_1 n}\end{aligned} \tag{58}$$

where the second and third terms on the RHS of (57) are upper bounded by $De^{-\gamma_1 n}$ (there exist such $D > 0, \gamma_1 > 0$ by Lemma 3), and the first term on the RHS of (57) is upper bounded by the first term (i.e., the summation term) on the RHS of (58). Next, we show that each term in the summation decays exponentially with n to get the desired result.

Let $\tilde{t}_1^r, \tilde{t}_2^r, ..., \tilde{t}_{\tilde{N}_r(n)}^r$ be the indices for the time instants in which cell $r \neq j, m$ has been probed and cells j, m have not been probed by time n. Let

$$\zeta \triangleq \frac{1-2\rho_m}{2(M-2)}. \tag{59}$$

Note that the event $S_j(\tilde{t}_{\zeta n}^r) \leq S_r(\tilde{t}_{\zeta n}^r)$ or $S_m(\tilde{t}_{\zeta n}^r) \leq S_r(\tilde{t}_{\zeta n}^r)$ must occur (otherwise, cell j or m will be probed). Hence[4],

$$\begin{aligned}&\mathbf{P}_m\left(\tilde{N}_r(n) > \tfrac{n(1-2\rho_m)}{M-2}, N_j(n) < \rho_m n, N_m(n) < \rho_m n\right)\\&= \sum_{q=0}^{n-\zeta n}\sum_{n'=0}^{\rho_m n} \mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{\zeta n+q} \ell_r(i)\right)\\&\quad+ \sum_{q=0}^{n-\zeta n}\sum_{n'=0}^{\rho_m n} \mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_m(i) \leq \sum_{i=1}^{\zeta n+q} \ell_r(i)\right).\end{aligned} \tag{60}$$

For upper bounding the first term on the RHS of (60) we write the sum LLRs as follows:

$$\begin{aligned}&\sum_{i=1}^{\zeta n+q} \ell_r(i) + \sum_{i=1}^{n'} -\ell_j(i)\\&= \sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} -\tilde{\ell}_j(i)\\&\qquad -D(f_r||g_r)\left(\zeta n+q\right) + D(f_{n'}||g_{n'})n'\\&\leq \sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} -\tilde{\ell}_j(i) - D(f_{\underline{j_m}}||g_{\underline{j_m}})\left(\zeta n+q-k_m n'\right),\end{aligned} \tag{61}$$

and by the definitions of ζ, k_m, ρ_m in (56) and (59), we have

$$\begin{aligned}&\zeta n+q-k_m n' \geq \zeta n+q-k_m n' - (k_m+1)\left(\rho_m n - n'\right)\\&= n\left(\zeta-(k_m+1)\rho_m\right) + q + n' \geq \frac{1}{4(M-2)}n + q + n'\\&\geq \frac{1}{4(M-2)}\left(n+q+n'\right),\end{aligned}$$

for all $n' \leq \rho_m n$. Therefore,

$$\sum_{i=1}^{\zeta n+q} \ell_r(i) + \sum_{i=1}^{n'} -\ell_j(i) \geq 0 \tag{62}$$

implies

$$\sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} -\tilde{\ell}_j(i) \geq C_1\left(n+q+n'\right), \tag{63}$$

where

$$C_1 = \frac{D(f_{\underline{j_m}}||g_{\underline{j_m}})}{4(M-2)} > 0. \tag{64}$$

Applying the Chernoff bound yields:

$$\begin{aligned}&\mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{\zeta n+q} \ell_r(i)\right)\\&\leq \mathbf{P}_m\left(\sum_{i=1}^{\zeta n+q} \tilde{\ell}_r(i) + \sum_{i=1}^{n'} -\tilde{\ell}_j(i) \geq C_1\left(n+q+n'\right)\right)\\&\leq \left[\mathbf{E}_m\left(e^{s\tilde{\ell}_r(1)}\right)\right]^{\zeta n+q}\left[\mathbf{E}_m\left(e^{s(-\tilde{\ell}_j(1))}\right)\right]^{n'}\\&\quad\times e^{-sC_1\left(n+q+n'\right)}\\&= \left[\mathbf{E}_m\left(e^{s\left(\tilde{\ell}_r(1)-C_1\right)}\right)\right]^{\zeta n+q}\\&\quad\times\left[\mathbf{E}_m\left(e^{s\left(-\tilde{\ell}_j(1)-C_1\right)}\right)\right]^{n'} \times e^{-sC_1(n-\zeta n)}.\end{aligned} \tag{65}$$

for all $s > 0$.

Since $\mathbf{E}_m(\tilde{\ell}_r(1)-C_1) = -C_1 < 0$ and $\mathbf{E}_m(-\tilde{\ell}_j(1)-C_1) = -C_1 < 0$ are strictly negative, by applying a similar argument as at the end of the proof of Lemma 3, there exist $s > 0$ and $\gamma_2 > 0$ such that $\mathbf{E}_m\left(e^{(s\tilde{\ell}_r(1)-C_1)}\right)$, $\mathbf{E}_m\left(e^{s(-\tilde{\ell}_j(1)-C_1)}\right)$ and e^{-sC_1} are strictly less than $e^{-\gamma_2} < 1$. Hence,

$$\mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{\zeta n+q} \ell_r(i)\right) \leq e^{-\gamma_2\left(n+q+n'\right)}, \tag{66}$$

and

$$\begin{aligned}&\sum_{q=0}^{n-\zeta n}\sum_{n'=0}^{\rho_m n} \mathbf{P}_m\left(\sum_{i=1}^{n'} \ell_j(i) \leq \sum_{i=1}^{\zeta n+q} \ell_r(i)\right)\\&\leq e^{-\gamma_2 n}\sum_{q=0}^{n-\zeta n} e^{-\gamma_2 q}\sum_{n'=0}^{\rho_m n} e^{-\gamma_2 n'} \leq C_2 e^{-\gamma_2 n},\end{aligned} \tag{67}$$

where $C_2 = \left(1-e^{-\gamma_2}\right)^{-2}$.

A similar Chernoff bounding technique can be applied to upper bound the second term on the RHS of (60). ■

[4] For the ease of presentation, throughout the proof we assume that ζn, $\rho_m n$ are integers. This assumption does not affect the exponential decay of the Chernoff bound but only the exact value of $C > 0$ in (54) (since $\alpha n - 1 \leq \lfloor \alpha n \rfloor \leq \lceil \alpha n \rceil \leq \alpha n + 1$ holds for all $\alpha \geq 0$ for all $n = 0, 1, ...$).

Next, we consider two cases:

1) the *balanced case*, referring to the case when $\min_{j\neq m} D(f_j||g_j) \geq \frac{1}{\sum_{j\neq m} 1/D(f_j||g_j)}$;
2) the *unbalanced case*, when the above inequality is reversed.

The reason for referring to the first case as the balanced case is that DGFi policy is able to balance the detection time so that the difference between the largest sum LLR and the sum LLRs of any other cell exceeds the threshold $-\log c$ approximately at the same time. As a result, the rate function is determined by a certain averaging among the KL divergences of the heterogeneous processes. On the other hand, when the smallest KL divergence is too small, then too many measurements are required to be gathered from that cell. In that case, the difference between the largest sum LLR and the sum LLR gathered from the cell with the smallest KL divergence exceeds the threshold $-\log c$ significantly after the difference between the largest sum LLR and the sum LLR gathered from any other cell does. As a result, the rate function is dominated by the smallest KL divergence.

For the unbalanced case, the proof follows directly from subsection B.2. Thus, here it remains to show the proof for the balanced case.

Definition 2: τ_2 is defined as the smallest integer such that $\sum_{i=\tau_1+1}^{n} \ell_{j_n}(i)\mathbf{1}_{j_n}(i) \leq \log c$ for some $j_n \neq m$ for all $n \geq \tau_2 \geq \tau_1$.
We also define $n_2 \triangleq \tau_2 - \tau_1$ as the total amount of time between τ_1 and τ_2.

In Lemma 5 we show that the probability that n_2 is greater than n decays exponentially with n when n is greater than $-\log c/I_m$. Later, we will show that the asymptotic search time is dominated by n_2, which together with Lemma 5 yields the desired search time $-\log c/I_m$ under hypothesis H_m.

Lemma 5: If the selection rule of DGFi is implemented indefinitely, then for every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m\left(n_2 > n\right) \leq Ce^{-\gamma n} \quad \forall n > -(1+\epsilon)\log c/I_m \ , \tag{68}$$

for all $m = 1, 2, ..., M$.

Proof: First, we consider the case where $I_m > D(g_m||f_m)$. Note that cell m is not observed for all $n \geq \tau_1$ in this case. Define $N'_j(\tau_1+t) = \sum_{i=\tau_1+1}^{\tau_1+t} 1_j(i)$ and $j^*(\tau_1+t) = \arg\max_j N'_j(\tau_1+t)D(f_j||g_j)$. Thus,

$$\begin{aligned}
&\mathbf{P}_m(n_2 > n) \\
&\leq \mathbf{P}_m(\sup_{t\geq n} \sum_{i=\tau_1+1}^{\tau_1+t} \ell_{j^*(\tau_1+t)}(i)1_{j^*(\tau_1+t)}(i) \geq \log c).
\end{aligned} \tag{69}$$

Since Kt is the total number of obeservation from τ_1 to τ_1+t, by the definition of $j^*(t)$ we have

$$\begin{aligned}
Kt &= \sum_{j\neq m} N'_j(\tau_1+t) = \sum_{j\neq m} \frac{N'_j(\tau_1+t)D(f_j||g_j)}{D(f_j||g_j)} \\
&\leq \sum_{j\neq m} \frac{N'_{j^*(\tau_1+t)}(\tau_1+t)D(f_{j^*(\tau_1+t)}||g_{j^*(\tau_1+t)})}{D(f_j||g_j)}.
\end{aligned} \tag{70}$$

Let $\epsilon_1 = I_m\epsilon/(1+\epsilon)$. Since $I_m = \sum_{j\neq m} K/D(f_j||g_j)$, we have

$$\epsilon_1 = \frac{\epsilon K}{(1+\epsilon)\sum_{j\neq m} 1/D(f_j||g_j)}. \tag{71}$$

Then,

$$\begin{aligned}
&\sum_{i=\tau_1+1}^{\tau_1+t} \ell_{j^*(\tau_1+t)}(i)1_{j^*(\tau_1+t)}(i) - \log c \\
&= \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i)1_{j^*(\tau_1+t)}(i) \\
&\quad -N'_{j^*(\tau_1+t)}(\tau_1+t)D(f_{j^*(\tau_1+t)}||g_{j^*(\tau_1+t)}) - \log c \\
&\leq \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i)1_{j^*(\tau_1+t)}(i) \\
&\quad -\frac{Kt}{\sum_{j\neq m} 1/D(f_j||g_j)} - \log c \\
&\leq \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i)1_{j^*(\tau_1+t)}(i) - tI_m + tI_m/(1+\epsilon) \\
&\leq \sum_{i=\tau_1+1}^{\tau_1+t} \tilde{\ell}_{j^*(\tau_1+t)}(i)1_{j^*(\tau_1+t)}(i) - t\epsilon_1
\end{aligned} \tag{72}$$

for all $t \geq n > -(1+\epsilon)\log c/I_m$. By applying the Chernoff bound, it can be shown that there exists $\gamma_1 > 0$ such that $P_m(\sum_{\tau_1+1}^{\tau_1+t} -\tilde{\ell}_{j^*(\tau_1+t)}(i) \geq t\epsilon_1) < e^{-\gamma_1 t}$ for all $t \geq n > -(1+\epsilon)\log c/I_m$. Hence, there exist $C_1 > 0$ and $\gamma_1 > 0$ such that $P_m(n_2 > n) \leq C_1 e^{-\gamma_1 n}$ for all $n > -(1+\epsilon)\log c/I_m$. A similar argument applies for case where $I_m \leq D(g_m||f_m)$. ■

In what follows we define a r.v. $\mathrm{DR}(t)$ as the dynamic range between sum LLRs of empty cells, $\max_{j\neq m} S_j(t) - \min_{j\neq m} S_j(t)$. Note that the dynamic range at time τ_2 (which is the time in which sufficient information has been gathered to distinguish H_m from at least one false hypothesis) can be viewed as a measure of the amount of information remains to gather in order to distinguish H_m from any other false hypothesis. Lemma 6 below shows that the dynamic range at time τ_2 is sufficiently small.

Definition 3: The dynamic range between sum LLRs of empty cells at time t is defined as follows:

$$\mathrm{DR}(t) \triangleq \max_{j\neq m} S_j(t) - \min_{j\neq m} S_j(t) \ . \tag{73}$$

Lemma 6: If the selection rule of DGFi is implemented indefinitely. Then, for every fixed $\epsilon_1 > 0, \epsilon_2 > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\begin{aligned}
&\mathbf{P}_m\left(\mathrm{DR}(\tau_2) > \epsilon_1 n\right) \leq Ce^{-\gamma n} \\
&\qquad\qquad \forall n > -(1+\epsilon_2)\log c/I_m \ ,
\end{aligned} \tag{74}$$

for all $m = 1, 2, ..., M$.

Proof: The proof follows directly by applying Lemma 5 and substituting τ_2 in subsection B.2. ■

Definition 4: τ_3^j is defined as the smallest integer such that $S_j(n) \geq -\log c$ for all $n \geq \tau_2$. We also define $\tau_3 = \max_j \tau_3^j$.

Note that by the definitions of the last passage times we have $\tau_3^j \geq \tau_2$.

Remark 2: Using some algebraic manipulations, it can be verified that $\Delta S_{m,j}(n) \geq -\log c$ for all $j \neq m$ for all $n \geq \tau_3^j$. Since $\tau_3 = \max_{j \neq m} \tau_3^j$ we have $\Delta S(n) = S_m(n) - S_{m^{(2)}(n)}(n) \geq -\log c$ for all $n \geq \tau_3$. Recall that DGFi stops the test once $\Delta S(n)$ first occurs. Thus, in the sequel we will use τ_3 as an upper bound on the actual stopping time τ.

Definition 5: $n_3 \triangleq \tau_3 - \tau_2$ is defined as the total amount of time between τ_2 and τ_3.

In Lemma 7 we show that n_3 is sufficiently small with high probability. We will use this result to show that the asymptotic expected search time is not affected by n_3.

Lemma 7: If the selection rule of DGFi is implemented indefinitely, then for every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m\left(n_3 > n\right) \leq Ce^{-\gamma n} \quad \forall n > -\epsilon \log c / I_m \,, \tag{75}$$

for all $m = 1, 2, ..., M$.

Proof: To prove the Lemma, we first define N_3^j as the total number of observations that the decision maker collected from cell j between τ_2 and τ_3^j. Since $n_3 \leq \sum_j N_3^j$, we only need to show that $\mathbf{P}_m(N_3^j > n)$ decays exponentially with n. We can write $\mathbf{P}_m(N_3^j > n)$ as follows:

$$\begin{aligned} \mathbf{P}(N_3^j > n) \leq \mathbf{P}_m(DR(\tau_2) > n\frac{\min_j D(f_j||g_j)}{2}) \\ +\mathbf{P}_m(N_3^j > n | DR(\tau_2) \leq n\frac{\min_j D(f_j||g_j)}{2}) \end{aligned} \tag{76}$$

Lemma 6 provides the desired decay for the first term on the RHS. We next show the desired decay for the second term. Let $t_1, t_2, \cdots$ denote the time indices when cell j is observed between τ_2 and τ_3^j. We can write:

$$\begin{aligned} &\mathbf{P}_m(N_3^j > n | DR(\tau_2) \leq n\frac{\min_j D(f_j||g_j)}{2}) \\ &\leq \mathbf{P}_m(\inf_{r>n} \sum_{i=1}^{r} -\ell_j(t_i) < n\frac{\min_j D(f_j||g_j)}{2}) \\ &\leq \mathbf{P}_m(\sum_{i=1}^{r} \tilde{\ell}_j(t_i) > r\frac{\min_j D(f_j||g_j)}{2}). \end{aligned} \tag{77}$$

Using the Chernoff bound and the i.i.d. property of $\tilde{\ell}_j(t_i)$ yields:

$$\mathbf{P}_m(\sum_{i=1}^{n} \tilde{\ell}_j(t_i) > n\frac{\min_j D(f_j||g_j)}{2}) < C_3 e^{-\gamma n} \tag{78}$$

for some C_3, γ_3 which completes the proof. ■

The following Lemma provides an upper bound on the detection time when DGFi policy implemented (i.e., the cells are probed based on the selection rule and the test terminates based on the stopping rule).

Lemma 8: If DGFi policy is implemented, then the expected detection time τ is upper bounded by:

$$\mathbf{E}_m(\tau) \leq -\left(1 + o(1)\right) \frac{\log(c)}{I_m} \,, \tag{79}$$

for $m = 1, ..., M$.

Proof: Since the actual detection time under DGFi is upper bounded by: $\tau \leq \tau_3 = \tau_1 + n_2 + n_3$, combining Lemmas 4, 5 and 7 proves the statement. ■

B. Analyzing the Dynamic of Empty Cells under DGFi

In this appendix we analyze the sum LLRs dynamics at empty cells under DGFi, used to prove the theorem in subsection A. To analyze the sum LLR of empty cells, we introduce the following (slightly different) active hypothesis testing problem. It should be noted that in what follows we slightly change the notations for the new problem setting for convenience and differentiating it from the original problem.

At each time, only k cells can be observed from cells $1, 2, \cdots, m$. When cell j is observed at time n, an observation $x_j(n)$ is drawn independently from previous times and $x_j(n)$ follows distribution f_j. We also assume that $g_j, j = 1, 2, \cdots, m$ are m known distributions.

For the ease of the presentation when analyzing the sum LLRs of empty cells. we remove the subscript m from $\mathbf{P}_m()$ when referring to the probability measure.

Let $1_j(n)$ be the indicator function, where $1_j(n) = 1$ if cell j is observed at time n, and $1_j(n) = 0$ otherwise.

Let

$$l_j(n) \triangleq \log \frac{f_j(x_j(n))}{g_j(x_j(n))} \tag{80}$$

and

$$S_j(n) \triangleq \sum_{t=1}^{n} l_j(t) 1_j(t) \tag{81}$$

Note that we are now focusing on empty cells. Thus, for convenience the LLR is defined as negative LLR defined in the original problem. The sum LLR is defined accordingly.

We know that

$$E[l_j(n)] = \int_x f_j(x) \log \frac{f_j(x)}{g_j(x)} dx = D(f_j||g_j). \tag{82}$$

Thus, here the sum LLR of an empty cell j is a random walk with a positive increment $D(f_j||g_j)$. Similarly, we define

$$\tilde{l}_j(n) \triangleq l_j(n) - D(f_j||g_j), \tag{83}$$

which is a zero mean r.v.. Without loss of generality, we assume $D(f_1||g_1) \leq D(f_2||g_2) \leq \cdots \leq D(f_m||g_m)$. We also define

$$\bar{v} \triangleq \frac{1}{\sum_j 1/D(f_j||g_j)}. \tag{84}$$

Let $r^{(i)}(n)$ denotes the cell index with the i^{th} smallest sum LLR collected from this cell up to time n. We define the following random variables:

$$U(n) \triangleq \max_j S_j(n), \; L(n) \triangleq \min_j S_j(n), \tag{85}$$

$$DR_j^k(n) \triangleq S_{r^{(k)}(n)}(n) - S_{r^{(j)}(n)}(n), \tag{86}$$

and

$$DR(n) \triangleq DR_1^m(n) = U(n) - L(n), \tag{87}$$

Also, let

$$N_j(t) = \sum_{i=1}^{t} 1_j(i), \tag{88}$$

$$\underline{j}(t) = \arg\min_j N_j(t) D(f_j||g_j), \tag{89}$$

$$\bar{j}(t) = \arg\max_j N_j(t) D(f_j||g_j). \tag{90}$$

Remark 3: Note that we defined the LLR in this appendix as the negative LLR which was defined in the original problem. Thus, the corresponding DGFi policy in this appendix chooses the k cells with smallest sum LLRs (in contrast to the selection of the empty cells with the top sum LLRs as done in the original problem).

Definition 6: The modified selection rule of DGFi for the active hypothesis testing problem defined in this appendix is given by: $\phi(n) = \{r^{(1)}(n), r^{(2)}(n), \cdots, r^{(k)}(n)\}$.

Next, we provide the outline for the next lemmas. Lemma 9 states that the smallest observed sum LLR is sufficiently small as requireed with high probability. Lemma 10 states that the largest observed sum LLR is sufficiently large as required with high probability. Lemma 11 shows that under (the modified) DGFi policy, the difference between the largest sum LLR and the $(m-k+1)^{th}$ largest sum LLR is sufficiently small as required with high probability. Whether the smallest sum LLR is approximately equal to the largest sum LLR depends on which one of balanced or unbalanced cases is valid. For the balanced case, Lemma 12 claims that the dynamic range is small under DGFi policy. Hence, DGFi can balance the search time among all the processes so that the search time is a certain averaging of their KL divergences. For the unbalanced case, Lemma 13 states that the sum LLRs of the cell with the smallest KL divergence cannot be too small (which will determine the rate function function for the search in this case) with high probability. Lemma 14 shows that the sum LLR of other cells are larger than that of the cell with the smallest KL divergence. Finally, Lemma 15 upper bounds the asymptotic search time.

Lemma 9: For any selection rule, $\forall t, \forall \epsilon > 0$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(L(t) > t \cdot k\bar{v} + n\epsilon) < Ce^{-\gamma n} \quad \forall n > t. \tag{91}$$

Proof: Note that

$$\mathbf{P}(L(t) > t \cdot k\bar{v} + n\epsilon) \leq \mathbf{P}(S_{\underline{j}(t)}(t) > t \cdot k\bar{v} + n\epsilon), \tag{92}$$

and

$$S_{\underline{j}(t)}(t) = N_{\underline{j}(t)}(t) D(f_{\underline{j}(t)}||g_{\underline{j}(t)}) + \sum_{i=1}^{t} \tilde{l}_{\underline{j}(t)}(i) 1_{\underline{j}(t)}(i). \tag{93}$$

Since kt is the total number of observations by time t, by the definition of $\underline{j}(t)$ we have

$$\begin{aligned} kt &= \sum_j N_j(t) = \sum_j \frac{N_j(t) D(f_j||g_j)}{D(f_j||g_j)} \\ &\geq \sum_j \frac{N_{\underline{j}(t)}(t) D(f_{\underline{j}(t)}||g_{\underline{j}(t)})}{D(f_j||g_j)}. \end{aligned} \tag{94}$$

Hence,

$$N_{\underline{j}(t)}(t) D(f_{\underline{j}(t)}||g_{\underline{j}(t)}) \leq kt \cdot \frac{1}{\sum_j 1/D(f_j||g_j)} = t \cdot k\bar{v}. \tag{95}$$

Therefore,

$$S_{\underline{j}(t)}(t) > t \cdot k\bar{v} + n\epsilon \tag{96}$$

implies

$$\sum_{i=1}^{t} \tilde{l}_{\underline{j}(t)}(i) 1_{\underline{j}(t)}(i) > n\epsilon. \tag{97}$$

Since $\tilde{l}_{\underline{j}(t)}(t)$ is a zero mean r.v. with a bounded moment generating function, applying the Chernoff inequality completes the proof. ∎

Lemma 10: For any selection rule, $\forall t, \forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(U(t) < t \cdot k\bar{v} - n\epsilon) < Ce^{-\gamma n} \quad \forall n > t. \tag{98}$$

Proof: Note that

$$\mathbf{P}(U(t) > t \cdot k\bar{v} - n\epsilon) \leq \mathbf{P}(S_{\bar{j}(t)}(t) < t \cdot k\bar{v} - n\epsilon), \tag{99}$$

and

$$S_{\bar{j}(t)}(t) = N_{\bar{j}(t)}(t) D(f_{\bar{j}(t)}||g_{\bar{j}(t)}) + \sum_{i=1}^{t} \tilde{l}_{\bar{j}(t)}(i) 1_{\bar{j}(t)}(i). \tag{100}$$

Since kt is the total number of observations by time t, by the definition of $\bar{j}(t)$ we have

$$\begin{aligned} kt &= \sum_j N_j(t) = \sum_j \frac{N_j(t) D(f_j||g_j)}{D(f_j||g_j)} \\ &\leq \sum_j \frac{N_{\bar{j}(t)}(t) D(f_{\bar{j}(t)}||g_{\bar{j}(t)})}{D(f_j||g_j)}. \end{aligned} \tag{101}$$

Hence,

$$N_{\bar{j}(t)}(t) D(f_{\bar{j}(t)}||g_{\bar{j}(t)}) \geq kt \cdot \frac{1}{\sum_j 1/D(f_j||g_j)} = t \cdot k\bar{v}. \tag{102}$$

Therefore,

$$S_{\bar{j}(t)}(t) < t \cdot k\bar{v} - n\epsilon \tag{103}$$

implies

$$\sum_{i=1}^{t} \tilde{l}_{\underline{j}(t)}(i) 1_{\underline{j}(t)}(i) < -n\epsilon. \tag{104}$$

Since $\tilde{l}_{\underline{j}(t)}(t)$ is a zero mean r.v. with a bounded moment generating function, applying the Chernoff inequality completes the proof. ∎

Lemma 11: For DGFi selection rule, $\forall t, \forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(DR_k^m(t) > D(f_m||g_m) + n\epsilon) < Ce^{-\gamma n} \quad \forall n > t. \tag{105}$$

Proof: We prove by induction with respect to t. When $t = 1$, $DR_k^m(1) > D(f_m||g_m) + n\epsilon$ indicates that

$$\begin{aligned} &D(f_m||g_m) + n\epsilon < l_j(1) = \tilde{l}_j(1) + D(f_j||g_j) \\ &\le \tilde{l}_j(1) + D(f_m||g_m) \end{aligned} \tag{106}$$

which indicates that

$$\tilde{l}_j(1) > n\epsilon.$$

Using the Chernoff bound completes the induction base.

If the statement is true for $t-1$, then for t we have

$$\begin{aligned} &\mathbf{P}(DR_k^m(t) > D(f_m||g_m) + n\epsilon) \\ =&\mathbf{P}(DR_k^m(t) > D(f_m||g_m) + n\epsilon, r^{(m)}(t) = r^{(m)}(t-1)) \\ &+ \mathbf{P}(DR_k^m(t) > D(f_m||g_m) + n\epsilon, r^{(m)}(t) \neq r^{(m)}(t-1)). \end{aligned} \tag{107}$$

For the first term on the RHS, we have

$$\begin{aligned} &\mathbf{P}(DR_k^m(t) > D(f_m||g_m) + n\epsilon, r^{(m)}(t) = r^{(m)}(t-1)) \\ &\le \mathbf{P}(DR_k^m(t-1) > D(f_m||g_m) + \frac{n\epsilon}{2}, r^{(m)}(t) = r^{(m)}(t-1) \\ &\text{or } l_{r^{(k)}(t-1)}(t) < -\frac{n\epsilon}{2}, r^{(m)}(t) = r^{(m)}(t-1)) \\ &\le \mathbf{P}(DR_k^m(t-1) > D(f_m||g_m) + \frac{n\epsilon}{2}) \\ &\quad + \mathbf{P}(l_{r^{(k)}(t-1)}(t) < -\frac{n\epsilon}{2}) \le C_1 e^{-\gamma_1 n}, \end{aligned} \tag{108}$$

where the first term can be bounded using assumptions on $t-1$ and the second term can be bounded using the Chernoff bound.

For the second term on the RHS of (107), we have

$$\begin{aligned} &\mathbf{P}(DR_k^m(t) > D(f_m||g_m) + n\epsilon, r^{(m)}(t) \neq r^{(m)}(t-1)) \\ \le&\mathbf{P}(l_{r^{(m)}(t)}(t) > D(f_m||g_m) + n\epsilon) \\ \le&\mathbf{P}(\tilde{l}_{r^{(m)}(t)}(t) > n\epsilon) < C_2 e^{-\gamma_2 n} \end{aligned} \tag{109}$$

Combining (107), (108), (109) completes the proof. ■

1) The Balanced Case:

Lemma 12: Under the DGFi selection rule, if

$$D(f_1||g_1) \ge k \cdot \bar{v} \tag{110}$$

holds, then we have the following statements:

1) $\forall t, \forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(U(t) > t \cdot k\bar{v} + kD(f_m||g_m) + n\epsilon) < Ce^{-\gamma n} \quad \forall n > t. \tag{111}$$

2) $\forall t, \forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(L(t) < t \cdot k\bar{v} - kD(f_m||g_m) - n\epsilon) < Ce^{-\gamma n} \quad \forall n > t. \tag{112}$$

3) $\forall t, \forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(DR(t) > k \cdot D(f_m||g_m) + n\epsilon) < Ce^{-\gamma n} \quad \forall n > t. \tag{113}$$

Proof: We prove by induction with respect to k. For $k = 1$, statement 3 follows directly from Lemma 11. For statement 1,

$$\begin{aligned} &\mathbf{P}(U(t) > t \cdot \bar{v} + D(f_m||g_m) + n\epsilon) \\ \le&\mathbf{P}(L(t) > t \cdot \bar{v} + n\epsilon \text{ or } DR(t) > D(f_m||g_m) + n\epsilon) \\ \le&\mathbf{P}(L(t) > t \cdot \bar{v} + n\epsilon) + \mathbf{P}(DR(t) > D(f_m||g_m) + n\epsilon), \end{aligned} \tag{114}$$

which can be bounded by Lemma 9 and 11. Similarly, we can prove statement 2 using Lemmas 10 and 11.

If the statement is true for $k-1$, for k we first prove statement 3. For any fixed t, let $\bar{j} = \arg\min_j S_j(t)$, and let t_0 be the smallest integer such that $\bar{j} \in \phi(\tau), \forall t_0 < \tau \le t$. From Lemma 11 it follows that $\forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(DR_k^m(t_0) > D(f_m||g_m) + n\epsilon) < Ce^{-\gamma n}, \quad \forall n > t_0.$$

Since $\bar{j} \notin \phi(t_0)$, we have:

$$\mathbf{P}(S_{\bar{j}}(t_0) - U(t_0) < -D(f_m||g_m) - n\epsilon) < Ce^{-\gamma n}, \forall n > t_0. \tag{115}$$

Since that $\bar{j} \in \phi(\tau), \forall t_0 < \tau \le t$ we have:

$$\begin{aligned} &\mathbf{P}(S_{\bar{j}}(t) - U(t_0) < (t - t_0)D(f_{\bar{j}}||g_{\bar{j}}) - D(f_m||g_m) - n\epsilon) \\ &\quad < Ce^{-\gamma n}, \qquad \forall n > t \end{aligned} \tag{116}$$

by applying the Chernoff bound.

Next, consider a subproblem where cell $\bar{j}$ is removed and only $k-1$ cells can be selected. Let $S'_j(n)$ be the sum LLR in this subproblem. Then, by statement 1 with assumption on $k-1$ we have that $\forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\begin{aligned} &P\left(U(t - t_0) > (t - t_0) \cdot (k-1)\bar{v}' \right. \\ &\qquad \left. + (k-1)D(f_m||g_m) + n\epsilon\right) \\ &< Ce^{-\gamma n}, \ \forall n > t - t_0 \end{aligned} \tag{117}$$

where

$$\bar{v}' = \frac{1}{\sum_{j \neq \bar{j}} 1/D(f_j||g_j)}.$$

Now, for the original problem, we have $S_j(t) = S_j(t_0) + S'_j(t - t_0) \le U(t_0) + S'_j(t - t_0)$. By (117) we have

$$\begin{aligned} &\mathbf{P}(U(t) - U(t_0) > (t - t_0) \cdot (k-1)\bar{v}' \\ &\qquad + (k-1)D(f_m||g_m) + n\epsilon) \\ &< Ce^{-\gamma n} \quad \forall n > t \end{aligned} \tag{118}$$

since $D(f_1||g_1) \ge k\bar{v}$ implies $D(f_{\bar{j}}||g_{\bar{j}}) \ge (k-1)\bar{v}'$. By (116) and (118) we have that $\forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(U(t) - S_{\bar{j}}(t) > k \cdot D(f_m||g_m) + n\epsilon) < Ce^{-\gamma n} \quad \forall n > t, \tag{119}$$

which proves statement 3 for k as desired. Then, statements 1 and 2 can be proved using Lemma 9 and 10 with statement 3 similar to the case with $k = 1$. ■

2) The Unbalanced Case:

Lemma 13: Under the DGFi selection rule, if

$$D(f_1||g_1) < k \cdot \bar{v} \tag{120}$$

then $\forall t, \forall \epsilon$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}(S_1(t) < tD(f_1||g_1) - n\epsilon) < Ce^{-\gamma n} \quad \forall n > t. \tag{121}$$

Proof: Define t_0 as the smallest integer such that cell 1 is observed at time i for all $t_0 < i \leq t$. Then, by our selection rule, cell 1 is the one of the top $m-k$ sum LLRs at time t_0. Then, by applying $t = t_0$ to Lemma 11 we have

$$\mathbf{P}(U(t_0) - S_1(t_0) > n\epsilon) < C_1 e^{-\gamma_1 n} \quad \forall n > t_0 \tag{122}$$

for some C_1, γ_1. Substituting $t = t_0$ in Lemma 10 we have:

$$\mathbf{P}(U(t_0) < t_0 \cdot k\bar{v} - n\epsilon) < C_2 e^{-\gamma_2 n} \quad \forall n > t_0 \tag{123}$$

for some C_2, γ_2. Hence,

$$\mathbf{P}(S_1(t_0) < t_0 \cdot k\bar{v} - n\epsilon) < C_3 e^{-\gamma_3 n} \quad \forall n > t_0 \tag{124}$$

for some C_3, γ_3. Then, by the definition of t_0 and using the Chernoff bound we have

$$\begin{aligned}\mathbf{P}(S_1(t) - S_1(t_0) < (t-t_0)D(f_1||g_1) - n\epsilon) < C_4 e^{-\gamma_4 n} \\ \forall n > (t - t_0).\end{aligned} \tag{125}$$

Since $k\bar{v} > D(f_1||g_1)$, we have:

$$\mathbf{P}(S_1(t) < tD(f_1||g_1) - n\epsilon) < C_5 e^{-\gamma_5 n} \quad \forall n > t \tag{126}$$

as desired. ■

Definition 7: Define $\tilde{\tau}_2 = -\log c / D(f_1||g_1)$.

Lemma 14: For every fixed $\epsilon > 0$, there exists $C > 0$ and $\gamma > 0$, such that for all j we have:

$$\mathbf{P}(S_1(\tilde{\tau}_2) - S_j(\tilde{\tau}_2) > \epsilon n) \leq Ce^{-\gamma n}, \quad \forall n > \tilde{\tau}_2. \tag{127}$$

Proof: For fixed j, define t_0^j as the smallest integer such that $S_1(n) > S_j(n)$ for all $t_0^j < i \leq \tilde{\tau}_2$. By definition, $S_1(t_0^j) \leq S_j(t_0^j)$. Then, by our selection rule, for all $t_0^j < i \leq \tilde{\tau}_2$, whenever cell 1 is observed, cell j must be observed based on their ranking of sum LLRs. Note that $D(f_1||g_1) \leq D(f_j||g_j)$. Thus,

$$\begin{aligned}&\sum_{i=t_0^j}^{\tilde{\tau}_2} l_j(i)1_j(i) - \sum_{i=t_0^j}^{\tilde{\tau}_2} l_1(i)1_1(i) \\ &= \sum_{i=t_0^j}^{\tilde{\tau}_2} \tilde{l}_j(i)1_j(i) - \sum_{i=t_0^j}^{\tilde{\tau}_2} \tilde{l}_1(i)1_1(i) \\ &\quad + D(f_j||g_j)\sum_{i=t_0^j}^{\tilde{\tau}_2} 1_j(i) - D(f_1||g_1)\sum_{i=t_0^j}^{\tilde{\tau}_2} 1_1(i) \\ &\geq \sum_{i=t_0^j}^{\tilde{\tau}_2} \tilde{l}_j(i)1_j(i) - \sum_{i=t_0^j}^{\tilde{\tau}_2} \tilde{l}_1(i)1_1(i),\end{aligned} \tag{128}$$

which indicates that the LHS has positive means. By applying the Chernoff bound and using the i.i.d. property of $\tilde{l}_j(t_i)$ we have:

$$\mathbf{P}(S_1(\tilde{\tau}_2) - S_1(t_0^j) - (S_j(\tilde{\tau}_2) - S_j(t_0^j)) > \epsilon n) \leq Ce^{-\gamma n}, \tag{129}$$

for some C, γ. Since $S_1(t_0^j) \leq S_j(t_0^j)$, we have:

$$\begin{aligned}&\mathbf{P}(S_1(\tilde{\tau}_2) - S_j(\tilde{\tau}_2) > \epsilon n) \\ &\leq \mathbf{P}(S_1(\tilde{\tau}_2) - S_1(t_0^j) - (S_j(\tilde{\tau}_2) - S_j(t_0^j)) > \epsilon n) \\ &\leq Ce^{-\gamma n}, \quad \forall n > \tilde{\tau}_2\end{aligned} \tag{130}$$

as desired. ■

Definition 8: Define $\tilde{\tau}_3^j$ as the smallest integer such that $S_j(n) \geq -\log c$ for all $n \geq \tilde{\tau}_3^j$. We also define $\tilde{\tau}_3 = \max_j \tilde{\tau}_3^j$.

Definition 9: $\tilde{n}_3 \triangleq \tilde{\tau}_3 - \tilde{\tau}_2$ denotes the total amount of time between $\tilde{\tau}_2$ and $\tilde{\tau}_3$.

Lemma 15: For every fixed $\epsilon > 0$, there exists $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}(\tilde{n}_3 > n) < Ce^{-\gamma n}, \quad \forall n > -\epsilon \log c / D(f_1||g_1). \tag{131}$$

Proof: By substituting $t = \tilde{\tau}_2$ in Lemma 13 we have:

$$\mathbf{P}(S_1(\tilde{\tau}_2) < -\log c - n\epsilon) < C_1 e^{-\gamma_1 n} \quad \forall n > \tilde{\tau}_2 \tag{132}$$

for some C_1, γ_1. By applying Lemma 14, we have:

$$\begin{aligned}\mathbf{P}(S_j(\tilde{\tau}_2) < -\log c - n\epsilon) < C_2 e^{-\gamma_2 n} \\ \forall n > \tilde{\tau}_2, j = 1, 2, \cdots, m\end{aligned} \tag{133}$$

for some $C_2, \gamma_2 > 0$.

Let N_3^j denote that total number of observations, taken from cell j between $\tilde{\tau}_2$ and $\tilde{\tau}_3^j$. Since $\tilde{n}_3 \leq \sum N_3^j$, it suffices to show that $\mathbf{P}(N_3^j > n)$ decays exponentially with n. Note that

$$\begin{aligned}&\mathbf{P}(N_3^j > n) \\ &\leq \mathbf{P}\left(S_j(\tilde{\tau}_2) < -\log c - n\frac{D(f_1||g_1)}{2}\right) \\ &\quad + \mathbf{P}\left(N_3^j > n | S_j(\tilde{\tau}_2) \geq -\log c - n\frac{D(f_1||g_1)}{2}\right).\end{aligned} \tag{134}$$

By (133) it remains to show that the second term decays exponentially with n. Let $t_1, t_2, \cdots$ denote the time indices when cell j is observed between $\tilde{\tau}_2$ and $\tilde{\tau}_3^j$. Then,

$$\begin{aligned}&\mathbf{P}(N_3^j > n | S_j(\tilde{\tau}_2) \geq -\log c - n\frac{D(f_1||g_1)}{2}) \\ \leq &\mathbf{P}\left(\inf_{r>n} \sum_{i=1}^{r} l_j(t_i) < n\frac{D(f_1||g_1)}{2}\right) \\ \leq &\mathbf{P}\left(\sum_{i=1}^{r} \tilde{l}_j(t_i) > r\frac{D(f_1||g_1)}{2}\right).\end{aligned}$$

Applying the Chernoff bound and using the i.i.d. property of $\tilde{l}_j(t_i)$ across time we have

$$\mathbf{P}\left(\sum_{i=1}^{r} \tilde{l}_j(t_i) > r\frac{D(f_1||g_1)}{2}\right) < C_3 e^{-\gamma n} \tag{135}$$

for some C_3, γ_3 which completes the proof. ■

C. The Asymptotic Lower Bound on the Bayes Risk

In this appendix, we show that the asymptotic Bayes risk that can be achieved by any policy is lower bounded by $-c\log c/I^*$.

Lemma 16: Assume that $\alpha_j(\Gamma) = O(-c\log c)$ for all $j = 1, ..., M$. Let $0 < \epsilon < 1$. Then:

$$\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c \mid \Gamma\right) = O(-c^{\epsilon}\log c)\ , \quad (136)$$

for all $m = 1, ..., M$.

Proof: Note that:

$$\begin{aligned}&\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c|\Gamma\right)\\&= \mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right)\\&+\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\ ,\ \delta \neq m|\Gamma\right)\\&\leq \mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right) + \alpha_m(\Gamma),\end{aligned} \quad (137)$$

where $\alpha_m(\Gamma) = O(-c\log c)$ by assumption. In what follows, we upper bound

$$\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right).$$

Similar to [2, Lemma 4] we can show that for all $j \neq m$ there exists $G > 0$ such that:

$$\begin{aligned}&-Gc\log c \geq \mathbf{P}_j\left(\delta \neq j|\Gamma\right) \geq \mathbf{P}_j\left(\delta = m|\Gamma\right)\\&\geq \mathbf{P}_j\left(\Delta S_{m,j}(\tau) \leq -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right)\\&\geq c^{1-\epsilon}\mathbf{P}_m\left(\Delta S_{m,j}(\tau) < -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right)\ ,\end{aligned} \quad (138)$$

where the last inequality holds by changing the measure as in [2, Lemma 4]. Thus,

$$\begin{aligned}&\mathbf{P}_m\left(\Delta S_{m,j}(\tau) < -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right)\\&\qquad = O\left(-c^{\epsilon}\log c\right) \quad \forall j \neq m\ .\end{aligned} \quad (139)$$

As a result,

$$\begin{aligned}&\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right)\\&\leq \sum_{j\neq m}\mathbf{P}_m\left(\Delta S_{m,j}(\tau) < -(1-\epsilon)\log c\ ,\ \delta = m|\Gamma\right)\\&\qquad = O\left(-c^{\epsilon}\log c\right)\ .\end{aligned} \quad (140)$$

Finally,

$$\mathbf{P}_m\left(\Delta S_m(\tau) < -(1-\epsilon)\log c|\Gamma\right) = O\left(-c^{\epsilon}\log c\right). \quad (141)$$

■

Lemma 17: Assume that

$$D(g_m||f_m) \geq \frac{1}{\sum_{j\neq m}\frac{1}{D(f_j||g_j)}}. \quad (142)$$

Then, the function:

$$d(t) \triangleq t\left[D(g_m||f_m) + \frac{\frac{n}{t}-1}{\sum_{j\neq m}\frac{1}{D(f_j||g_j)}}\right] \quad (143)$$

is monotonically increasing with t for $0 \leq t \leq n$.

Proof: Differentiation $d(t)$ with respect to t yields:

$$\frac{\partial d(t)}{\partial t} = D(g_m||f_m) - \frac{1}{\sum_{j\neq m}\frac{1}{D(f_j||g_j)}} \geq 0\ ,$$

which completes the proof. ■

For the next lemma we define

$$j^*(t) \triangleq \arg\min_{j\neq m} N_j(t)D(f_j||g_j), \quad (144)$$

and

$$W_m^*(t) \triangleq \sum_{i=1}^{t}\tilde{\ell}_m(i)\mathbf{1}_m(i) - \sum_{i=1}^{t}\tilde{\ell}_{j^*(t)}(i)\mathbf{1}_{j^*(t)}(i), \quad (145)$$

which is a sum of zero-mean r.v.

Lemma 18: For every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_m\left(\max_{1\leq t\leq n} W_m^*(t) \geq n\epsilon|\Gamma\right) \leq Ce^{-\gamma n} \quad (146)$$

for all $m = 1, ..., M$ and for any policy Γ.

Proof: We upper bound (146) by summing over any possible values that $N_m(t), N_{j^*(t)}(t)$ can take and using the Chernoff bound:

$$\begin{aligned}&\mathbf{P}_m\left(\max_{1\leq t\leq n} W_m^*(t) \geq n\epsilon|\Gamma\right)\\&= \sum_{t=1}^{n}\sum_{i=0}^{t}\sum_{j=0}^{t}\\&\quad\mathbf{P}_m\left(\sum_{r=1}^{t}\tilde{\ell}_m(r)\mathbf{1}_m(r) + \sum_{r=1}^{t}-\tilde{\ell}_{j^*(t)}(r)\mathbf{1}_{j^*(t)}(r) \geq n\epsilon,\right.\\&\qquad\qquad\left. N_m(t) = i, N_{j^*(t)} = j|\Gamma\right)\\&\leq \sum_{t=1}^{n}\sum_{i=0}^{t}\sum_{j=0}^{t}\left[\mathbf{E}_m\left(e^{s(\tilde{\ell}_m(1)-\epsilon/2)}\right)\right]^i \times\\&\qquad\left[\mathbf{E}_m\left(e^{s(-\tilde{\ell}_{j^*(t)}(1)-\epsilon/2)}\right)\right]^j \times\\&\qquad\exp\left\{-s\frac{\epsilon}{2}(2n-i-j)\right\}\ ,\end{aligned} \quad (147)$$

for all $s > 0$.

Since $\mathbf{E}_m(\tilde{\ell}_m(1) - \epsilon/2) = -\epsilon/2 < 0$ and $\mathbf{E}_m(-\tilde{\ell}_{j^*(t)}(1) - \epsilon/2) = -\epsilon/2 < 0$ are strictly negative, using a similar argument as at the end of the proof of Lemma 3, there exist $s > 0$ and $\gamma' > 0$ such that $\mathbf{E}_m\left(e^{s(\tilde{\ell}_m(1)-\epsilon/2)}\right)$, $\mathbf{E}_m\left(e^{s(-\tilde{\ell}_{j^*(t)}(1)-\epsilon/2)}\right)$ and $e^{-s\epsilon/2}$ are strictly less than $e^{-\gamma'} < 1$. Since $2n - i - j \geq 0$, there exist $C > 0$ and $\gamma > 0$, such that summing over t, i, j yields (146). ■

Lemma 19: For any fixed $\epsilon > 0$,

$$\mathbf{P}_m\left(\max_{1\leq t\leq n}\Delta S_m(t) \geq n\left(I_m + \epsilon\right)\ |\ \Gamma\right) \to 0 \quad \text{as } n \to \infty\ , \quad (148)$$

for all $m=1,...,M$ and for any policy Γ.

Proof: We next show exponential decay of (148) (which is stronger than the polynomial decay shown under the binary composite hypothesis testing case in [2, Lemma 5]). Let

$$\Delta S_m^*(t) \triangleq S_m(t) - S_{j^*(t)}(t).$$

Note that $\Delta S_m(t) \leq \Delta S_m^*(t)$ for all m and t. As a result,

$$\begin{aligned}
&\mathbf{P}_m\left(\max_{1\leq t\leq n} \Delta S_m(t) \geq n\left(I_m+\epsilon\right)|\Gamma\right) \\
&\leq \mathbf{P}_m\left(\max_{1\leq t\leq n} \Delta S_m^*(t) \geq n\left(I_m+\epsilon\right)|\Gamma\right).
\end{aligned} \tag{149}$$

We next prove the lemma for the case where $I_m = F_m(K)$. Proving the lemma for the cases where $I_m = D(g_m||f_m) + F_m(K-1)$ applies with minor modifications.

Note that:

$$\begin{aligned}
\Delta S_m^*(t) &= W_m^*(t) + N_m(t) D(g_m||f_m) \\
&\quad + N_{j^*(t)}(t) D(f_{j^*(t)}||g_{j^*(t)}) \\
&\leq W_m^*(t) + N_m(t) \cdot \frac{1}{\sum_{j\neq m} 1/D(f_j||g_j)} \\
&\quad + N_{j^*(t)}(t) D(f_{j^*(t)}||g_{j^*(t)}).
\end{aligned} \tag{150}$$

Since that $j^*(t) = \arg\min_{j\neq m} N_j(t) D(f_j||g_j)$ and $Kt - N_m(t)$ is the total number of observations taken from $M-1$ cells $j\neq m$, we have:

$$\sum_{j\neq m} \frac{N_{j^*(t)} D(f_{j^*(t)}||g_{j^*(t)})}{D(f_j||g_j)} \leq Kt - N_m(t) \leq Kn - N_m(t). \tag{151}$$

Hence,

$$\begin{aligned}
\Delta S_m^*(t) &\leq W_m^*(t) + Kn \tfrac{1}{\sum_{j\neq m} 1/D(f_j||g_j)} \\
&= W_m^*(t) + nI_m \ .
\end{aligned} \tag{152}$$

Therefore,

$$\Delta S_m^*(t) \geq n\left(I_m+\epsilon\right)$$

implies

$$W_m^*(t) \geq n\epsilon.$$

By Lemma 18 we have:

$$\begin{aligned}
&\mathbf{P}_m\left(\max_{1\leq t\leq n} \Delta S_m(t) \geq n\left(I_m+\epsilon\right)\right) \\
&\leq \mathbf{P}_m\left(\max_{1\leq t\leq n} W_m^*(t) \geq n\epsilon\right) \\
&\leq Ce^{-\gamma n} \to 0 \text{ as } n\to\infty.
\end{aligned} \tag{153}$$

■

Finally, we show that the Bayes risk cannot be made smaller than $\frac{-c\log(c)}{I_m}$:

Lemma 20: Any policy Γ that satisfies $R_j(\Gamma) = O(-c\log c)$ for all $j=1,...,M$ must satisfy:

$$R_m(\Gamma) \geq -\left(1+o(1)\right) \frac{c\log(c)}{I_m} \ . \tag{154}$$

for all $m=1,...,M$.

Proof: For any $\epsilon>0$ let $n_c = -(1-\epsilon)\dfrac{\log c}{I_m+\epsilon}$. Note that

$$\begin{aligned}
&\mathbf{P}_m\left(\tau \leq n_c \mid \Gamma\right) \\
&= \mathbf{P}_m\left(\tau \leq n_c\ ,\ \Delta S_m(\tau) \geq -\left(1-\epsilon\right)\log c \mid \Gamma\right) \\
&+ \mathbf{P}_m\left(\tau \leq n_c\ ,\ \Delta S_m(\tau) < -\left(1-\epsilon\right)\log c \mid \Gamma\right) \\
&\leq \mathbf{P}_m\left(\max_{t\leq n_c} \Delta S_m(t) \geq -\left(1-\epsilon\right)\log c \mid \Gamma\right) \\
&+ \mathbf{P}_m\left(\Delta S_m(\tau) < -\left(1-\epsilon\right)\log c \mid \Gamma\right).
\end{aligned} \tag{155}$$

Both terms on the RHS approaches zero as $c\to 0$ by Lemmas 16, 19. Hence,

$$\begin{aligned}
\mathbf{E}_m(\tau|\Gamma) &\geq \sum_{n=n_c+1}^{\infty} n\mathbf{P}_m\left(\tau = n|\Gamma\right) \\
&\geq n_c \mathbf{P}_m\left(\tau \geq n_c+1|\Gamma\right) \to n_c \quad \text{as} \quad c\to 0
\end{aligned} \tag{156}$$

Since $\epsilon > 0$ is arbitrarily small we have $\mathbf{E}_m(\tau|\Gamma) \geq -\left(1+o(1)\right)\log(c)/I_m$. As a result, $R_m(\Gamma) \geq c\mathbf{E}_m(\tau|\Gamma) \geq -\left(1+o(1)\right) c\log(c)/I_m$. ■

Appendix C: proof of Theorem 2

We now focus on proving asymptotic optimality for $L>1$, and $K=1$. For $L>1$, we define τ_1 as the smallest integer such that $S_m(n) > S_j(n)$ for all $m\in D$, $j\neq D$ and $n\geq\tau_1$. Note that when $K=1$ and $n\geq\tau_1$ the decision maker always probe the consistent cell (target or not depending on the order of G_D and F_D) for making the difference between the L^{th} and $(L+1)^{th}$ largest sum LLRs greater than the threshold $-\log c$. As a result, the decision maker can always balance the detection time so that the difference between the largest sum LLR and the sum LLRs of any other cell exceeds the threshold $-\log c$ approximately at the same time as $c\to 0$. Thus, proving the asymptotic optimality of DGFi for $L>1$ and $K=1$ follows similar arguments as in the balanced case in the proof of Theorem 1 given in Appendix B, and we focus here only on the key modifications. Let

$$\Delta S_{\mathcal{D}}(n) \triangleq \min_{m\in\mathcal{D}, j\notin\mathcal{D}} \Delta S_{m,j}(n), \tag{157}$$

where $\Delta S_{m,j}(n)$ is defined in (44). Without loss of generality we prove the theorem when set $\mathcal{D}$ contains all the targets. We define

$$\tilde{\ell}_k(i) = \begin{cases} \ell_k(i) - D(g_k||f_k)\ , \text{ if } k\in\mathcal{D}, \\ \\ \ell_k(i) + D(f_k||g_k)\ , \text{ if } k\notin\mathcal{D}, \end{cases} \tag{158}$$

which is a zero-mean r.v.

We start with sowing the upper bound on the Bayes risk obtained by DGFi. Similar to Lemma 2, we can show that the error probability under DGFi is $O(c)$. Specifically, we can show that the error probability is upper bounded by:

$$P_e \leq (M-L)L\cdot c\ . \tag{159}$$

We can show this by letting $\alpha_{\mathcal{D}} = \mathbf{P}_{\mathcal{D}}(\delta \neq \mathcal{D})$ and $\alpha_{\mathcal{D},j} = \mathbf{P}_{\mathcal{D}}(j \in \delta)$ for all $j \notin \mathcal{D}$, where the subscript D denotes the measure when set D contains all the targets. Thus, $\alpha_{\mathcal{D}} \leq \sum_{j \notin \mathcal{D}} \alpha_{\mathcal{D},j}$. By the stopping rule, accepting $j \in \delta$ implies $\Delta S_{j,m} \geq -\log c$ for some $m \in \mathcal{D}$. Hence, for all $j \notin \mathcal{D}$ we have:

$$\begin{aligned} \alpha_{\mathcal{D},j} &= \mathbf{P}_{\mathcal{D}}\left(j \in \mathcal{D}\right) \\ &\leq \textstyle\sum_{m \in \mathcal{D}} \mathbf{P}_{\mathcal{D}}\left(\Delta S_{j,m}(\tau) \geq -\log c\right) \\ &\leq \sum_{m \in \mathcal{D}} c \mathbf{P}_{\mathcal{D} \cup j \setminus m}\left(\Delta S_{j,m}(\tau) \geq -\log c\right) \leq L \cdot c, \end{aligned} \tag{160}$$

where we changed the measure in the second inequality. As a result,

$$\alpha_{\mathcal{D}} \leq \sum_{j \notin \mathcal{D}} \alpha_{\mathcal{D},j} \leq (M-L) L \cdot c,$$

which yields (159).

Here we consider the case where $I_{\mathcal{D}} = G_{\mathcal{D}}$, the case $I_{\mathcal{D}} = F_{\mathcal{D}}$ applies with minor modifications. For showing that τ_1 is sufficiently small we need to show first the following Lemmas:

Lemma 21: For all $j \notin D$, $\forall 0 < q < 1$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}(N_j(n) > qn) < Ce^{-\gamma n} \tag{161}$$

Proof: For each j, define $t^j(n)$ as the time when cell j is observed for the n^{th} time. By DGFi selection rule, if cell j is observed at time t, then there exists $m \in \mathcal{D}$ such that $S_j(t) \geq S_m(t)$. Hence,

$$\begin{aligned} &\mathbf{P}_{\mathcal{D}}(N_j(n) > qn) \\ &\leq \sum_{t=1}^{n} \mathbf{P}_{\mathcal{D}}(N_j(t) > qn, \exists m \in \mathcal{D} : S_j(t) > S_m(t)) \\ &\qquad \times \mathbf{P}_{\mathcal{D}}(t^j(\lceil qn \rceil) = t). \end{aligned} \tag{162}$$

It suffices to show that there exist constants C, γ such that

$$\mathbf{P}_{\mathcal{D}}(N_j(t) > qn, \exists m \in \mathcal{D} : S_j(t) > S_m(t)) \leq Ce^{-\gamma n} \tag{163}$$

for all $t \leq n$.

First we have

$$\begin{aligned} &\mathbf{P}_{\mathcal{D}}(N_j(t) > qn, \exists m \in \mathcal{D} : S_j(t) > S_m(t)) \\ &\leq \sum_{m \in \mathcal{D}} \mathbf{P}_{\mathcal{D}}(N_j(t) > qn, S_j(t) > S_m(t)). \end{aligned} \tag{164}$$

Fix m, then we have

$$\begin{aligned} &\mathbf{P}_{\mathcal{D}}(N_j(t) > qn, S_j(t) > S_m(t)) \\ &\leq \sum_{r=\lceil qn \rceil}^{n} \sum_{k=0}^{n} \mathbf{P}_{\mathcal{D}}\left(\sum_{i=1}^{n} \ell_j(i) + \sum_{k=1}^{k} -\ell_m(i) \geq 0\right) \\ &\leq C_m e^{-\gamma_m n}. \end{aligned} \tag{165}$$

The last inequality can be shown using the Chernoff bound as in Lemma 3.

To show (163), we let $C = \sum_m C_m, \gamma = \min_m \gamma_m$, which completes the proof. ∎

Lemma 22: For all $m \in D$, and $\epsilon > 0$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}\left(N_m(n) > \frac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n\right) \leq Ce^{-\gamma n} \tag{166}$$

Proof: For each m, define $t^m(n)$ as the time when cell m is observed for the n^{th} time. By DGFi selection rule, if cell m is observed at time t, either there exists $j \notin D$ such that $S_j(n) > S_m(n)$ or $S_{m'}(n) > S_m(n)$ for all $m' \in D$. Similar to (162), it suffices to show that

$$\begin{aligned} &\mathbf{P}_{\mathcal{D}}\left(N_m(t) > \tfrac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \exists j \notin \mathcal{D} : S_j(t) > S_m(t)\right) \\ &\leq Ce^{-\gamma n} \end{aligned} \tag{167}$$

and

$$\begin{aligned} &\mathbf{P}_{\mathcal{D}}\left(N_m(t) > \tfrac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \forall m' \in \mathcal{D} : S_{m'}(t) > S_m(t)\right) \\ &\leq Ce^{-\gamma n} \end{aligned} \tag{168}$$

for all $t < n$.

Since (167) can be shown similarly as in (163), it remains to show (168). By the definition of $G_{\mathcal{D}}$, if $N_m(t) > \frac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n$, there exists $m' \in D$ and $\epsilon' > 0$ such that $N_{m'}(t) < \frac{G_{\mathcal{D}}}{D(g'_m||f'_m) + \epsilon'} \cdot t$. Hence,

$$\begin{aligned} &\mathbf{P}_{\mathcal{D}}\Bigg(N_m(t) > \frac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \\ &\qquad\qquad \forall m' \in D : S_{m'}(t) > S_m(t)\Bigg) \\ &\leq \sum_{m' \in D} \mathbf{P}_{\mathcal{D}}\Bigg(N_m(t) > \frac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \\ &\quad S_{m'}(t) > S_m(t), N'_m(t) < \frac{G_{\mathcal{D}}}{D(g'_m||f'_m) + \epsilon'} \cdot t\Bigg). \end{aligned} \tag{169}$$

Fix m', and let $s_1 = \frac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon}$, $s_2 = \frac{G_{\mathcal{D}}}{D(g_{m'}||f_{m'}) + \epsilon}$. Then, we have

$$\begin{aligned} &\mathbf{P}_{\mathcal{D}}\Bigg(N_m(t) > \frac{G_{\mathcal{D}}}{D(g_m||f_m) - \epsilon} \cdot n, \\ &S_{m'}(t) > S_m(t), N'_m(t) < \frac{G_{\mathcal{D}}}{D(g_{m'}||f_{m'}) + \epsilon'} \cdot t\Bigg) \\ &\leq \sum_{r=\lceil s_1 n \rceil}^{n} \sum_{k=0}^{\lfloor s_2 t \rfloor} \mathbf{P}_{\mathcal{D}}\left(\sum_{i=1}^{r} -\ell_m(i) + \sum_{i=1}^{k} \ell_{m'}(i) \geq 0\right) \\ &\leq \sum_{r=\lceil s_1 n \rceil}^{n} \sum_{k=0}^{\lfloor s_2 t \rfloor} \mathbf{P}_{\mathcal{D}}\Bigg(\sum_{i=1}^{r} D(g_m||f_m) - \epsilon - \ell_m(i) \\ &+ \sum_{i=1}^{k} \ell_{m'}(i) - D(g_{m'}||f_{m'}) - \epsilon' \geq 0\Bigg) \\ &\leq \sum_{r=\lceil s_1 n \rceil}^{n} \sum_{k=0}^{\lfloor s_2 t \rfloor} \left[\mathbf{E}_{\mathcal{D}}\left(e^{s(-\tilde{\ell}_m(1) - \epsilon)}\right)\right]^r \left[\mathbf{E}_{\mathcal{D}}\left(e^{s(\ell_{m'}(1) - \epsilon')}\right)\right]^k \\ &\leq C_{m'} e^{-\gamma_{m'} n} \end{aligned} \tag{170}$$

The last inequality can be shown using the Chernoff bound as in Lemma 3. To show (169), we let $C = \sum_{m'} C_{m'}, \gamma = \min_{m'} \gamma_{m'}$, which completes the proof. ∎

Lemma 23: For all $m \in D$, $\forall \epsilon > 0$, there exist $C, \gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}\left(N_m(n) < (\frac{G_{\mathcal{D}}}{2D(g_m||f_m)})n\right) \leq Ce^{-\gamma n}. \qquad (171)$$

Proof: By choosing q_j and ϵ'_m in Lemma 21 and Lemma 22 such that $\sum_j q_j + \sum'_m \epsilon'_m = \frac{G_{\mathcal{D}}}{2D(g_m||f_m)}$, we have

$$\begin{aligned}&\mathbf{P}_{\mathcal{D}}\left(N_m(n) < (\frac{G_{\mathcal{D}}}{2D(g_m||f_m)})n\right)\\ &\leq \sum_{j\notin\mathcal{D}} \mathbf{P}_{\mathcal{D}}\left(N_j(n) > q_j n\right)\\ &+ \sum_{m'\in\mathcal{D}} \mathbf{P}_{\mathcal{D}}\left(N'_m(n) > (\frac{G_{\mathcal{D}}}{D(g'_m||f'_m)} + \epsilon'_m)n\right) \leq C_{m'}e^{-\gamma n}\end{aligned} \qquad (172)$$

as desired. ∎

Next, similar to Lemma 4, we can show that the probability that τ_1 is greater than n decreases exponentially with n. This result is used when evaluating the asymptotic expected search time to show that it is not affected by τ_1. We can show this by noting that

$$\begin{aligned}\mathbf{P}_{\mathcal{D}}\left(\tau_1 > n\right) &\leq \mathbf{P}_{\mathcal{D}}\left(\max_{j\notin\mathcal{D}, m\in\mathcal{D}} \sup_{t\geq n} \left(S_j(t) - S_m(t)\right) \geq 0\right)\\ &\leq \sum_{j\notin\mathcal{D}, m\in\mathcal{D}} \sum_{t=n}^{\infty} \mathbf{P}_{\mathcal{D}}\left(S_j(t) \geq S_m(t)\right) \ .\end{aligned} \qquad (173)$$

Following (173), it suffices to show that $\mathbf{P}_{\mathcal{D}}\left(S_j(n) \geq S_m(n)\right)$ decays exponentially with n. Note that

$$\begin{aligned}&\mathbf{P}_{\mathcal{D}}\left(S_j(n) \geq S_m(n)\right)\\ &\leq \mathbf{P}_{\mathcal{D}}\left(S_j(n) \geq S_m(n), N_m(n) \geq (\frac{G_{\mathcal{D}}}{2D(g_m||f_m)})n\right)\\ &\quad + \mathbf{P}_{\mathcal{D}}\left(N_m(n) < (\frac{G_{\mathcal{D}}}{2D(g_m||f_m)})n\right)\end{aligned} \qquad (174)$$

The first term decays exponentially with n by Lemma 3 (with minor modifications). The second term decays exponentially with n by Lemma 23.

Note that we obtained that the expectation of τ_1 is bounded, and we can use similar arguments as in the balanced case of Theorem 1 in Appendix B to obtain the detection rate I_D for $n \geq \tau_1$. Combining these results yields that the expected detection time τ under the DGFi policy is upper bounded by:

$$\mathbf{E}_{\mathcal{D}}(\tau) \leq -\left(1 + o(1)\right) \frac{\log(c)}{I_{\mathcal{D}}} \ , \qquad (175)$$

for $m = 1, ..., M$.

Finally, showing that the asymptotic Bayes risk is lower bounded by $-c\log c/I^*_L$ follows a similar outline as in Appendix B. Specifically, similar to Lemma 16, if $\alpha_{\mathcal{D}}(\Gamma) = O(-c\log c)$ for all $\mathcal{D}$, and we let $0 < \epsilon < 1$, then:

$$\mathbf{P}_{\mathcal{D}}\left(\Delta S_m(\tau) < -\left(1-\epsilon\right)\log c \mid \Gamma\right) = O(-c^{\epsilon}\log c) \ , \qquad (176)$$

for all $\mathcal{D}$ and $m \in \mathcal{D}$. Then, we define:

$$j^*(t) \triangleq \arg\min_{j\notin\mathcal{D}} N_j(t)D(f_j||g_j), \qquad (177)$$

$$m^*(t) \triangleq \arg\min_{m\in\mathcal{D}} N_{m^*(t)}(t)D(g_m||f_m), \qquad (178)$$

and

$$W^*_{\mathcal{D}}(t) \triangleq \sum_{i=1}^{t} \tilde{\ell}_{m^*(t)}(i)\mathbf{1}_{m^*(t)}(i) - \sum_{i=1}^{t} \tilde{\ell}_{j^*(t)}(i)\mathbf{1}_{j^*(t)}(i), \qquad (179)$$

where $W^*_{\mathcal{D}}(t)$ is a sum of zero-mean r.v. Using these definitions, similar to Lemma 18, we can show that for every fixed $\epsilon > 0$ there exist $C > 0$ and $\gamma > 0$ such that

$$\mathbf{P}_{\mathcal{D}}\left(\max_{1\leq t\leq n} W^*_{\mathcal{D}}(t) \geq n\epsilon|\Gamma\right) \leq Ce^{-\gamma n} \qquad (180)$$

for all $\mathcal{D}$ and for any policy Γ.

Next, similar to Lemma 19 we can show that for any fixed $\epsilon > 0$,

$$\mathbf{P}_{\mathcal{D}}\left(\max_{1\leq t\leq n} \Delta S_{\mathcal{D}}(t) \geq n\left(I_{\mathcal{D}} + \epsilon\right) \mid \Gamma\right) \to 0 \quad \text{as } n \to \infty \ , \qquad (181)$$

for all $\mathcal{D}$ and for any policy Γ.

Finally, similar to Lemma 20, we can show that any policy Γ that satisfies $R_{\mathcal{D}}(\Gamma) = O(-c\log c)$ for all $\mathcal{D}$ must satisfy:

$$R_{\mathcal{D}}(\Gamma) \geq -\left(1 + o(1)\right) \frac{c\log(c)}{I_{\mathcal{D}}} \ . \qquad (182)$$

for all $\mathcal{D}$.

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